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Direct product of infinite family of B-Algebras

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Abstract. The concept of the direct product of finite family of B-algebras is introduced by Lingcong and Endam [J. A. V. Lingcong and J. C. Endam, Direct product of B-algebras, *Int. J. Algebra*, 10(1):33-40, 2016.]. In this paper, we introduce the concept of the direct product of infinite family of B-algebras, we call the external direct product, which is a generalization of the direct product in the sense of Lingcong and Endam. Also, we introduce the concept of the weak direct product of B-algebras. Finally, we provide several fundamental theorems of (anti-)B-homomorphisms in view of the external direct product B-algebras.

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1. Introduction and Preliminaries

Imai and Iséki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras [8, 9]. In 2002, Neggers and Kim [17] constructed a new algebraic structure. They took some properties from BCI and BCK-algebras be called a B-algebra. A B-algebra $X = (X; *, 0)$ is an algebra of type $(2, 0)$, that is, a nonempty set X together with a binary operation $*$ and a constant 0 satisfying some axioms.

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B-algebras and some of their properties have been discussed, e.g., some axiomatizations of B-algebras by Walendziak in 2006 [22], medial B-algebras by Kim in 2014 [11], fuzzy order relative to fuzzy B-algebras by Gonzaga, Jr. and Vilela in 2019 [7], B-ideals in a topological B-algebra and the uniform B-topological space by Belleza and Vilela in 2020 [3]. In 2021, Gan et al. [6] guaranteed the existences of both direct limits and inverse limits in the categories of quantum B-algebras with morphisms of exact ones or spectral ones, etc.

The concept of the direct product [19] was first defined in the group and obtained some properties. For example, a direct product of the group is also a group, and a direct product of the abelian group is also an abelian group. Then, direct product groups are applied to other algebraic structures. In 2016, Lingcong and Endam [12] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra, and some of the related properties were investigated. Also, they introduced two canonical mappings of the direct product of B-algebras and we obtained some of their properties [13]. In the same year, Endam and Teves [5] defined the direct product of BF-algebras, 0-commutative BF-algebras, and BF-homomorphism and obtained related properties. In 2018, Abebe [1] introduced the concept of the finite direct product of BRK-algebras and proved that the finite direct product of BRK-algebras is a BRK-algebra. In 2019, Widianto et al. [23] defined the direct product of BG-algebras, 0-commutative BG-algebras, and BG-homomorphism, including related properties of BG-algebras. In 2020, Setiani et al. [19] defined the direct product of BP-algebras, which is equivalent to B-algebras. They obtained the relevant property of the direct product of BP-algebras and then defined the direct product of BP-algebras as applied to finite sets of BP-algebras, finite family 0-commutative BP-algebras, and finite family BP-homomorphisms. In 2021, Kavitha and Gowri [10] defined the direct product of GK algebra. They derived the finite form of the direct product of GK algebra and function as well. They investigated and applied the concept of the direct product of GK algebra in GK function and GK kernel and obtained interesting results.

In this paper, we introduce the concept of the direct product of infinite family of B-algebras, we call the external direct product, which is a generalization of the direct product in the sense of Lingcong and Endam [12]. Moreover, we introduce the concept of the weak direct product of B-algebras. Finally, we discuss several (anti-)B-homomorphism theorems in view of the external direct product B-algebras.

First of all, we start with the definitions and examples of B-algebras as well as other relevant definitions for the study in this paper as follows:

Definition 1. *[17] A B-algebra $P = (P; *, 0)$ is an algebra of type $(2, 0)$, that is, a nonempty set P together with a binary operation $*$ and a constant 0 satisfying the following axioms:*

$$(\forall x \in P)(x * x = 0), \tag{B-1}$$

$$(\forall x \in P)(x * 0 = x), \tag{B-2}$$

$$(\forall x, y, z \in P)((x * y) * z = x * (z * (0 * y))). \tag{B-3}$$

Example 1. *Let $P = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table as follows:*

$*$	0	1	2	3	4	5
0	0	1	4	5	2	3
1	1	0	5	4	3	2
2	2	3	0	1	4	5
3	3	2	1	0	5	4
4	4	5	2	3	0	1
5	5	4	3	2	1	0

Then $P = (P; *, 0)$ is a B-algebra.

Definition 2. *[17] A B-algebra $(P; *, 0)$ is said to be commutative if*

$$(\forall x, y \in P)(x * (0 * y) = y * (0 * x)).$$

Example 2. *From Example 1, we have $P = (P; *, 0)$ is commutative.*

Definition 3. *[16] A nonempty subset N of a B-algebra $P = (P; *, 0)$ is said to be a B-subalgebra of P if*

$$(\forall x, y \in N)(x * y \in N).$$

Definition 4. *[2] A nonempty subset I of a B-algebra $P = (P; *, 0)$ is called a B-ideal of P if it satisfies following conditions:*

$$0 \in I, \tag{BI-1}$$
$$(\forall x, y \in P)((x * y \in I, y \in I) \Rightarrow x \in I). \tag{BI-2}$$

By (B-1), we have every B-subalgebra of a B-algebra satisfies (BI-1).

