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Semi-Regularization Topological Spaces

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Abstract. If $(X , \mathcal{T})$ is a topological space, then the semi-regularization topology $\mathcal{T}_s$ on X of $\mathcal{T}$ is the coarser topology on X generated by the family of all open domains of $(X , \mathcal{T})$ where a subset U is called an open domain if $U = \text{int}(\overline{U})$. In this paper, we study the semi-regularity of some generated spaces and some properties of weaker version of normality of the semi-regularization space $(X , \mathcal{T}_s)$ of a space $(X , \mathcal{T})$.

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If $(X , \mathcal{T})$ is a topological space, then the semi-regularization topology $\mathcal{T}_s$ on X of $\mathcal{T}$ is the coarser topology on X generated by the family of all open domains of $(X , \mathcal{T})$ where a subset U is called an open domain if $U = \text{int}(\overline{U})$. In this paper, we study some properties of weaker version of normality of the semi-regularization space $(X , \mathcal{T}_s)$ of a space $(X , \mathcal{T})$. Also, we study the semi-regularity of some generated spaces. This paper may considered as a continuation of the study of Mršević, Reilly and Vamanamurthy in [10].

Throughout this paper, we denote the set of positive integers by $\mathbb{N}$, the rationals by $\mathbb{Q}$, the irrationals by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff space ($T_{3\frac{1}{2}}$) is a T_1 completely regular space. We do not assume T_2 in the definition of compactness and countable compactness. For a subset A of a space X, $\text{int}A$ and $\overline{A}$ denote the interior and the closure of A, respectively. If two topologies $\mathcal{T}$ and $\mathcal{T}'$ on a set X are considered, we denote the interior of A in $(X , \mathcal{T})$ by $\text{int}_{\mathcal{T}}A$ and $\text{int}_{\mathcal{T}'}A$ for the interior of A in $(X , \mathcal{T}')$. We denote the closure of A in $(X , \mathcal{T}')$ by $\overline{A}^{\mathcal{T}'}$ and, similarly, $\overline{A}^{\mathcal{T}}$ denotes the closure of A in $(X , \mathcal{T})$. We denote an order pair by $\langle x, y \rangle$.

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1. Semi-Regularity

We have to start by recalling some basic definitions.

Definition 1. *A subset A of a space X is called closed domain* [6], called also *regular closed, κ-closed* [9], if $A = \overline{\text{int}A}$. A subset A of a space X is called *open domain* [6], called also *regular open, κ-open,* if $A = \text{int}(\overline{A})$.

It is easy to see that *a subset is an open domain if and only if it is the interior of a closed set, a subset is a closed domain if and only if it is the closure of an open set, the complement of a closed domain is an open domain* and *the complement of an open domain is a closed domain* [6]. Now, let $(X, \mathcal{T})$ be a topological space and let OD denotes the family of all open domains in $(X, \mathcal{T})$. Since X is an open domain and an intersection of two open domains is an open domain [6, 1.1.C], then we have the following definition [6].

Definition 2. *If $(X, \mathcal{T})$ is a topological space, then the semi-regularization topology $\mathcal{T}_s$ on X of $\mathcal{T}$ is the coarser topology on X generated by the family of all open domains of $(X, \mathcal{T})$. $(X, \mathcal{T})$ is called semi-regular* if $\mathcal{T} = \mathcal{T}_s$. $(X, \mathcal{T}_s)$ is called the *semi-regularization topological space of $(X, \mathcal{T})$,* see [10].

Since any open domain is an open set, then for any space $(X, \mathcal{T})$, we have that $\mathcal{T}_s$ is coarser than $\mathcal{T}$, that is, $\mathcal{T}_s \subseteq \mathcal{T}$. Note that if $\emptyset \neq U \subseteq X$, then $U \in \mathcal{T}_s$ if and only if $U = \bigcup_{\alpha\in\Lambda} V_\alpha$ with V_α is an open domain in $(X, \mathcal{T})$ for each $\alpha \in \Lambda$. Equivalently $U \in \mathcal{T}_s$ if and only if for each $x \in U$ there exists an open domain G in $(X, \mathcal{T})$ such that $x \in G \subseteq U$, [11].

