EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 2, 2022, 774-783
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Results about P-Normality

Lutfi Kalantan[1], Mai Mansouri[1,*]

[1] *King Abdulaziz University, Department of Mathematics, P.O.Box 80203, Jeddah 21589, Saudi Arabia*

Abstract. A topological space X is called *P-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq X$. In this paper we present some new results on P-normality. We study the invariance and inverse invariance of P-normality as a topological property. We also investigate the Alexandroff Duplicate of a P-normal space, the closed extension of a P-normal space, the discrete extension of a P-normal space and the Dowker topological space. Furthermore, we introduce a new property related to P-normality which we call strong P-normality.

2020 Mathematics Subject Classifications: 54D15, 54C10

Key Words and Phrases: Normal, P-normal, L-normal, C-normal, Strong P-noramlity, Alexandroff Duplicate, Invariance, Closed extension, Discrete extension, Paracompact, Product

---

1. Introduction

We introduced P-normality in our previous paper [10]. The purpose of this paper is to study some new results about P-normality. We investigate some types of invariance. We also discuss the Alexandroff Duplicate, the closed extension space and the discrete extension space of a P-normal space. We examine whether P-normality is preserved in these spaces or not. Finally, we define a new topological property called strong P-normality. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff space is a T_1 completely regular space. We do not assume T_2 in the definition of compactness, paracompactness and countable compactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. The first infinite ordinal is ω_0, the first uncountable ordinal is ω_1, and the successor cardinal of ω_1 is ω_2. We begin by recalling the following definitions:

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i2.4387

Email addresses: LNKalantan@hotmail.com (L. Kalantan), mfmansouri1@kau.edu.sa (M. Mansouri)

Recall that a topological space $(X , \mathcal{T})$ is paracompact if any open cover has a locally finite open refinement. For a subspace A of X, A is paracompact if $(A , \mathcal{T}_A)$ is paracompact, i.e., any open (open in the subspace) cover of A has a locally finite open (open in the subspace) refinement. We do not assume T_2 in the definition.

Definition 1. *A topological space X is called P-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq X$ [10].

2. New Results on P-normality

In [10], we proved P-normality is a topological property, studied its independence of other topological properties, and investigated whether or not P-normality was an additive property, a multiplicative property and a hereditary property. Until now, we still don't know if P-normality is hereditary with respect to closed subspaces. But under some conditions it is hereditary with respect to compact subspaces. Before we state such conditions we introduce some results:

Proposition 1. *If X is a T_1 space, $f : X \longrightarrow Y$ is a one to one and onto map, and the restriction of f on any finite subset of X is a homeomorphism. Then Y is also T_1.*

Proof. Since X is T_1 and f is a bijection then Y has more than one element. Let $a, b \in Y$ be arbitrary such that $a \neq b$. Then there exist unique $c, d \in X$ such that $f(c) = a$ and $f(d) = b$ and $c \neq d$. Now, $\{c, d\} \subseteq X$ is a finite subset of X. So $f|_{\{c,d\}} : \{c, d\} \longrightarrow \{a, b\}$ is a homeomorphism. Now, $a = f(c)$ and c is isolated in $\{c, d\}$ because $\{c, d\} \subseteq X$ is discrete. Also, $b = f(d)$ and d is isolated in $\{c, d\}$. So there exist Y-open subset U containing a such that $U \cap \{a, b\} = \{a\}$, and there exists Y-open subset V containing b such that $V \cap \{a, b\} = \{b\}$. Then $b \notin U \ni a$ and $a \notin V \ni b$. Which implies that Y is T_1.

Corollary 1. *If X is a T_1, P-normal space then the witness Y is T_4.*

Using the previous proposition we can state the following theorems:

Corollary 2. *If X is T_1 and the only paracompact subspaces of X are the finite subspaces, then X is P-normal.*

Theorem 1. *Let X be a T_1, Fréchet P-normal space. Then, any compact subspace of X is P-normal.*

Proof. Let Y and f be a witness space and function respectively of the P-normality of X. Since X is T_1, then Y is T_4 by the above corollary. Now, f is continuous since X is Fréchet by [10, Theorem 5]. Let $A \subseteq X$ be any compact subset of X. The continuous image of a compact subset is compact so $f(A) \subseteq Y$ is compact in Y. Moreover, since Y is T_2 that means $f(A)$ is closed in Y. Y is normal and normality is hereditary with respect to closed sets so $f(A)$ is normal and hence will be a witness for the P-normality of A. Let $g = f|_A : A \longrightarrow f(A)$. Let $C \subseteq A$ be any paracompact subspace of A. Since a subspace of a subspace is a subspace, then C is paracompact in X. Therefore, $g|_C = f|_{A_{|_C}} = f|_C$ is a homeomorphism. Which means A, the arbitrary compact subset of X, is P-normal and we are done.