Definition 5. *[16] A nonempty subset N of a B-algebra $P = (P; *, 0)$ is said to be normal of P if*

$$(\forall x, y, a, b \in P)(x * y, a * b \in N \Rightarrow (x * a) * (y * b) \in N).$$

Theorem 1. *[16] Every normal subset of a B-algebra is a B-subalgebra and hence, it satisfies* (BI-1)*.*

The concept of B-homomorphisms was also introduced by Neggers and Kim [16]. Let $A = (A; *_A, 0_A)$ and $B = (B; *_B, 0_B)$ be B-algebras. A map $\varphi : A \to B$ is called a *B-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(x) *_B \varphi(y)),$$

an *anti-B-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(y) *_B \varphi(x)).$$

The *kernel* of φ, denoted by $\ker\varphi$, is defined to be the $\{x \in A \mid \varphi(x) = 0_B\}$. The $\ker\varphi$ is a normal B-subalgebra of A, and $\ker\varphi = \{0_A\}$ if and only if φ is injective. A (anti-)B-homomorphism φ is called a (anti-)B-monomorphism, (anti-)B-epimorphism, or (anti-)B-isomorphism if φ is injective, surjective, or bijective, respectively.

Theorem 2. *[16] Let N be a nonempty subset of a B-algebra $P = (P; *, 0)$. Then the following statements are equivalent:*

(i) N is a B-subalgebra of P.

(ii) $x * (0 * y), 0 * y \in N$ for all $x, y \in N$.

2. External Direct Product of B-algebras

Lingcong and Endam [12] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra, and some of the related properties were investigated as follows:

Definition 6. *[12] Let $(P_i; *_i)$ be an algebra for each $i \in \{1, 2, ..., k\}$. Define the direct product of algebras $P_1, P_2, ..., P_k$ to be the structure $(\prod_{i=1}^{k} P_i; \otimes)$, where*

$$\prod_{i=1}^{k} P_i = P_1 \times P_2 \times ... \times P_k = \{(p_1, p_2, ..., p_k) \mid p_i \in P_i \ \forall i = 1, 2, ..., k\}$$

and whose operation $\otimes$ is given by

$$(p_1, p_2, ..., p_k) \otimes (q_1, q_2, ..., q_k) = (p_1 *_1 q_1, p_2 *_2 q_2, ..., p_k *_k q_k)$$

for all $(p_1, p_2, ..., p_k), (q_1, q_2, ..., q_k) \in \prod_{i=1}^{k} P_i$.

Theorem 3. *[12] $(P_i; *_i, 0_i)$ is a B-algebra for all $i = 1, 2, ..., k$ if and only if*

$$(\prod_{i=1}^{k} P_i; \otimes, (0_1, 0_2, ..., 0_k))$$

is a B-algebra, where the binary operation $\otimes$ is defined in Definition 6.

Now, we extend the concept of the direct product to infinite family of B-algebras and provide some of its properties.

Definition 7. *Let P_i be a nonempty set for each $i \in I$. Define the external direct product of sets P_i for all $i \in I$ to be the set $\prod_{i \in I} P_i$, where*

$$\prod_{i \in I} P_i = \{f : I \to \bigcup_{i \in I} P_i \mid f(i) \in P_i \ \forall i \in I\}.$$

For convenience, we define an element of $\prod_{i \in I} P_i$ with a function $(p_i)_{i \in I} : I \to \bigcup_{i \in I} P_i$, where $i \mapsto p_i \in P_i$ for all $i \in I$.

Definition 8. *Let $P_i = (P_i; *_i)$ be an algebra for all $i \in I$. Define the binary operation $\otimes$ on the external direct product $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes)$ as follows:*

$$(\forall (p_i)_{i\in I}, (q_i)_{i\in I} \in \prod_{i\in I} P_i)((p_i)_{i\in I} \otimes (q_i)_{i\in I} = (p_i *_i q_i)_{i\in I}). \tag{2.1}$$

We shall show that $\otimes$ is a binary operation on $\prod_{i\in I} P_i$. Let $(p_i)_{i\in I}, (q_i)_{i\in I} \in \prod_{i\in I} P_i$. Since $*_i$ is a binary operation on P_i for all $i \in I$, we have $p_i *_i q_i \in P_i$ for all $i \in I$. Then $(p_i *_i q_i)_{i\in I} \in \prod_{i\in I} P_i$ such that

$$(p_i)_{i\in I} \otimes (q_i)_{i\in I} = (p_i *_i q_i)_{i\in I}.$$

Let $(p_i)_{i\in I}, (q_i)_{i\in I}, (p'_i)_{i\in I}, (q'_i)_{i\in I} \in \prod_{i\in I} P_i$ be such that $(p_i)_{i\in I} = (q_i)_{i\in I}$ and $(p'_i)_{i\in I} = (q'_i)_{i\in I}$. We shall show that $(p_i)_{i\in I} \otimes (p'_i)_{i\in I} = (q_i)_{i\in I} \otimes (q'_i)_{i\in I}$. Then

$$p_i = q_i \text{ for all } i \in I \text{ and } p'_i = q'_i \text{ for all } i \in I.$$

Since $*_i$ is a binary operation on P_i for all $i \in I$, we have $p_i *_i p'_i = q_i *_i q'_i$ for all $i \in I$. Thus

$$\begin{aligned}(p_i)_{i\in I} \otimes (p'_i)_{i\in I} &= (p_i *_i p'_i)_{i\in I}\\ &= (q_i *_i q'_i)_{i\in I}\\ &= (q_i)_{i\in I} \otimes (q'_i)_{i\in I}.\end{aligned}$$

Hence, $\otimes$ is a binary operation on $\prod_{i\in I} P_i$.

Let $P_i = (P_i; *_i, 0_i)$ be a B-algebra for all $i \in I$. For $i \in I$, let $p_i \in P_i$. We define the function $f_{p_i} : I \to \bigcup_{i\in I} P_i$ as follows:

$$(\forall j \in I)\left(f_{p_i}(j) = \begin{cases} p_i & \text{if } j = i\\ 0_j & \text{otherwise}\end{cases}\right). \tag{2.2}$$

Then $f_{p_i} \in \prod_{i\in I} P_i$.