If $\mathcal{CF}$ is the finite complement topology on an infinite set, then $\mathcal{CF}_s = \mathcal{I}$, where $\mathcal{I}$ is the indiscrete topology. If $\mathcal{CC}$ is the countable complement topology on an uncountable set, then $\mathcal{CC}_s = \mathcal{I}$. If $X = \{ \langle x, y\rangle : y \geq 0 \}$, the closed upper half plan, then the semi-regularization topology of the Half-Disc topology on X [12, Example 78], is the usual metric topology $\mathcal{U}$ on X. If X is regular, then it is semi-regular [6, 1.1.8]. The converse is not always true. As an example the simplified Arens square [12, Example 81].

2. Semi-regularity of generated spaces

There are many ways of generating new spaces from old ones. In this section, we study the semi-regularity of the Alexandroff duplicate, the closed extension, the discrete extension, and the open extension of a given space X.

Recall that the Alexandroff Duplicate space $A(X)$ of a space X is defined as follows: Let X be any topological space. Let $X' = X \times \{1\}$, so X' is just a copy of X. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1\rangle$ in X' by x' and for a subset $B \subseteq X$, let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus E) : U$ is open in X with $x \in U$ and E is a finite subset of $X'\}$. Then $\mathcal{B} = \{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$

will generate a unique topology on $A(X)$ such that $\mathcal{B}$ is its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of* X [2, 5].

Our goal is to show that *"if X is semi-regular, then so is its Alexandroff duplicate $A(X)$"*. In order to show this we will follow five steps expressed in the following Lemmas and Theorem 1. As a notation we will call a subset which is closed and open by *clopen.*

Lemma 1. *For each $x' \in X'$, $\{x'\}$ is clopen in $A(X)$.*

Proof. Let $x' \in X'$ be arbitrary. We need only to show that $\{x'\}$ is closed. So, let $a \in A(X) \setminus \{x'\}$ be arbitrary. If $a \in X'$, then $\{a\}$ is an open neighborhood of a in $A(X)$ with $\{a\} \subset A(X) \setminus \{x'\}$. If $a \in X$, pick any open neighborhood $U \subseteq X$ of a. Then $U \cup (U' \setminus \{x'\})$ is an open neighborhood of a in $A(X)$ with $U \cup (U' \setminus \{x'\}) \subseteq A(X) \setminus \{x'\}$. Thus $A(X) \setminus \{x'\}$ is open in $A(X)$. Therefore, $\{x'\}$ is closed.

Lemma 2. *Let $(X, \mathcal{T})$ be any topological space. If C is clopen in X and B is an open domain in X, then $B \setminus C$ is an open domain in X.*

Proof. we want to show that, $\mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}}) = \mathrm{B} \setminus \mathrm{C}$. We always have $B \setminus C \subseteq \overline{B \setminus C}$, by taking the interior in both sides we get $\mathrm{int}(\mathrm{B} \setminus \mathrm{C}) \subseteq \mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}})$. But $\mathrm{int}(\mathrm{B} \setminus \mathrm{C}) = \mathrm{int}(\mathrm{B} \cap (\mathrm{X} \setminus \mathrm{C})) = \mathrm{int}(\mathrm{B}) \cap \mathrm{int}(\mathrm{X} \setminus \mathrm{C}) = \mathrm{B} \cap (\mathrm{X} \setminus \mathrm{C}) = \mathrm{B} \setminus \mathrm{C}$, because C is clopen and B is an open domain so B is an open set, thus $B \setminus C$ is an open set. Thus, $B \setminus C \subseteq \mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}})$. Now, we always have $\mathrm{int}(\overline{B \setminus C}) \subseteq \overline{B \setminus C}$. Thus $\mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}}) \subseteq \overline{\mathrm{B} \setminus \mathrm{C}} = \overline{\mathrm{B} \cap (\mathrm{X} \setminus \mathrm{C})} \subseteq \overline{\mathrm{B}} \cap \overline{(\mathrm{X} \setminus \mathrm{C})} = \overline{\mathrm{B}} \cap (\mathrm{X} \setminus \mathrm{C})$, because C is clopen in X. $\mathrm{int}(\overline{B \setminus C}) \subseteq \overline{B} \cap (X \setminus C)$ then, $\mathrm{int}(\mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}})) \subseteq \mathrm{int}(\overline{\mathrm{B}} \cap (\mathrm{X} \setminus \mathrm{C}))$ thus, $\mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}}) \subseteq \mathrm{int}(\overline{\mathrm{B}}) \cap \mathrm{int}(\mathrm{X} \setminus \mathrm{C})$ hence, $\mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}}) \subseteq \mathrm{B} \cap (\mathrm{X} \setminus \mathrm{C}) = \mathrm{B} \setminus \mathrm{C}$, because B is an open domain and C is clopen in X. Thus, $\mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}}) \subseteq \mathrm{B} \setminus \mathrm{C}$. Therefore, $\mathrm{int}(\overline{\mathrm{B} \setminus \mathrm{C}}) = \mathrm{B} \setminus \mathrm{C}$. Thus, $B \setminus C$ is an open domain .