In the same way we proved the previous theorem, we can deduce the following corollary about P-normality being hereditary with respect to countably compact subspaces with additional conditions. recall that a C-closed space Y is a T_2 space where every countably compact subset $A \subseteq Y$ is closed [9].

Corollary 3. *Let X be a T_1, Fréchet P-normal space. Let Y a witness of the P-normality of X be a C-closed space. Then, any countably compact subspace of X is P-normal.*

Recall that a topological space (X , $\mathcal{T}$) is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on X such that (X , $\mathcal{T}'$) is T_4 [3].

Theorem 2. *Let X be a T_1, Fréchet P-normal space, then X is epinormal.*

By the above discussion we have seen that if X is T_1 and P-normal then Y, the witness of P-normality, is T_4. Now, since X is Fréchet that means the witness function f is continuous by [10, Theorem 5]. This allows us to consider Y as a coarser space of X [7, 2.4].

Note that since in this case Y is coarser than X and Y is T_4 hence T_2 then X has to be T_2. Which means, if a space X is not T_2, but T_1 and Fréchet then it cannot be P-normal.

3. Invariance

We begin by studying the invariant properties of P-normality. P-normality is not invariant in general.

Example 1. *In [10] we showed that the Dieudonné plank $(X,\mathcal{T})$ is not P-normal. Now, consider $(X,\mathcal{T}')$, where $\mathcal{T}'$ is generated by making any element on the right side of the plank A isolated. Consider $id_X : (X,\mathcal{T}') \longrightarrow (X,\mathcal{T})$. Since $\mathcal{T}$ is coarser than $\mathcal{T}'$ then $id_X : (X,\mathcal{T}') \longrightarrow (X,\mathcal{T})$ is continious, one to one and onto. $(X,\mathcal{T}')$ is P-normal being T_2-paracompact i.e normal but the Dieudonné plank $(X,\mathcal{T})$ is not.*

By a theorem of Ponomarev [7, 4.2.D] which says: " a T_0 space X is first countable if and only if X is a continuous image of a metrizable space under an open mapping". Consider $(\mathbb{R}, \mathcal{RS})$, which we have shown is not P-normal in [10], but it is Tychonoff and first countable. So there exists a metrizable space X and an open function $g : X \longrightarrow (\mathbb{R}, \mathcal{RS})$. Since any metrizable space is P-normal, then this example shows that P-normality is not open invariant and hence cannot be quotient invariant either.

Example 2.

The function $f : (\mathbb{R}, \mathcal{RS}) \longrightarrow (\{0,1\}, \mathfrak{D})$ *defined by*

$$f(x) = \begin{cases} 0 & ;\ if\ \ x < 0 \\ 1 & ;\ if\ \ x \geq 0 \end{cases}$$

is both closed and open. Where $\mathcal{D}$ *is the discrete topology. Now, as we previously mentioned* $(\mathbb{R}, \mathcal{RS})$ *is not* P*-normal but* $(\{0,1\}, \mathfrak{D})$ *is* P*-normal since it is normal. This example shows that* P*-normality is not inverse invariant in general. Moreover, it is not inverse open invariant nor inverse closed invariant.*