Lemma 1. *Let $P_i = (P_i; *_i, 0_i)$ be a B-algebra for all $i \in I$. For $i \in I$, let $p_i, q_i \in P_i$. Then $f_{p_i} \otimes f_{q_i} = f_{p_i *_i q_i}$.*

Proof. Now,

$$(\forall j \in I)\left((f_{p_i} \otimes f_{q_i})(j) = \begin{cases} p_i *_i q_i & \text{if } j = i\\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right).$$

By (B-1), we have

$$(\forall j \in I)\left((f_{p_i} \otimes f_{q_i})(j) = \begin{cases} p_i *_i q_i & \text{if } j = i\\ 0_j & \text{otherwise}\end{cases}\right).$$

By (2.2), we have $f_{p_i} \otimes f_{q_i} = f_{p_i *_i q_i}$.

The following theorem shows that the direct product of B-algebras in term of infinite family of B-algebras is also a B-algebra which is more generalized than Theorem 3.

Theorem 4. *$P_i = (P_i; *_i, 0_i)$ is a B-algebra for all $i \in I$ if and only if $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$ is a B-algebra, where the binary operation $\otimes$ is defined in Definition 8.*

Proof. Assume that $P_i = (P_i; *_i, 0_i)$ is a B-algebra for all $i \in I$.

(B-1) Let $(p_i)_{i\in I} \in \prod_{i\in I} P_i$. Since P_i satisfies (B-1), we have $p_i *_i p_i = 0_i$ for all $i \in I$. Thus

$$(p_i)_{i\in I} \otimes (p_i)_{i\in I} = (p_i *_i p_i)_{i\in I} = (0_i)_{i\in I}.$$

(B-2) Let $(p_i)_{i\in I} \in \prod_{i\in I} P_i$. Since P_i satisfies (B-2), we have $p_i *_i 0_i = p_i$ for all $i \in I$. Thus

$$(p_i)_{i\in I} \otimes (0_i)_{i\in I} = (p_i *_i 0_i)_{i\in I} = (p_i)_{i\in I}.$$

(B-3) Let $(p_i)_{i\in I}, (q_i)_{i\in I}, (r_i)_{i\in I} \in \prod_{i\in I} P_i$. Since P_i satisfies (B-3), we have $(p_i *_i q_i) *_i r_i = p_i *_i (r_i *_i (0_i *_i q_i))$ for all $i \in I$. Thus

$$\begin{aligned}
((p_i)_{i\in I} \otimes (q_i)_{i\in I}) \otimes (r_i)_{i\in I} &= (p_i *_i q_i)_{i\in I} \otimes (r_i)_{i\in I} \\
&= ((p_i *_i q_i) *_i r_i)_{i\in I} \\
&= (p_i *_i (r_i *_i (0_i *_i q_i)))_{i\in I} \\
&= (p_i)_{i\in I} \otimes (r_i *_i (0_i *_i q_i))_{i\in I} \\
&= (p_i)_{i\in I} \otimes ((r_i)_{i\in I} \otimes (0_i *_i q_i)_{i\in I}) \\
&= (p_i)_{i\in I} \otimes ((r_i)_{i\in I} \otimes ((0_i)_{i\in I} \otimes (q_i)_{i\in I})).
\end{aligned}$$

Hence, $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$ is a B-algebra.

Conversely, assume that $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$ is a B-algebra, where the binary operation $\otimes$ is defined in Definition 8. Let $i \in I$.

(B-1) Let $p_i \in P_i$. Then $f_{p_i} \in \prod_{i\in I} P_i$, which is defined by (2.2). Since $\prod_{i\in I} P_i$ satisfies (B-1), we have $f_{p_i} \otimes f_{p_i} = (0_i)_{i\in I}$. Now,

$$(\forall j \in I)\left((f_{p_i} \otimes f_{p_i})(j) = \begin{cases} p_i *_i p_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

this implies that $p_i *_i p_i = 0_i$.

(B-2) Let $p_i \in P_i$. Then $f_{p_i} \in \prod_{i\in I} P_i$, which is defined by (2.2). Since $\prod_{i\in I} P_i$ satisfies (B-2), we have $f_{p_i} \otimes (0_i)_{i\in I} = f_{p_i}$. Now,

$$(\forall j \in I)\left((f_{p_i} \otimes (0_i)_{i\in I})(j) = \begin{cases} p_i *_i 0_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

this implies that $p_i *_i 0_i = p_i$.

(B-3) Let $p_i, q_i, r_i \in P_i$. Then $f_{p_i}, f_{q_i}, f_{r_i} \in \prod_{i\in I} P_i$, which is defined by (2.2). Since $\prod_{i\in I} P_i$ satisfies (B-3), we have $(f_{p_i} \otimes f_{q_i}) \otimes f_{r_i} = f_{p_i} \otimes (f_{r_i} \otimes ((0_i)_{i\in I} \otimes f_{q_i}))$. Now,

$$(\forall j \in I)\left((f_{p_i} \otimes f_{q_i}) \otimes f_{r_i})(j) = \begin{cases} (p_i *_i q_i) *_i r_i & \text{if } j = i \\ (0_j *_j 0_j) *_j 0_j & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(f_{p_i} \otimes (f_{r_i} \otimes ((0_i)_{i\in I} \otimes f_{q_i})))(j) = \begin{cases} p_i *_i (r_i *_i (0_i *_i q_i)) & \text{if } j = i \\ 0_j *_j (0_j *_j (0_j *_j 0_j)) & \text{otherwise} \end{cases}\right),$$

this implies that

$$(p_i *_i q_i) *_i r_i = p_i *_i (r_i *_i (0_i *_i q_i)).$$

Hence, $P_i = (P_i; *_i, 0_i)$ is a B-algebra for all $i \in I$.