Notice that, if U is any non-empty open set in X, then $U \cup (U' \setminus \emptyset) = U \cup U'$ is a basic open neighborhood in $A(X)$ of any $x \in U$. So, we establish that the following lemma.

Lemma 3. *If U is an open set in X, then $U \cup U'$ is an open set in $A(X)$*

Theorem 1. *If U is an open domain in X, then $U \cup U'$ is an open domain in A(X).*

Proof. Let U be an open domain in X. We show that $U \cup U' = \mathrm{int}_{A(X)}(\overline{U \cup U'}^{A(X)})$ first, we show that $\mathrm{int}_{A(X)}(\overline{U \cup U'}^{A(X)}) \subseteq U \cup U'$. Let $x \in \mathrm{int}_{A(X)}(\overline{U \cup U'}^{A(X)})$ be arbitrary, then $x \in \overline{U \cup U'}^{A(X)}$. There are only two cases.
Case 1: $x \in X'$. So, $\{x\}$ is an open set in $A(X)$ with $x \in \{x\}$ and $\{x\} \cap (U \cup U') \neq \emptyset$, then $x \in U' \subseteq U \cup U'$. Thus, $x \in U \cup U'$.
Case 2: $x \in X$. Since $x \in \mathrm{int}_{A(X)}(\overline{U \cup U'}^{A(X)})$, then there exist an open set V in X with $x \in V$ such that $x \in V \cup (V' \setminus E) \subseteq \overline{U \cup U'}^{A(X)} = \overline{U}^{A(X)} \cup \overline{U'}^{A(X)}$ where E is a finite subset of X'. Thus, $x \in V \subseteq \overline{U}^{A(X)}$, but $\overline{U}^{A(X)} = \overline{U}^{X}$. So, $x \in V \subseteq \overline{U}^{X}$, by taking the interior of both sides with respect to X we get, $\mathrm{int}_X(V) \subseteq \mathrm{int}_X(\overline{U}^{X})$, then $V \subseteq U$ as V is an open set in X and U is an open domain in X. Thus, $x \in U \subseteq U \cup U'$. Hence

$\text{int}_{A(X)}(\overline{U \cup U'}^{A(X)}) \subseteq U \cup U'$.
Now, we show that $U \cup U' \subseteq \text{int}_{A(X)}(\overline{U \cup U'}^{A(X)})$. Note that we always have $U \cup U' \subseteq \overline{U \cup U'}^{A(X)}$. Since U is an open domain in X, then U is an open set in X , so by Lemma 3, $U \cup U'$ is an open set in $A(X)$. Then by taking the interior of both sides with respect to $A(X)$ we get, $U \cup U' = \text{int}_{A(X)}(U \cup U') \subseteq \text{int}_{A(X)}(\overline{U \cup U'}^{A(X)})$.
Hence, $U \cup U' = \text{int}_{A(X)}(\overline{U \cup U'}^{A(X)})$. Therefore, $U \cup U'$ is an open domain in $A(X)$.

As an immediate consequence of the Theorem 1 is the following Corollary 1.