4. Generating Spaces and P-normality

In this section, we start with the study of the Alexandroff Duplicate space of a P-normal space. Let us first recall the definition of the Alexandroff Duplicate topological space. Let X be an infinite topological space. We denote the family of all finite subsets of X by $[\,X\,]^{<\omega_0}$, i.e.,

$$[\,X\,]^{<\omega_0} = \{\,E \subset X : E \text{ is finite }\}.$$

Put $X' = X \times \{1\}$. So, X' is just a copy of X and we have $X \cap X' = \emptyset$. The ground set of the Alexandroff duplicate space $A(X)$ of X is $A(X) = X \cup X'$. To simplify the symbols, we do the following: For an element $x \in X$, we denote the element $\langle x, 1\rangle$ in X' by x' and for any subset $B \subseteq X$, put $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, put $\mathcal{B}(x') = \{\{x'\}\}$, so any element in X' will be isolated in $A(X)$. For each $x \in X$, put $\mathcal{B}(x) = \{U \cup (\,U' \setminus\ E'\,) : U \text{ is open in } X \text{ with } x \in U \text{ and } E \in [\,X\,]^{<\omega_0}\,\}$. Then $\mathcal{B} = \{\,\mathcal{B}(y) : y \in A(X)\,\}$ generates a unique topology on $A(X)$ such that $\mathcal{B}$ is its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of* X, see [2] and [6]. Observe that for any open set U in X, we have that $U \cup U'$ is open in $A(X)$ and for any $x \in U$, we have $U \cup (\,U' \setminus \{\,x'\,\}\,)$ is a basic open neighborhood of x.

Theorem 3. *If* X *is* P*-normal, then so is its Alexandroff Duplicate* $A(X)$*.*

Proof. Let X be any P-normal space. Pick a normal space Y and a bijective function $f : X \longrightarrow Y$ such that $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each paracompact subspace $C \subseteq X$. Consider the Alexandroff Duplicate spaces $A(X)$ and $A(Y)$ of X and Y respectively. Y is normal then so is $A(Y)$ [2].

Let us define $g : A(X) \longrightarrow A(Y)$ by $g(a) = f(a)$ if $a \in X$. If $a \in X'$, let b be the unique element in X such that $b' = a$, then define $g(a) = (f(b))'$. Then g is a bijective function. Now, a subspace $C \subseteq A(X)$ is paracompact in $A(X)$ if and only if $C \cap X$ is paracompact in X. To prove this, assume that C is paracompact in $A(X)$. Let $\mathcal{U} = \{U_\alpha \subseteq C \cap X : U_\alpha$ is open in $C \cap X$ for each $\alpha \in \lambda\}$ be any open cover (open in $C \cap X$) for $C \cap X$. That means for each $\alpha \in \lambda$ there exists $V_\alpha \subseteq X$ open in X such that $U_\alpha = V_\alpha \cap (C \cap X)$. Now, for every $\alpha \in \lambda$, V_α is open in X and therefore $V_\alpha \cup V'_\alpha$ is open in $A(X)$. So $(V_\alpha \cup V'_\alpha) \cap C$ is open in C. This implies $(V_\alpha \cup V'_\alpha) \cap C = (V_\alpha \cap C) \bigcup (V'_\alpha \cap C)$ is open in C. Take the unions of these sets