We call the B-algebra $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$ in Theorem 4 the external direct product B-algebra induced by a B-algebra $P_i = (P_i; *_i, 0_i)$ for all $i \in I$.

Theorem 5. *Let $P_i = (P_i; *_i, 0_i)$ be a B-algebra for all $i \in I$. Then P_i is commutative for all $i \in I$ if and only if $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$ is commutative, where the binary operation $\otimes$ is defined in Definition 8.*

Proof. By Theorem 4, we have $P_i = (P_i; *_i, 0_i)$ is a B-algebra for all $i \in I$ if and only if $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$ is a B-algebra, where the binary operation $\otimes$ is defined in Definition 8. We are left to prove that P_i is commutative for all $i \in I$ if and only if $\prod_{i\in I} P_i$ is commutative.

Assume that P_i is commutative for all $i \in I$. Let $(p_i)_{i\in I}, (q_i)_{i\in I} \in \prod_{i\in I} P_i$. Since P_i is commutative for all $i \in I$, we have $p_i *_i (0_i *_i q_i) = q_i *_i (0_i *_i p_i)$ for all $i \in I$. Thus

$$\begin{aligned} (p_i)_{i\in I} \otimes ((0_i)_{i\in I} \otimes (q_i)_{i\in I}) &= (p_i)_{i\in I} \otimes (0_i *_i q_i)_{i\in I} \\ &= (p_i *_i (0_i *_i q_i))_{i\in I} \\ &= (q_i *_i (0_i *_i p_i))_{i\in I} \\ &= (q_i)_{i\in I} \otimes (0_i *_i p_i)_{i\in I} \\ &= (q_i)_{i\in I} \otimes ((0_i)_{i\in I} \otimes (p_i)_{i\in I}). \end{aligned}$$

Hence, $\prod_{i\in I} P_i$ is commutative.

Conversely, assume that $\prod_{i\in I} P_i$ is commutative. Let $i \in I$. Let $p_i, q_i \in P_i$. Then $f_{p_i}, f_{q_i} \in \prod_{i\in I} P_i$, which is defined by (2.2). Since $\prod_{i\in I} P_i$ is commutative, we have $f_{p_i} \otimes ((0_i)_{i\in I} \otimes f_{q_i}) = f_{q_i} \otimes ((0_i)_{i\in I} \otimes f_{p_i})$. Now,

$$(\forall j \in I)\left((f_{p_i} \otimes ((0_i)_{i\in I} \otimes f_{q_i}))(j) = \begin{cases} p_i *_i (0_i *_i q_i) & \text{if } j = i \\ 0_j *_j (0_j *_j 0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left((f_{q_i} \otimes ((0_i)_{i\in I} \otimes f_{p_i}))(j) = \begin{cases} q_i *_i (0_i *_i p_i) & \text{if } j = i \\ 0_j *_j (0_j *_j 0_j) & \text{otherwise} \end{cases}\right),$$

this implies that

$$p_i *_i (0_i *_i q_i) = q_i *_i (0_i *_i p_i).$$

Hence, P_i is commutative for all $i \in I$.

Next, we introduce the concept of the weak direct product of infinite family of B-algebras and obtain some of its properties as follows:

Definition 9. *Let $P_i = (P_i; *_i, 0_i)$ be a B-algebra for all $i \in I$. Define the weak direct product of a B-algebra P_i for all $i \in I$ to be the structure $\prod^{\mathrm{w}}_{i\in I} P_i = (\prod^{\mathrm{w}}_{i\in I} P_i; \otimes)$, where*

$$\prod^{\mathrm{w}}_{i\in I} P_i = \{(p_i)_{i\in I} \in \prod_{i\in I} P_i \mid p_i \neq 0_i, \textit{ where the number of such } i \textit{ is finite}\}.$$

Then $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} P_i \subseteq \prod_{i\in I} P_i$.

Theorem 6. *Let $P_i = (P_i; *_i, 0_i)$ be a B-algebra for all $i \in I$. Then $\prod^{\mathrm{w}}_{i\in I} P_i$ is a B-subalgebra of the external direct product B-algebra $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} P_i \neq \emptyset$. Let $(p_i)_{i\in I}, (q_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} P_i$, where $I_1 = \{i \in I \mid p_i \neq 0_i\}$ and $I_2 = \{i \in I \mid q_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus

$$(\forall j \in I)\left(((p_i)_{i\in I} \otimes (q_i)_{i\in I})(j) = \begin{cases} p_j *_j 0_j & \text{if } j \in I_1 - I_2 \\ p_j *_j q_j & \text{if } j \in I_1 \cap I_2 \\ 0_j *_j q_j & \text{if } j \in I_2 - I_1 \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (B-1) and (B-2), we have

$$(\forall j \in I)\left(((p_i)_{i\in I} \otimes (q_i)_{i\in I})(j) = \begin{cases} p_j & \text{if } j \in I_1 - I_2 \\ p_j *_j q_j & \text{if } j \in I_1 \cap I_2 \\ 0_j *_j q_j & \text{if } j \in I_2 - I_1 \\ 0_j & \text{otherwise} \end{cases}\right).$$

This implies that the number of such $((p_i)_{i\in I} \otimes (q_i)_{i\in I})(j)$ is not more than $|I_1 \cup I_2|$, that is, it is finite. Thus $(p_i)_{i\in I} \otimes (q_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} P_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} P_i$ is a B-subalgebra of $\prod_{i\in I} P_i$.