Corollary 1. *If U is an open domain in X, then $U \cup (U' \setminus E)$ is an open domain in $A(X)$ where E is a finite subset of X' .*

Proof. Let U be any open domain in X. By Lemma 1, we have any singleton in X' is clopen in $A(X)$, then E is clopen in $A(X)$ because finite union of closed sets is closed and arbitrary union of open sets is open. Also, by Theorem 1, we have $U \cup U'$ is an open domain in $A(X)$. But, $U \cup (U' \setminus E) = (U \cup U') \setminus E$ is an open domain in $A(X)$ by Lemma 2.

Theorem 2. *If $(X , \mathcal{T})$ is semi-regular, then so is its Alexandroff duplicate $A(X)$.*

Proof. Let $(X , \mathcal{T})$ be any semi-regular topological space. Let $\emptyset \neq W$ be any open set in $A(X)$ and let $x \in W$ be arbitrary. To show that $A(X)$ is semi-regular, it is enough to exhibit an open domain subset G in $A(X)$ such that $x \in G \subseteq W$. For such an x, we have only two cases.
Case 1: $x \in X'$. Since any clopen subset is an open domain, then by Lemma 1, there exist an open domain $G = \{x\}$ in $A(X)$ such that $x \in G \subseteq W$.
Case 2: $x \in X$. Since W is an open set in $A(X)$ with $x \in W$, then there exists an open set U in X with $x \in U$ and $U \cup (U' \setminus E) \subseteq W$, where E is a finite subset of X'. Now, since $(X , \mathcal{T})$ is semi-regular, then there exists an open domain V in X such that $x \in V \subseteq U$. Thus, $x \in (V \cup (V \setminus E)) \subseteq (U \cup (U \setminus E)) \subseteq W$ By Corollary 1, we get $V \cup (V' \setminus E) = G$ is an open domain in $A(X)$ such that $x \in G \subseteq W$.
Therefore, $A(X)$ is semi-regular.

Definition 3. *Let $(X , \mathcal{T})$ be a topological space and let p be an object not in X, that is, $p \notin X$. Put $X^p = X \cup \{ p \}$. Define a topology $\mathcal{T}^\star$ on X^p by $\mathcal{T}^\star = \{ \emptyset \} \cup \{ U \cup \{ p \} : U \in \mathcal{T} \}$. The space $(X^p , \mathcal{T}^\star)$ is called the closed extension space* of $(X , \mathcal{T})$, see [12, Example 12].

Consider the particular point topology $\mathcal{T}_p = \{ W \subseteq X^p : p \in W \}$ on X^p, [12, Example 10]. It is easy to see that $\mathcal{T}^\star$ is coarser than $\mathcal{T}_p$, that is, $\mathcal{T}^\star \subseteq \mathcal{T}_p$. Notice that the closed extension $(X^p , \mathcal{T}^\star)$ of a space $(X , \mathcal{T})$ is not semi-regular regardless wither $(X , \mathcal{T})$ is semi-regular or not.

Example 1. Let $(X , \mathcal{T})$ be the Simplified Arens Square topological space, [12, Example 81]. So, $X = \{ \langle 0,0\rangle, \langle 1,0\rangle \} \cup \{ \langle x,y\rangle : 0 < x,y < 1 \}$. The topology $\mathcal{T}$ on X is generated by the following neighborhood system: For each $\langle x,y\rangle \in \{ \langle x,y\rangle : 0 < x,y < 1 \}$, let $\mathfrak{B}(\langle x,y\rangle) = \{ B_d(\langle x,y\rangle;\epsilon) \subset S : \epsilon > 0 \}$ where d is the usual metric on $\mathbb{R}^2$ and $B_d(\langle x,y\rangle;\epsilon)$ is the open ball centered at $\langle x,y\rangle$ of radius $\epsilon > 0$ so that ϵ is small enough to make the open ball $B_d(\langle x,y\rangle;\epsilon)$ is contained in $\{ \langle x,y\rangle : 0 < x,y < 1 \}$. Let $\mathfrak{B}(\langle 0,0\rangle) = \{ U_n(\langle 0,0\rangle) : n \in \mathbb{N} \}$, where for each $n \in \mathbb{N}$, we have
$U_n(\langle 0,0\rangle) = \{ \{ \langle 0,0\rangle \} \bigcup \{ \langle x,y\rangle \in S : 0 < x < \frac{1}{2} \text{ and } 0 < y < \frac{1}{n} \}$. Let $\mathfrak{B}(\langle 1,0\rangle) = \{ U_n(\langle 1,0\rangle) : n \in \mathbb{N} \}$, where for each $n \in \mathbb{N}$, we have
$U_n(\langle 1,0\rangle) = \{ \{ \langle 1,0\rangle \} \bigcup \{ \langle x,y\rangle \in S : \frac{1}{2} < x < 1 \text{ and } 0 < y < \frac{1}{n} \}$. In [12, Example 81], it was shown that the Simplified Arens Square space $(X , \mathcal{T})$ is semi-regular.