: $(\cup_{\alpha\in\lambda}(V_\alpha\cap C))\bigcup(\cup_{\alpha\in\lambda}(V'_\alpha\cap C))$. For each $x'\in C\setminus(\cup_{\alpha\in\lambda}(V'_\alpha\cap C))$ consider the singleton $\{x'\}$. Consider $\mathcal{W}=\{(V_\alpha\cap C)\bigcup(V'_\alpha\cap C),\{x'\}:\alpha\in\lambda$, and $x'\in C\setminus(\cup_{\alpha\in\lambda}(V'_\alpha\cap C))\}$. This is an open cover of C in $A(X)$ (open in C). Since C is assumed to be paracompact in $A(X)$ then this cover $\mathcal{W}$ has a locally finite open refinement (in C). That is, there exists $\{G_s : s\in\mathcal{S}\}$ such that for each $s\in\mathcal{S}$, G_s is open in C and for each $s\in\mathcal{S}$ there exists $\alpha\in\lambda$ such that either $G_s\subseteq(V_\alpha\cap C)\bigcup(V'_\alpha\cap C)$ or $G_s=\{x'\}$ for some $x'\in C\setminus(\cup_{\alpha\in\lambda}(V'_\alpha\cap C))\}$. Let $\mathcal{S}'=\{s\in\mathcal{S}:G_s\subseteq V_\alpha\cap C\}$. Then $\{G_s : s\in\mathcal{S}'\}$ is a sub-family of $\{G_s : s\in\mathcal{S}\}$ which is locally finite, hence $\{G_s : s\in\mathcal{S}'\}$ is locally finite as well. So $\{G_s : s\in\mathcal{S}'\}$ is a locally finite open (open in $C\cap X$) refinement of $\mathcal{U}$. On the other hand, assume $C\cap X$ is a paracompact subset in X. Let $\mathcal{G}=\{G_\alpha\cap C:\alpha\in\lambda;G_\alpha\subseteq A(X)$ open in $A(X)$ for each $\alpha\in\lambda\}$ be an arbitrary open (open in C) cover of C in $A(X)$. So $C\subseteq\bigcup\mathcal{G}$. Consider $\mathcal{G}_X=\{(G_\alpha\cap C)\cap X:\alpha\in\lambda\}=\{G_\alpha\cap(C\cap X):\alpha\in\lambda\}$.Then $C\cap X\subseteq\bigcup\mathcal{G}_X$, i.e, this is an open (open in $C\cap X$) cover for $C\cap X$. By assumption there exists $\{H_s : s\in\mathcal{S}\}$ locally finite open (open in $C\cap X$) refinement of $\mathcal{G}_X$. That is, for each $s\in\mathcal{S}$ there exists $\alpha_s\in\lambda$ such that $H_s\subseteq G_{\alpha_s}$. For every $s\in\mathcal{S}$ there exists $K_s\subseteq X$ open in X such that $K_s\cap(X\cap C)=H_s$. Consider $\{(K_s\cup K'_s)\cup C : s\in\mathcal{S}\}=\{(K_s\cap C)\bigcup(K'_s\cap C):s\in\mathcal{S}\}$. Since $\{K_s\cap C : s\in\mathcal{S}\}$ is locally finite then so is $\{(K_s\cap C)\bigcup(K'_s\cap C):s\in\mathcal{S}\}$. For every $x'\in(C\cap X')\setminus(\bigcap_{s\in\mathcal{S}}(K'_s\cap C))$ there exists $\alpha_{x'}\in\lambda$ such that $x'\in G_{\alpha'_x}$. Consider $\mathcal{K}=\{(K_s\cap C)\cup(K'_s\cap C),\{x'\}:s\in\mathcal{S};x'\in(C\cap X')\setminus(\bigcap_{s\in\mathcal{S}}(K'_s\cap C))\}$. $\mathcal{K}$ is a locally finite open refinement (open in C) of $\mathcal{G}$.

Now, let $C\subseteq A(X)$ be any paracompact subspace. We show $g_{|_C}:C\longrightarrow g(C)$ is a homeomorphism. Let $a\in C$ be arbitrary. If $a\in C\cap X'$, let $b\in X$ be the unique element such that $b'=a$. For the smallest basic open neighborhood $\{(f(b))'\}$ of the point $g(a)$ we have that $\{a\}$ is open in $C\cap X'$ and $g(\{a\})\subseteq\{(f(b))'\}$. If $a\in C\cap X$. Let W be any open set in Y such that $g(a)=f(a)\in W$. Consider $H=(W\cup(W'\setminus\{f(a)'\}))\cap g(C)$ which is a basic open neighborhood of $f(a)$ in $g(C)$. Since $f_{|_{C\cap X}}:C\cap X\longrightarrow f(C\cap X)$ is a homeomorphism, then there exists an open set U in X with $a\in U$ and $f_{|_{C\cap X}}(U\cap C)\subseteq W$. Now, $(U\cup(U'\setminus\{a'\}))\cap C=G$ is open in $C\cap X$ such that $a\in G$ and $g_{|_C}(G)\subseteq H$. Therefore, $g_{|_C}$ is continuous. Now, we show that $g_{|_C}$ is open. Let $K\cup(K'\setminus\{k'\})$, where $k\in K$ and K is open in X, be any basic open set in $A(X)$, then $(K\cap C)\cup((K'\cap C)\setminus\{k'\})$ is a basic open set in C. Since $X\cap C$ is compact in X, then $g_{|_C}(K\cap(X\cap C))=f_{|_{X\cap C}}(K\cap(X\cap C))$ is open in $Y\cap f(C\cap X)$ as $f_{|_{X\cap C}}$ is a homeomorphism. Thus $K\cap C$ is open in $Y\cap f(X\cap C)$. Also, $g((K'\cap C)\setminus\{k'\})$ is open in $Y'\cap g(C)$ being a set of isolated points. Thus $g_{|_C}$ is an open function. Therefore, $g_{|_C}$ is a homeomorphism.