Theorem 7. *Let $P_i = (P_i; *_i, 0_i)$ be a B-algebra and Q_i a subset of P_i for all $i \in I$. Then Q_i is a B-subalgebra of P_i for all $i \in I$ if and only if $\prod_{i\in I} Q_i$ is a B-subalgebra of the external direct product B-algebra $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that Q_i is a B-subalgebra of P_i for all $i \in I$. Then $0_i \in Q_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} Q_i \neq \emptyset$. Let $(p_i)_{i\in I}, (q_i)_{i\in I} \in \prod_{i\in I} Q_i$. Then $p_i, q_i \in Q_i$ for all $i \in I$. Thus $p_i *_i q_i \in Q_i$ for all $i \in I$, so $(p_i)_{i\in I} \otimes (q_i)_{i\in I} = (p_i *_i q_i)_{i\in I} \in \prod_{i\in I} Q_i$.

Hence, $\prod_{i\in I} Q_i$ is a B-subalgebra of $\prod_{i\in I} P_i$.

Conversely, assume that $\prod_{i\in I} Q_i$ is a B-subalgebra of $\prod_{i\in I} P_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} Q_i$, so $0_i \in Q_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $p_i, q_i \in Q_i$. Then $f_{p_i}, f_{q_i} \in \prod_{i\in I} Q_i$, which is defined by (2.2). Since $\prod_{i\in I} Q_i$ is a B-subalgebra of $\prod_{i\in I} P_i$, we have $f_{p_i} \otimes f_{q_i} \in \prod_{i\in I} Q_i$. Now,

$$(\forall j \in I)\left((f_{p_i} \otimes f_{q_i})(j) = \begin{cases} p_i *_i q_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

this implies that $p_i *_i q_i \in Q_i$.

Hence, Q_i is a B-subalgebra of P_i for all $i \in I$.

Theorem 8. *Let $P_i = (P_i; *_i, 0_i)$ be a B-algebra and Q_i a subset of P_i for all $i \in I$. Then Q_i is normal of P_i for all $i \in I$ if and only if $\prod_{i\in I} Q_i$ is normal of the external direct product B-algebra $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that Q_i is normal of P_i for all $i \in I$. Then $Q_i \neq \emptyset$ for all $i \in I$, so $\prod_{i\in I} Q_i \neq \emptyset$. Let $(p_i)_{i\in I}, (p'_i)_{i\in I}, (q_i)_{i\in I}, (q'_i)_{i\in I} \in \prod_{i\in I} P_i$ be such that $(p_i)_{i\in I} \otimes (q_i)_{i\in I}, (p'_i)_{i\in I} \otimes (q'_i)_{i\in I} \in \prod_{i\in I} Q_i$. Then $p_i *_i q_i, p'_i *_i q'_i \in Q_i$ for all $i \in I$. Since Q_i is normal of P_i, we have $(p_i *_i p'_i) *_i (q_i *_i q'_i) \in Q_i$ for all $i \in I$. Thus

$$((p_i)_{i\in I}\otimes(p'_i)_{i\in I})\otimes((q_i)_{i\in I}\otimes(q'_i)_{i\in I}) = (p_i*_ip'_i)_{i\in I}\otimes(q_i*_iq'_i)_{i\in I} = ((p_i*_ip'_i)*_i(q_i*_iq'_i))_{i\in I} \in \prod_{i\in I} Q_i.$$

Hence, $\prod_{i\in I} Q_i$ is normal of $\prod_{i\in I} P_i$.

Conversely, assume that $\prod_{i\in I} Q_i$ is normal of $\prod_{i\in I} P_i$. Then $\prod_{i\in I} Q_i \neq \emptyset$, so $Q_i \neq \emptyset$ for all $i \in I$. By Theorem 1, we have $(0_i)_{i\in I} \in \prod_{i\in I} Q_i$. Thus $0_i \in Q_i$ for all $i \in I$. Let $i \in I$ and let $p_i, q_i, p'_i, q'_i \in P_i$ be such that $p_i *_i q_i, p'_i *_i q'_i \in Q_i$. Then $f_{p_i}, f_{q_i}, f_{p'_i}, f_{q'_i} \in \prod_{i\in I} P_i$, which is defined by (2.2). Now,

$$(\forall j \in I)\left((f_{p_i} \otimes f_{q_i})(j) = \begin{cases} p_i *_i q_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (B-1), we have

$$(\forall j \in I)\left((f_{p_i} \otimes f_{q_i})(j) = \begin{cases} p_i *_i q_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right),$$

this implies that $f_{p_i} \otimes f_{q_i} \in \prod_{i\in I} Q_i$. Similarly, $f_{p'_i} \otimes f_{q'_i} \in \prod_{i\in I} Q_i$. Since $\prod_{i\in I} Q_i$ is normal of $\prod_{i\in I} P_i$, we have $(f_{p_i} \otimes f_{p'_i}) \otimes (f_{q_i} \otimes f_{q'_i}) \in \prod_{i\in I} Q_i$. Now,

$$(\forall j \in I)\left(((f_{p_i} \otimes f_{p'_i}) \otimes (f_{q_i} \otimes f_{q'_i}))(j) = \begin{cases} (p_i *_i p'_i) *_i (q_i *_i q'_i) & \text{if } j = i \\ (0_j *_j 0_j) *_j (0_j *_j 0_j) & \text{otherwise} \end{cases}\right),$$

this implies that $(p_i *_i p_i') *_i (q_i *_i q_i') \in Q_i$.

Hence, Q_i is normal of P_i for all $i \in I$.

Theorem 9. *Let $P_i = (P_i; *_i, 0_i)$ be a B-algebra and Q_i a subset of P_i for all $i \in I$. Then Q_i is a B-ideal of P_i for all $i \in I$ if and only if $\prod_{i\in I} Q_i$ is a B-ideal of the external direct product B-algebra $\prod_{i\in I} P_i = (\prod_{i\in I} P_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that Q_i is a B-ideal of P_i for all $i \in I$.