Let U be any non-empty proper open subset of X, then $U \cup \{p\}$ is an open set in X^p such that $U \neq X^p$. Now, $\overline{U \cup \{p\}}^{\mathcal{T}^\star} = \overline{U}^{\mathcal{T}^\star} \cup \overline{\{p\}}^{\mathcal{T}^\star} = \overline{U}^{\mathcal{T}^\star} \cup X^p = X^p$ because $\{p\}$ is dense in $(X^p , \mathcal{T}^\star)$. Hence, $\operatorname{int}_{\mathcal{T}^\star}(\overline{U \cup \{p\}}^{\mathcal{T}^\star}) = \operatorname{int}_{\mathcal{T}^\star} (X^p) = X^p \neq U \cup \{p\}$. Thus the only open domains in $(X^p , \mathcal{T}^\star)$ are X^p and $\emptyset$, then $\mathcal{T}^\star_s = \mathcal{I}$ on X^p, where $\mathcal{I}$ is the indiscrete topology. Therefore, the closed extension topological space $(X^p , \mathcal{T}^\star)$ of the Simplified Arens Square space $(X , \mathcal{T})$ is not semi-regular. ■

Definition 4. *Let M be a non-empty proper subset of a topological space $(X , \mathcal{T})$. Define a new topology $\mathcal{T}_{(M)}$ on X as follows: $\mathcal{T}_{(M)} = \{U \cup K : U \in \mathcal{T} \text{ and } K \subseteq X \setminus M \}$. $(X , \mathcal{T}_{(M)})$ is called a discrete extension* of $(X , \mathcal{T})$ and we denote $(X , \mathcal{T}_{(M)})$, simply, by X_M [1], see also [6, Example 5.1.22].

Observe that if U is an open set in X, then U is also open in X_M because we can write $U = U \cup \emptyset$. The space X_M has the following neighborhood system: For each $x \in X \setminus M$, let $\mathcal{B}(x) = \{\{x\}\}$ and for each $x \in M$, let $\mathcal{B}(x) = \{ U \in \mathcal{T} : x \in U \}$. If X is a semi-regular topological space and $\emptyset \neq M \subset X$, then the discrete extension X_M may not be semi-regular as can be shown in the following example.

Example 2. Consider, $(\mathbb{R} , \mathcal{I})$ where $\mathcal{I}$ is the indiscrete topology. It is clear that $(\mathbb{R} , \mathcal{I})$ is semi-regular. Put $M = \mathbb{R} \setminus \{0\}$. Then, the discrete extension X_M can be describe as follows: $\mathcal{B}(0) = \{ \{0\} \}$ and for each $x \neq 0$, $\mathcal{B}(x) = \{ \mathbb{R} \}$. X_M is not semi-regular because $\{0\}$ is an open set in X_M, but $\operatorname{int}_{X_M} (\overline{\{0\}}^{X_M}) = \operatorname{int}_{X_M} (\mathbb{R}) = \mathbb{R} \neq \{0\}$. Thus, $\{0\}$ is not an open domain in X_M. Hence, $0 \in \{0\}$ with $\{0\}$ is an open set and there is no open domain G in X_M satisfies $0 \in G \subseteq \{0\}$. Therefore, X_M is not semi-regular. ■

Lemma 4. *Let $(X , \mathcal{T})$ be a topological space. Let M be any non-empty proper subset of X. Then, for any open domain U in X, U is an open domain in X_M.*

Proof. Let U be any open domain in X, we always have $U \subseteq \overline{U}^{X_M}$. By taking the interior of both sides with respect to X_M we get, $\operatorname{int}_{X_M}(U) \subseteq \operatorname{int}_{X_M}(\overline{U}^{X_M})$. But since U is an open domain in X, then U is an open set in X. Thus, U is an open set in X_M. Hence, $\operatorname{int}_{X_M}(U) = U$, therefore $U \subseteq \operatorname{int}_{X_M}(\overline{U}^{X_M}) \ldots \star$.