Next, we present a result about *Dowker* topological spaces. This result may seem repeated as it was mentioned about L-normality in [11] but it is so interesting that we mention it again with regards to P-normality. Recall that a *Dowker space* is a T_4 space whose product with I, $I=[0,1]$ with its usual metric, is not normal. M. E. Rudin used the existence of a Suslin line to obtain a Dowker space which is hereditarily separable and first countable [12]. Using CH, I. Juhász, K. Kunen, and M. E. Rudin constructed a first countable hereditarily separable real compact Dowker space [8]. Weiss constructed a first countable separable locally compact Dowker space whose existence is consistent with MA

+ ¬ CH [14]. We already know that such spaces are consistent examples of Dowker spaces whose product with I are not L-normal [11]. This means that they cannot be P-normal either; since any regular P-normal space is L-normal [10].

We move on to studying the P-normality of the closed extension. Let $(X , \mathcal{T})$ be a topological space and let p be an object not in X, i.e., $p \notin X$. Put $X^p = X \cup \{ p \}$. Define a topology $\mathcal{T}^\star$ on X^p by $\mathcal{T}^\star = \{ \emptyset \} \cup \{ U \cup \{ p \} : U \in \mathcal{T} \}$. The space $(X^p , \mathcal{T}^\star)$ is called the *closed extension space* of $(X , \mathcal{T})$, [13, Example 12].

Since characterizing all paracompact subspaces [10] is a core subject in the notion of P-normality, we will start with characterizing all paracompact subspaces of the closed extension space $(X^p , \mathcal{T}^\star)$ of a given space $(X , \mathcal{T})$.

Proposition 2. *Let $(X , \mathcal{T})$ be a topological space. Consider the closed extension space $(X^p , \mathcal{T}^\star)$ of $(X , \mathcal{T})$. Let $A \subseteq X^p$.*

If $p \notin A$, then A is a paracompact subset in $(X^p , \mathcal{T}^\star)$ if and only if A is a paracompact subset in $(X , \mathcal{T})$.

If $p \in A$, then A is a paracompact subset in $(X^p , \mathcal{T}^\star)$ if and only if $A \setminus \{p\}$ is a compact subset in $(X , \mathcal{T})$.

The proof of this proposition can be found in [5]. A space is called *ultra-connected* if any two non-empty closed sets intersect [13]. Since any normal space is P-normal (just by taking in Definition 1, $Y = X$ and f to be the identity function) then by [5, Theorem 1.4] , we get the following theorem: If $(X , \mathcal{T})$ is ultra-connected, then its closed extension $(X^p , \mathcal{T}^\star)$ is P-normal [5]. Recall that a topological space X is called C-*normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$ [4]. Now, in [5] it was proved that the closed extension space $(X^p , \mathcal{T}^\star)$ is not C-normal if $(X , \mathcal{T})$ is not ultra-connected. In [10], we showed that P-normality implies C-normality. Combining all the information above together we get:

Theorem 4. *If $(X , \mathcal{T})$ is not ultra-connected, then the closed extension space $(X^p , \mathcal{T}^\star)$ is not P-normal .*

Now, we discuss a new result about P-normality and whether it's preserved or not in the discrete extension space. To do this let us recall the definition of the discrete extension space: Let M be a non-empty proper subset of a topological space $(X , \mathcal{T})$. Define a new topology $\mathcal{T}_{(M)}$ on X as follows: $\mathcal{T}_{(M)} = \{U \cup K : U \in \mathcal{T} \text{ and } K \subseteq X \setminus M \}$. $(X , \mathcal{T}_{(M)})$ is called a *discrete extension* of $(X , \mathcal{T})$ and we denote it by X_M [13], see also [7, 5.1.22]. We will now show that P-normality is not preserved by a discrete extension. That is, the discrete extension of a P-normal space need not be P-normal.