(BI-1) By (BI-1), we have $0_i \in Q_i$ for all $i \in I$. Then $(0_i)_{i\in I} \in \prod_{i\in I} Q_i$.

(BI-2) Let $(p_i)_{i\in I}, (q_i)_{i\in I} \in \prod_{i\in I} P_i$ be such that $(p_i)_{i\in I} \otimes (q_i)_{i\in I} \in \prod_{i\in I} Q_i$ and $(q_i)_{i\in I} \in \prod_{i\in I} Q_i$. Then $(p_i *_i q_i)_{i\in I} \in \prod_{i\in I} Q_i$. Thus $p_i *_i q_i \in Q_i$ and $q_i \in Q_i$, it follows from (BI-2) that $p_i \in Q_i$ for all $i \in I$. Thus $(p_i)_{i\in I} \in \prod_{i\in I} Q_i$.

Hence, $\prod_{i\in I} Q_i$ is a B-ideal of $\prod_{i\in I} P_i$.

Conversely, assume that $\prod_{i\in I} Q_i$ is a B-ideal of $\prod_{i\in I} P_i$. Then $\prod_{i\in I} Q_i \neq \emptyset$, so $Q_i \neq \emptyset$ for all $i \in I$. Let $i \in I$.

(BI-1) By (BI-1), we have $(0_i)_{i\in I} \in \prod_{i\in I} Q_i$. Then $0_i \in Q_i$.

(BI-2) Let $p_i, q_i \in P_i$ be such that $p_i *_i q_i \in Q_i$ and $q_i \in Q_i$. By (BI-1), we have $0_i \in Q_i$ for all $i \in I$. Then $f_{p_i} \in \prod_{i\in I} P_i$ and $f_{p_i *_i q_i}, f_{q_i} \in \prod_{i\in I} Q_i$, which are defined by (2.2). By Lemma 1, we have $f_{p_i} \otimes f_{q_i} = f_{p_i *_i q_i} \in \prod_{i\in I} Q_i$. By (BI-2), we have $f_{p_i} \in \prod_{i\in I} Q_i$. By (2.2), we have $p_i \in Q_i$.

Hence, Q_i is a B-ideal of P_i for all $i \in I$.

Moreover, we discuss several homomorphism theorems in view of the external direct product of B-algebras.

Definition 10. *Let $P_i = (P_i; *_i)$ and $Q_i = (Q_i; \circ_i)$ be algebras and $\psi_i : P_i \to Q_i$ be a function for all $i \in I$. Define the function $\psi : \prod_{i\in I} P_i \to \prod_{i\in I} Q_i$ given by*

$$(\forall (p_i)_{i\in I} \in \prod_{i\in I} P_i)(\psi(p_i)_{i\in I} = (\psi_i(p_i))_{i\in I}). \tag{2.3}$$

We shall show that $\psi : \prod_{i\in I} P_i \to \prod_{i\in I} Q_i$ is a function. Let $(p_i)_{i\in I} \in \prod_{i\in I} P_i$. Since $\psi_i : P_i \to Q_i$ is a function and $p_i \in P_i$ for all $i \in I$, we have $\psi_i(p_i) \in Q_i$ for all $i \in I$. Thus $(\psi_i(p_i))_{i\in I} \in \prod_{i\in I} Q_i$. such that $\psi(p_i)_{i\in I} = (\psi_i(p_i))_{i\in I}$. Let $(p_i)_{i\in I}, (p_i')_{i\in I} \in \prod_{i\in I} P_i$ be such that $(p_i)_{i\in I} = (p_i')_{i\in I}$. Then $p_i = p_i'$ for all $i \in I$, so $\psi_i(p_i) = \psi_i(p_i')$. Thus

$$\psi(p_i)_{i\in I} = (\psi_i(p_i))_{i\in I} = (\psi_i(p_i'))_{i\in I} = \psi(p_i')_{i\in I}.$$

Therefore, $\psi : \prod_{i\in I} P_i \to \prod_{i\in I} Q_i$ is a function.

Theorem 10. *Let $P_i = (P_i; *_i)$ and $Q_i = (Q_i; \circ_i)$ be algebras and $\psi_i : P_i \to Q_i$ be a function for all $i \in I$.*

(i) ψ_i is injective for all $i \in I$ if and only if ψ is injective which is defined in Definition 10,

(ii) ψ_i *is surjective for all* $i \in I$ *if and only if* ψ *is surjective,*

(iii) ψ_i *is bijective for all* $i \in I$ *if and only if* ψ *is bijective.*

Proof. (i) Assume that ψ_i is injective for all $i \in I$. Let $(p_i)_{i\in I}, (q_i)_{i\in I} \in \prod_{i\in I} P_i$ be such that $\psi(p_i)_{i\in I} = \psi(q_i)_{i\in I}$. Then $(\psi_i(p_i))_{i\in I} = (\psi_i(q_i))_{i\in I}$. Thus $\psi_i(p_i) = \psi_i(q_i)$ for all $i \in I$. Since ψ_i is injective for all $i \in I$, we have $p_i = q_i$ for all $i \in I$. Thus $(p_i)_{i\in I} = (q_i)_{i\in I}$. Hence, ψ is injective.

Conversely, assume that ψ is injective. Let $i \in I$. Let $p_i, p'_i \in P_i$ be such that $\psi_i(p_i) = \psi_i(p'_i)$.