Now, let $x \in \operatorname{int}_{X_M}(\overline{U}^{X_M})$ be arbitrary, then $x \in (\overline{U}^{X_M})$. There are only two cases.
Case 1: $x \in X \setminus M$. Since $\{x\}$ is an open neighborhood of x in X_M satisfies $\{x\} \cap U \neq \emptyset$,

then $x \in U$.
Case 2: $x \in M$. Since $x \in \text{int}_{X_M}(\overline{U}^{X_M})$, then there exist an open set V in X such that $x \in V \subseteq \overline{U}^{X_M} \subseteq \overline{U}^{X}$ and the last inclusion is true because the topology on X is coarser than the topology on X_M. Therefore, we have $x \in V \subseteq \overline{U}^{X}$, then by taking the interior of both sides with respect to X we have, $x \in \text{int}_X V = V \subseteq \text{int}_X(\overline{U}^{X}) = U$ because U is an open domain in X and V is an open set in X. Hence, $x \in U$, thus $\text{int}_{X_M}(\overline{U}^{X_M}) \subseteq U$ ... $\star\star$.
By $\star$ and $\star\star$ we get $U = \text{int}_{X_M}(\overline{U}^{X_M})$. Therefore, U is an open domain in X_M.

In the next theorem, we will use the following fact which was proved in [1]: "*If X is T_1, then so is X_M for any non-empty proper subset M of X*".

Theorem 3. *If X is T_1 and semi-regular, then for any non-empty proper subset M of X, we have that the discrete extension X_M of X is semi-regular.*

Proof. Assume the hypotheses. Let W be an arbitrary non-empty open set in X_M. Let $x \in W$ be arbitrary. There are only two cases.
Case 1: $x \in X \setminus M$. Then we have $\{x\}$ is an open neighborhood of x in X_M. Since X is T_1, then X_M is also T_1. Thus $\{x\}$ is also closed in X_M. Hence $\{x\}$ is clopen in X_M, thus $\{x\}$ is an open domain in X_M such that $x \in \{x\} \subseteq W$.
Case 2: $x \in M$. Since X is semi-regular, then there is a base for X consisting of open domains. Thus, there exists an open domain V in X such that $x \in V \subseteq W$. By Lemma 4, we get V is an open domain in X_M. Therefore, X_M is semi-regular.

Definition 5. *Let $(X, \mathcal{T})$ be a topological space and let p be an object not in X, that is, $p \notin X$. Put $X^p = X \cup \{p\}$. Define a topology $\mathcal{T}'$ on X^p by $\mathcal{T}' = \{X^p\} \cup \{U : U \in \mathcal{T}\} = \{X^p\} \cup \mathcal{T}$. The space $(X^p, \mathcal{T}')$ is called the open extension space* of $(X, \mathcal{T})$, see [12, Example 16].

Observe that $(X, \mathcal{T})$ and $(X^p, \mathcal{T}')$ have the same open sets except for X^p. Also, if U is an open domain in $(X, \mathcal{T})$, then U is an open domain in $(X^p, \mathcal{T}')$ because $\overline{U}^{X^p} = \overline{U}^{X} \cup \{p\}$ as the only open neighborhood of p in $(X^p, \mathcal{T}')$ is X^p itself. Thus, $\text{int}_{X^p}(\overline{U}^{X^p}) = \text{int}_{X^p}(\overline{U}^{X} \cup \{p\}) = \text{int}_X(\overline{U}^{X}) = U$. It is easy to see that if U is an open domain in $(X^p, \mathcal{T}')$ such that $p \notin U$, then U is an open domain in $(X, \mathcal{T})$.