Example 3. *We know that $(\mathbb{R}, \mathcal{RS})$ where $\mathcal{RS}$ is the rational sequence topology on $\mathbb{R}$, is a Tychonoff locally compact non compact space [13, Example 65]. Thus $\mathbb{R}$ with the* *rational sequence topology has a one-point compactification. Let* $X = \mathbb{R} \cup \{p\}$*, where* $p \notin \mathbb{R}$*, be a one-point compactification of* $\mathbb{R}$*. Since* X *is* T_2*-compact, then it is* T_4*, hence* P*-normal [10]. Now, take the discrete extension of* X *denoted by* $X_{\mathbb{R}}$*. Observe that in* $X_{\mathbb{R}}$*, the singleton* $\{p\}$ *is closed-and-open.* $X_{\mathbb{R}}$ *is first countable and Tychonoff because* $\mathbb{R}$ *with the rational sequence topology is, thus* $X_{\mathbb{R}}$ *is of countable tightness.* $X_{\mathbb{R}}$ *is also separable because* $(\mathbb{R}, \mathcal{RS})$ *is separable and* $\mathbb{Q} \cup \{p\}$ *is a countable dense subset of* $X_{\mathbb{R}}$ *[1]. Now,* $\mathbb{R}$ *with the rational sequence topology is not normal. Since* $\mathbb{R}$ *is closed in* $X_{\mathbb{R}}$*, we conclude that* $X_{\mathbb{R}}$ *is not normal. Using the theorem: "If* Y *is* T_3*, separable,* P*-normal and of countable tightness, then* Y *is normal."* [10], we conclude that $X_{\mathbb{R}}$ cannot be P-normal. ■

This example shows that [1, Theorem 12] is not true for P-normality. That is if Y is a Tychonoff space, then a discrete extension X_M of any compactification X of Y need not be P-normal.

5. Strong P-Normality

Definition 2. *A topological space* X *is called strongly* P*-normal* if there exists a bijective function $f : X \longrightarrow I$, where $I = [\,0\,,\,1\,]$ the closed unit interval considered with its usual metric topology, such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq X$.

It is clear from the definition that any strongly P-normal space is P-normal. The converse is not always true.

Example 4. $\omega_2 + 1$ *with its usual ordered topology is* P*-normal because it is normal being* T_2 *compact. But* $\omega_2 + 1$ *cannot be strongly* P*-normal because* $|[\,0\,,\,1\,]| = |\mathbb{R}| = \mathfrak{c} < \omega_2 = |\omega_2 + 1|$*.* ■

Example 5. $(\mathbb{R},\mathcal{U})$ *is not strongly* P*-normal.* $(\mathbb{R},\mathcal{U})$ *is homeomorphic to the open interval* $(0,1)$ *with the usual topology. So, if* $(\mathbb{R},\mathcal{U})$ *is strongly* P*-normal, then there will be a bijection* $f : (0,1) \longrightarrow [0,1]$ *such that* $f|_A$ *is a homeomorphism for every paracompact subset* $A \subseteq (0,1)$*. Since* $(0,1)$ *is Fréchet,* f *is continuous by [10, Theorem 5]. Since* f *is bijection, there is unique* $a, b \in (0,1)$ *such that* $f(a) = 0$ *and* $f(b) = 1$*. Assume without loss of generality that* $a < b$*. Then,* $(0,1) \setminus [a,b] \neq \emptyset$ *and clearly* f *is continuous on* $[a,b]$*. Now using the Intermediate Value, for every* $y \in (f(a), f(b)) = (0,1)$*, there exists* $x \in (a,b)$ *such that* $f(x) = y$*. Hence,* $f^{-1}([0,1]) \subseteq [a,b] \subset (0,1)$*. This implies that for every* $x \in (0,1) \setminus [a,b]$ *(which is non-empty),* x *has no image in* $[0,1]$ *which contradicts that* f *is a function. Hence there is no continuous bijection between* $\mathbb{R}$ *and* I*. Therefore,* $(\mathbb{R},\mathcal{U})$ *is not strongly* P*-normal.* ■

Theorem 5. *Strong* P*-normality is a topological property.*

Proof. Let X be any strongly P-normal space. Assume that $X \cong Z$, so there exists a homeomorphism $k : Z \longrightarrow X$. Since X is strongly P-normal then there exists a witness function $h : X \longrightarrow I$ which is a bijection with the restriction $h|_C : C \longrightarrow f(C)$ is a homeomorphism for any paracompact subspace C of X. . Then $h \circ k : Z \longrightarrow I$ satisfies the requirements.