Let $p_j = p'_j \in P_j$ for all $j \in I$ and $j \neq i$. Then $\psi_j(p_j) = \psi_j(p'_j) \in Q_j$. Let $h_{\psi_i(p_i)} : I \to \bigcup_{i\in I} Q_i$ and $h_{\psi_i(p'_i)} : I \to \bigcup_{i\in I} Q_i$ are functions defined by

$$(\forall j \in I)\left(h_{\psi_i(p_i)}(j) = \begin{cases} \psi_i(p_i) & \text{if } j = i \\ \psi_j(p_j) & \text{otherwise} \end{cases}\right) \tag{2.4}$$

and

$$(\forall j \in I)\left(h_{\psi_i(p'_i)}(j) = \begin{cases} \psi_i(p'_i) & \text{if } j = i \\ \psi_j(p'_j) & \text{otherwise} \end{cases}\right). \tag{2.5}$$

Then $h_{\psi_i(p_i)}, h_{\psi_i(p'_i)} \in \prod_{i\in I} Q_i$ such that $\psi(p_i)_{i\in I} = h_{\psi_i(p_i)} = h_{\psi_i(p'_i)} = \psi(p'_i)_{i\in I}$. Since ψ is injective, we have $(p_i)_{i\in I} = (p'_i)_{i\in I}$. Thus $p_i = p'_i$. Hence, ψ_i is injective for all $i \in I$.

(ii) Assume that ψ_i is surjective for all $i \in I$. Let $(q_i)_{i\in I} \in \prod_{i\in I} Q_i$. Then $q_i \in Q_i$ for all $i \in I$. Since ψ_i is surjective, there exists $p_i \in P_i$ such that $\psi_i(p_i) = q_i$ for all $i \in I$. Thus $(p_i)_{i\in I} \in \prod_{i\in I} P_i$ such that

$$\psi(p_i)_{i\in I} = (\psi_i(p_i))_{i\in I} = (q_i)_{i\in I}.$$

Hence, ψ is surjective.

Conversely, assume that ψ is surjective. Let $i \in I$. Let $k_i \in Q_i$.

Let $k_j \in Q_j$ for all $j \in I$ and $j \neq i$. Then $(k_i)_{i\in I} \in \prod_{i\in I} Q_i$. Since ψ is surjective, there exists $(p_i)_{i\in I} \in \prod_{i\in I} P_i$ such that

$$(k_i)_{i\in I} = \psi(p_i)_{i\in I} = (\psi_i(p_i))_{i\in I}.$$

Thus $k_i = \psi_i(p_i)$. Hence, ψ_i is surjective for all $i \in I$.

(iii) It is straightforward from (i) and (ii).

Theorem 11. *Let* $P_i = (P_i; *_i, 0_i)$ *and* $Q_i = (Q_i; \circ_i, 1_i)$ *be B-algebras and* $\psi_i : P_i \to Q_i$ *be a function for all* $i \in I$. *Then*

(i) ψ_i *is a B-homomorphism for all* $i \in I$ *if and only if* ψ *is a B-homomorphism which is defined in Definition 10,*

(ii) ψ_i *is a B-monomorphism for all* $i \in I$ *if and only if* ψ *is a B-monomorphism,*

(iii) ψ_i *is a* B*-epimorphism for all* $i \in I$ *if and only if* ψ *is a* B*-epimorphism,*

(iv) ψ_i *is a* B*-isomorphism for all* $i \in I$ *if and only if* ψ *is a* B*-isomorphism,*

(v) $\ker \psi = \prod_{i \in I} \ker \psi_i$ *and* $\psi(\prod_{i \in I} P_i) = \prod_{i \in I} \psi_i(P_i)$.

Proof. (i) Assume that ψ_i is a B-homomorphism for all $i \in I$. Let $(p_i)_{i \in I}, (p'_i)_{i \in I} \in \prod_{i \in I} P_i$. Then

$$\begin{aligned}\psi((p_i)_{i \in I} \otimes (p'_i)_{i \in I}) &= \psi(p_i *_i p'_i)_{i \in I} \\ &= (\psi_i(p_i *_i p'_i))_{i \in I} \\ &= (\psi_i(p_i) *_i \psi_i(p'_i))_{i \in I} \\ &= (\psi_i(p_i))_{i \in I} \otimes (\psi_i(p'_i))_{i \in I} \\ &= \psi(p_i)_{i \in I} \otimes \psi(p'_i)_{i \in I}.\end{aligned}$$

Hence, ψ is a B-homomorphism.

Conversely, assume that ψ is a B-homomorphism. Let $i \in I$. Let $p_i, q_i \in P_i$. Then $f_{p_i}, f_{q_i} \in \prod_{i \in I} P_i$, which is defined by (2.2). Since ψ is a B-homomorphism, we have $\psi(f_{p_i} \otimes f_{q_i}) = \psi(f_{p_i}) \otimes \psi(f_{q_i})$. Since

$$(\forall j \in I)\left((f_{p_i} \otimes f_{q_i})(j) = \begin{cases} p_i *_i q_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{p_i} \otimes f_{q_i})(j) = \begin{cases} \psi_i(p_i *_i q_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{2.6}$$

Since

$$(\forall j \in I)\left(\psi(f_{p_i})(j) = \begin{cases} \psi_i(p_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{q_i})(j) = \begin{cases} \psi_i(q_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{p_i}) \otimes \psi(f_{q_i}))(j) = \begin{cases} \psi_i(p_i) \circ_i \psi_i(q_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{2.7}$$