Theorem 4. *$(X, \mathcal{T})$ is semi-regular if and only if $(X^p, \mathcal{T}')$ is semi-regular.*

Proof. Assume that $(X, \mathcal{T})$ is semi-regular. To show that $(X^p, \mathcal{T}')$ is semi-regular, we only need to prove that $\mathcal{T}' \subseteq \mathcal{T}'_s$. Let $W \in \mathcal{T}'$ be an arbitrary such that $\emptyset \neq W \neq X^p$, then $W \in \mathcal{T}$. Since $(X, \mathcal{T})$ is semi-regular, then $\mathcal{T} = \mathcal{T}_s$. So, the family of all open domains in $(X, \mathcal{T})$ is a base for $(X, \mathcal{T})$. Thus W can be written as a union of open domains in $(X, \mathcal{T})$. So, W can be written as a union of open domains in $(X^p, \mathcal{T}')$. Thus $W \in \mathcal{T}'_s$. Thus $\mathcal{T}' \subseteq \mathcal{T}'_s$. Therefore $(X^p, \mathcal{T}')$ is semi-regular.

Conversely, Assume that $(X^p, \mathcal{T}')$ is semi-regular, that is, $\mathcal{T}' = \mathcal{T}'_s$. To show that $(X, \mathcal{T})$ is semi-regular, we only need to show that $\mathcal{T} \subseteq \mathcal{T}_s$. Let $W \in \mathcal{T}$ be arbitrary, then $p \notin W$. But $W \in \mathcal{T}'$ implies that W can be written as a union of open domains in

$(X^p, \mathcal{T}')$. Since any open domain in $(X^p, \mathcal{T}')$ which does not contain the element p is also an open domain in $(X, \mathcal{T})$, then $W \in \mathcal{T}_s$, implies that $\mathcal{T} \subseteq \mathcal{T}_s$ and hence $(X, \mathcal{T})$ is semi-regular.

3. New results about semi-regularization spaces

In this section, we study the relationship between a topological space $(X, \mathcal{T})$ and its semi-regularization space $(X, \mathcal{T}_s)$ regarding a topological property. We start with the property of scattered. Recall that a space X is *scattered* if any non-empty subset of X has an isolated point, that is, if $\emptyset \neq A \subseteq X$, then there exists an element $a \in A$ and there exists an open set U such that $a \in U$ and $U \cap A = \{a\}$. It is easy to see that if $(X, \mathcal{T}_s)$ is scattered, then so is $(X, \mathcal{T})$. This follows from the containment $\mathcal{T}_s \subseteq \mathcal{T}$. But the converse is not always true as can be shown in the following example.

Example 3. Consider $\mathbb{R}$ with the particular point topology $\mathcal{T}_0$ which is scattered, see [12, Example 10]. But the semi-regularization of $(\mathbb{R}, \mathcal{T}_0)$ is $(\mathbb{R}, \mathcal{I})$ where $\mathcal{I}$ is the indiscrete topology which is not scattered. ■

Definition 6. *A topological space $(X, \mathcal{T})$ is called epi-normal* if there exists a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4, see [3].

Lemma 5. *Let (Y, ν) be a regular space. If $f : (X, \mathcal{T}) \longrightarrow (Y, \nu)$ is continuous, then $f : (X, \mathcal{T}_s) \longrightarrow (Y, \nu)$ is continuous,* [8].

Theorem 5. *$(X, \mathcal{T})$ is epi-normal if and only if $(X, \mathcal{T}_s)$ is epi-normal.*

Proof. Assume that $(X, \mathcal{T})$ is epi-normal. Pick a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4. Consider the identity function $id_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T}')$ which is continuous since $\mathcal{T}' \subseteq \mathcal{T}$. Then, by Lemma 5, we have $id_X : (X, \mathcal{T}_s) \longrightarrow (X, \mathcal{T}')$ is continuous, hence $\mathcal{T}' \subseteq \mathcal{T}_s$. Thus, $(X, \mathcal{T}_s)$ is epi-normal.

Conversely, assume that $(X, \mathcal{T}_s)$ is epi-normal. Then there exist a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4. Since $\mathcal{T}_s \subseteq \mathcal{T}$, then result follows.