Theorem 6. *If X is Fréchet and strongly P-normal, then any function witnessing the strong P-normality of X is continuous.*

Proof. Assume that X is strongly P-normal and Fréchet. Let $f : X \longrightarrow I$ be a witness of the strong P-normality of X. Let $A \subseteq X$ and pick $y \in f(\overline{A})$. Pick the unique $x \in X$ such that $f(x) = y$. Thus $x \in \overline{A}$. Since X is Fréchet, there exist a sequence $(a_n) \subseteq A$ such that $a_n \longrightarrow x$. The subspace $B = \{x, a_n : n \in \mathbb{N}\}$ of X is paracompact being compact , thus $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism. Now, let $W \subseteq I$ be any open neighborhood of y, then $W \cap f(B)$ is open in the subspace $f(B)$ containing y. By continuity of the homeomorphism $f_{|_B}$, $f^{-1}(W \cap f(B)) = f^{-1}(W) \cap B$ is an open neighborhood of x in B. Then,$(f^{-1}(W) \cap B) \cap \{a_n : n \in \mathbb{N}\} \neq \emptyset$. So $(f^{-1}(W) \cap B) \cap A \neq \emptyset$. Therefore we have, $\emptyset \neq f((f^{-1}(W) \cap B) \cap A) \subseteq f(f^{-1}(W) \cap A) = W \cap f(A)$ then $W \cap f(A) \neq \emptyset$. Hence $y \in \overline{f(A)}$, thus $f(\overline{A}) \subseteq \overline{f(A)}$. Therefore, f is continuous.

Example 6. *It is clear that I with its usual metric topology is strongly P-normal. We show that the product $I \times I$ is not strongly P-normal.*

Proof. Suppose to the contrary that $I \times I$ is strongly P-normal. Pick a bijection $f : I \times I \longrightarrow I$ such that $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq I \times I$. Now $I \times I$ is first countable and hence Fréchet. This implies that f is continuous, see Theorem 6, which contradicts the fact that there exists no continuous bijection $f : I \times I \longrightarrow I$. Because if there were, then f would be a homeomorphism since $I \times I$ is compact and I is T_2. This is a contradiction since $I \times I$ is connected with no cut points $(I \times I) \setminus \{\langle x, y\rangle\}$ is connected for every point $\langle x, y\rangle \in I \times I$, while I is connected with cut points (Take any $x \in (0, 1) \subset I$, then $I \setminus \{x\} = [0, x) \cup (x, 1]$ where both $[0, x)$ and $(x, 1]$ are non-empty disjoint open subsets of I).

Therefore, $I \times I$ is not strongly P-normal. ■

So, a product of two strongly P-normal spaces may not be strongly P-normal. But, a product of two strongly P-normal spaces is P-normal. To show this, we start with a lemma.

Lemma 1. *If A is a paracompact subset of the product $X \times Z$, then $p_1(A)$ and $p_2(A)$ are both paracompact in X and Z respectively. Where p_1 and p_2 are the natural projection functions.*

Proof. Let A be a paracompact subset of the product $X \times Z$. Suppose that $p_1(A)$ is not paracompact subset in X, i.e., $p_1(A)$ as a subspace of X is not paracompact. Then there exist an open cover $\mathcal{U} = \{ U_\alpha \subseteq p_1(A) : U_\alpha$ is open in $p_1(A)$ for each $\alpha \in \Lambda \}$ for $p_1(A)$ such that any open (open in $p_1(A)$) refinement of $\mathcal{U}$ is not locally finite. Now, let $x \in p_1(A)$ and fix an $\alpha_x \in \Lambda$ such that $x \in U_{\alpha_x}$. For each $z \in Z$ such that there exists $x \in p_1(A)$ with $\langle x, z\rangle \in A$, let W_z be an open neighborhood of z in Z. Note that $A \subseteq p_1(A) \times p_2(A)$. Consider the family $\mathcal{K} = \{ (U)\alpha_x \times W_z) \cap A : \langle x, z\rangle \in A \}$ which is an open (open in A) cover for A.

Claim: $\mathcal{K}$ has no locally finite open refinement.