By (2.6) and (2.7), we have $\psi_i(p_i *_i q_i) = \psi_i(p_i) \circ_i \psi_i(q_i)$. Hence, ψ_i is a B-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 10 (i).
(iii) It is straightforward from (i) and Theorem 10 (ii).
(iv) It is straightforward from (i) and Theorem 10 (iii).
(v) Let $(p_i)_{i\in I} \in \prod_{i\in I} P_i$. Then

$$\begin{aligned}(p_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(p_i)_{i\in I} = (1_i)_{i\in I}\\ &\Leftrightarrow (\psi_i(p_i))_{i\in I} = (1_i)_{i\in I}\\ &\Leftrightarrow \psi_i(p_i) = 1_i \ \forall i \in I\\ &\Leftrightarrow p_i \in \ker\psi_i \ \forall i \in I\\ &\Leftrightarrow (p_i)_{i\in I} \in \prod_{i\in I}\ker\psi_i.\end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I}\ker\psi_i$. Now,

$$\begin{aligned}(q_i)_{i\in I} \in \psi(\prod_{i\in I} P_i) &\Leftrightarrow \exists (p_i)_{i\in I} \in \prod_{i\in I} P_i \text{ s.t. } (q_i)_{i\in I} = \psi(p_i)_{i\in I}\\ &\Leftrightarrow \exists (p_i)_{i\in I} \in \prod_{i\in I} P_i \text{ s.t. } (q_i)_{i\in I} = (\psi_i(p_i))_{i\in I}\\ &\Leftrightarrow \exists p_i \in P_i \text{ s.t. } q_i = \psi_i(p_i) \in \psi(P_i) \ \forall i \in I\\ &\Leftrightarrow (q_i)_{i\in I} \in \prod_{i\in I}\psi_i(P_i).\end{aligned}$$

Hence, $\psi(\prod_{i\in I} P_i) = \prod_{i\in I}\psi_i(P_i)$.

Finally, we discuss several anti-B-homomorphism theorems in view of the external direct product of B-algebras.

Theorem 12. *Let $P_i = (P_i; *_i, 0_i)$ and $Q_i = (Q_i; \circ_i, 1_i)$ be B-algebras and $\psi_i : P_i \to Q_i$ be a function for all $i \in I$. Then*

(i) ψ_i is an anti-B-homomorphism for all $i \in I$ if and only if ψ is an anti-B-homomorphism which is defined in Definition 10,

(ii) ψ_i is an anti-B-monomorphism for all $i \in I$ if and only if ψ is an anti-B-monomorphism,

(iii) ψ_i is an anti-B-epimorphism for all $i \in I$ if and only if ψ is an anti-B-epimorphism,

(iv) ψ_i is an anti-B-isomorphism for all $i \in I$ if and only if ψ is an anti-B-isomorphism.

Proof. (i) Assume that ψ_i is an anti-B-homomorphism for all $i \in I$. Let $(p_i)_{i\in I}, (p'_i)_{i\in I} \in \prod_{i\in I} P_i$. Then

$$\begin{aligned}\psi((p_i)_{i\in I} \otimes (p'_i)_{i\in I}) &= \psi(p_i *_i p'_i)_{i\in I}\\ &= (\psi_i(p_i *_i p'_i))_{i\in I}\\ &= (\psi_i(p'_i) *_i \psi_i(p_i))_{i\in I}\end{aligned}$$

$$= (\psi_i(p_i'))_{i\in I} \otimes (\psi_i(p_i))_{i\in I}$$
$$= \psi(p_i')_{i\in I} \otimes \psi(p_i)_{i\in I}.$$

Hence, ψ is an anti-B-homomorphism.

Conversely, assume that ψ is an anti-B-homomorphism. Let $i \in I$. Let $p_i, q_i \in P_i$. Then $f_{p_i}, f_{q_i} \in \prod_{i\in I} P_i$, which is defined by (2.2). Since ψ is an anti-B-homomorphism, we have $\psi(f_{p_i} \otimes f_{q_i}) = \psi(f_{q_i}) \otimes \psi(f_{p_i})$. Since

$$(\forall j \in I)\left((f_{p_i} \otimes f_{q_i})(j) = \begin{cases} p_i *_i q_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{p_i} \otimes f_{q_i})(j) = \begin{cases} \psi_i(p_i *_i q_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{2.8}$$

Since

$$(\forall j \in I)\left(\psi(f_{q_i})(j) = \begin{cases} \psi_i(q_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{p_i})(j) = \begin{cases} \psi_i(p_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{q_i}) \otimes \psi(f_{p_i}))(j) = \begin{cases} \psi_i(q_i) \circ_i \psi_i(p_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{2.9}$$

By (2.8) and (2.9), we have $\psi_i(p_i *_i q_i) = \psi_i(q_i) \circ_i \psi_i(p_i)$. Hence, ψ_i is an anti-B-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 10 (i).

(iii) It is straightforward from (i) and Theorem 10 (ii).

(iv) It is straightforward from (i) and Theorem 10 (iii).

3. Conclusions and Future Work

In this paper, we have introduced the concept of the direct product of infinite family of B-algebras, we call the external direct product, which is a generalization of the direct product in the sense of Lingcong and Endam [12]. We proved that the external direct product of B-algebras is also a B-algebra. Also, we have introduced the concept of the weak direct product of B-algebras. We proved that the weak direct product of B-algebras is a B-subalgebra of the external direct product B-algebras. Finally, we have provided several fundamental theorems of (anti-)B-homomorphisms in view of the external direct product B-algebras.

Based on the concept of the external direct product of B-algebras in this article, we can apply it to the study of the external direct product in other algebraic systems. Researching the external and weak direct products that we will study in the future will be UP-algebras.

The research topics of interest by our research team being studied in the external direct product of B-algebras are as follows:

(1) to study fuzzy set theory (with respect to a triangular norm) based on the concept of Somjanta et al. [20] and Thongarsa et al. [4, 21],

(2) to study bipolar fuzzy set theory based on the concept of Muhiuddin [14],

(3) to study interval-valued fuzzy set theory based on the concept of Muhiuddin et al. [15],

(4) to study interval-valued intuitionistic fuzzy set theory based on the concept of Senapati et al. [18].

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