Definition 7. *A topological space X is called submetrizable* if there exists a metric d on X such that $\mathcal{T}_d \subseteq \mathcal{T}$, [7].

Similar argument of the proof of Theorem 5 gives the following theorem.

Theorem 6. *$(X, \mathcal{T})$ is submetrizable if and only if $(X, \mathcal{T}_s)$ is submetrizable.*

Definition 8. *A topological space X is called C-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$, [4].

The following example shows that, if $(X, \mathcal{T}_s)$ is C-normal, then $(X, \mathcal{T})$ may not be C-normal.

Example 4. Consider $\mathbb{R}$ with the particular point topology $\mathcal{T}_0$ which is not C-normal see [4, Example 1.5]. But the semi-regularization topological space of $(\mathbb{R}, \mathcal{T}_0)$ is $(\mathbb{R}, \mathcal{I})$, where $\mathcal{I}$ is the indiscrete topology, which is a normal space, thus C-normal. ■

Lemma 6. *If X is T_1 and C-normal, then any witness Y of its C-normality is T_4.*

Proof. Assume that X is T_1 and C-normal. Pick a normal space Y and a bijective function $f : X \longrightarrow Y$ such that $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. Let x, y be any two distinct elements in Y. Since f is bijective, there are unique elements $a, b \in X$ such that $f(a) = x$ and $f(b) = y$ such that $a \neq b$. Consider $\{a, b\}$ which is a compact subset of X. This implies $f_{|_{\{a,b\}}} : \{a, b\} \longrightarrow \{x, y\}$ is a homeomorphism. But X is T_1, thus $\{a, b\}$ is a discrete subspace of X, hence $\{x, y\}$ is a discrete subspace of Y, then there are two open neighborhoods U and V of x and y respectively in Y such that $U \cap \{x, y\} = \{x\}$ and $V \cap \{x, y\} = \{y\}$ where $y \notin U$ and $x \notin V$. Thus Y is T_1 and given that Y is normal, thus Y is T_4.

Recall that a topological space X is called a *Fréchet space* if for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of points of A such that $a_n \longrightarrow x$, [6].

Lemma 7. *If X is Fréchet and C-normal, then any witness function of its C-normality is continuous.*

Proof. Assume that X is Fréchet and C-normal. Let $f : X \longrightarrow Y$ be a witness function of the C-normality of X. Let $A \subset X$ and let $y \in f(\overline{A})$ be arbitrary. Pick the unique element $x \in X$ such that $f(x) = y$. Thus $x \in \overline{A}$. Since X is a Fréchet space, then there exist a sequence $(a_n) \subseteq A$ such that (a_n) converges to x. The subspace $B = \{x, a_n : n \in \mathbb{N}\}$ of X is compact and thus $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism. Now, let $W \subseteq Y$ be any open neighborhood of y, then $W \cap f(B)$ is open in the subspace $f(B)$ containing y. Since $\overline{f(\{a_n : n \in \mathbb{N}\})} \subseteq \overline{f(B)} \cap f(A)$ and $W \cap f(B) \neq \emptyset$, then $W \cap f(A) \neq \emptyset$. Hence $y \in \overline{f(A)}$ and thus $f(\overline{A}) \subseteq \overline{f(A)}$. Therefore, f is Continuous.

Theorem 7. *If $(X, \mathcal{T})$ is Fréchet, T_1 and C-normal, then its semi-regularization topological space $(X, \mathcal{T}_s)$ is C-normal*

Proof. Assume the hypothesis. Pick a normal topological space $(Y, \mathcal{T}')$ and a bijective function $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ such that $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for any compact subspace A of X. As X is Fréchet, then by Lemma 7, f is continuous and by Lemma 6, we get $(Y, \mathcal{T}')$ is T_4. Pick the same bijection function $f : (X, \mathcal{T}_s) \longrightarrow (Y, \mathcal{T}')$ which is continuous by Lemma 5. Let B be any compact subset of $(X, \mathcal{T}_s)$, then $f_{|_B} : B \longrightarrow f(B)$ is bijective and continuous, thus by [6, Theorem 3.1.13] $f_{|_B}$ is a homeomorphism. Therefore, $(X, \mathcal{T}_s)$ is C-normal.

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