Proof of Claim: Suppose that $\mathcal{K}$ has a locally finite open refinement, say $\{ G_s \times H_s : s \in S \}$ (we can assume that this refinement is of the basic open set form in the product $X \times Z$), then the family $\{ G_s \cap p_a(A) : s \in S \}$ would be a locally finite open refinement of $\mathcal{U}$ which is a contradiction and Claim is proved.

So, $\mathcal{K}$ is an open (open in A) cover for A which has no locally finite open refinement and this contradicts that A is a paracompact subset in $X \times Z$. Therefore, $p_1(A)$ is a paracompact subset in X.

Similarly, $p_2(A)$ is a paracompact subset of Z.

Theorem 7. *If X and Z are both strongly P-normal, then $X \times Z$ is P-normal.*

Proof. Assume that X and Z are both strongly P-normal. Pick two bijection functions $f : X \longrightarrow I$ and $g : Z \longrightarrow I$ such $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq X$ and $g_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each paracompact subspace $A \subseteq Z$. $I \times I$ is normal being T_2 compact. Put $h = f \times g$, i.e., $h : X \times Z \longrightarrow I \times I$ is defined by $h(\langle x, z\rangle) = \langle f(x), g(z)\rangle$ for each $\langle x, z\rangle \in X \times Z$. It is clear that h is a bijection function. Let A be any paracompact subset of $X \times Z$. By Lemma 1, we have that $p_1(A)$ is a paracompact subset of X and $p_2(A)$ is a paracompact subset of Z. Thus $f_{|_{p_1(A)}} : p_1(A) \longrightarrow f(p_1(A))$ is a homeomorphism and $g_{|_{p_2(A)}} : p_2(A) \longrightarrow g(p_2(A))$ is a homeomorphism. Since a product of two homeomorphisms is a homeomorphism [7], we get that $h_{p_1(A)\times p_2(A)} = (f_{|_{p_1(A)}}) \times (g_{|_{p_2(A)}}) : p_1(A) \times p_2(A) \longrightarrow (f(p_1(A))) \times (g(p_2(A))) = h(p_1(A) \times p_2(A))$ is a homeomorphism. Since $A \subseteq p_1(A) \times p_2(A)$ and a restriction of a homeomorphism is a homeomorphism, we conclude that $h_A : A \longrightarrow h(A)$ is a homeomorphism. Therefore, $X \times Z$ is P-normal.

The following problems are still open:

1. Is P-normality hereditary with respect to closed sets?
2. Is there a Tychonoff P-normal space which is not normal?

References

[1] L Kalantan A Alawadi and M Saeed. On the discrete extension spaces. *Journal of Mathematical Analysis*, 9(2):150–157., 2018.

[2] P S Alexandroff and P S Urysohn. Mémoire sur les espaces topologiques compacts. *Verh. Konink. Acad. Wetensch. Amsterdam*, 14:1–96., 1929.

[3] S AlZahrani and L Kalantan. Epinormality. *Journal of Nonlinear Sciences & Applications*, 9(9):5398–5402., 2016.

[4] S AlZahrani and L Kalantan. C-normal topological property. *Filomat*, 31(2):407–411., 2017.

[5] S Al-Qarhi D Abuzaid and L Kalantani. On the closed extension spaces. *To Appear.*

[6] R Engelking. On the double circumference of alexandroff. *Bull. Acad. Pol. Sci. Ser. Astron. Math. Phys.*, 16(8):629–634., 1968.

[7] R Engelking. *General Topology.* PWN, Warszawa, 1977.

[8] K Kunen I Juhász and M E Rudin. Two more hereditarily separable non-Lindelöf spaces. *Canadian Journal of Mathematics*, 28(5):998–1005., 1976.

[9] M Ismail and P Nyikos. On spaces in which countably compact sets are closed and hereditary properties. *Topology and its Applications*, 11(3):281–292., 1980.

[10] L Kalantan and M Mansouri. P-normality. *Journal of Mathematical Analysis*, 12(6):1–8., 2021.

[11] L Kalantan and M Saeed. L-normality. *Topology Proceedings*, 50:141–149., 2017.

[12] M E Rudin. A separable dowker space. *Symposia Mathematica*, 16:125–132., 1973.

[13] L Steen and J A Seebach. *Counterexamples in Topology.* Dover Publications INC, USA, 1995.

[14] W Weiss. Small dowker spaces. *Pacific Journal of Mathematics*, 24(2):485–492., 1981.