EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 3, 2022, 830-840
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On $M-$quasi paranormal operators

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Abstract. In this paper we introduce a new class of operators called $M-$quasi paranormal operators. A bounded linear operator T in a complex Hilbert space $\mathcal{H}$ is said to be a $M-$quasi paranormal operator if it satisfies

$$\|T^2x\|^2 \leq M\|T^3x\| \cdot \|Tx\|,$$

$\forall x \in \mathcal{H}$, where M is a real positive number.
We prove basic properties, the structural and spectral properties of this class of operators.

2020 Mathematics Subject Classifications: 47B47, 47B20

Key Words and Phrases: $M-$quasi paranormal operator, $M-$quasi hyponormal operator, $M-$paranormal operator

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1. Introduction

Throughout this paper, let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra of all bounded operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\ker T$ the null space, by $T(\mathcal{H})$ the range of T. The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$. The closure of a set $\mathcal{A}$ will be denoted by $\overline{\mathcal{A}}$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be (see [6]):

- an isometry if $\|Tx\| = \|x\|, \forall x \in \mathcal{H}$;
- an unitary operator if $T^*T = TT^* = I$;
- a positive operator $T \geq O$, if $\langle Tx, x \rangle \geq 0, \forall x \in \mathcal{H}$.

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DOI: https://doi.org/10.29020/nybg.ejpam.v15i3.4392

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By $\sigma(T)$ we write the spectrum of T, the $r(T)$ is the spectral radius of operator T which is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. The $\sigma_a(T)$ is the approximate point spectrum of operator T and it is proved that if $\lambda \in \sigma_a(T)$, then exists sequence (x_n), where $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0, n \to +\infty$ (see [6]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be:

- a paranormal operator if $\|Tx\|^2 \leq \|T^2x\|, \forall x \in \mathcal{H}, \|x\| = 1$, or equivalently if $T^{*2}T^2 - 2kT^*T + k^2 \geq O, \forall k > 0$ (see [2] [4], [5], [11]);
- a $M-$paranormal operators if $M^2T^{*2}T^2 - 2kT^*T + k^2 \geq O, \forall k > 0$ and for a fixed real positive number M, or equivalently if $\|Tx\|^2 \leq M\|T^2x\|, \forall x \in \mathcal{H}, \|x\| = 1$ and for a fixed real positive number M (see [1], [3], [8]);
- a $M-$quasi hyponormal if $\|T^*Tx\| \leq M\|T^2x\|, \forall x \in \mathcal{H}$ and for a fixed real positive number M (see [9], [10]).

2. Main results

Analyzing the very good qualities of these classes of operators, $M-$paranormal and $M-$quasi hyponormal operators, we came to the idea to introduce a new class of operators $M-$quasi paranormal, which could include these classes of operators, to be their generality and possibly to satisfy some of their properties.

Definition 1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a $M-$quasi paranormal operator, for a fixed real positive number M if T satisfies*

$$\|T^2x\|^2 \leq M\|T^3x\| \cdot \|Tx\|,$$

$\forall x \in \mathcal{H}$.

From the definition we can prove the proper inclusions relation among the $M-$quasi hyponormal, $M-$paranormal and $M-$quasi paranormal operators as follows:

Proposition 1. *Let be $T \in \mathcal{L}(\mathcal{H})$.*

1. Every $M-$quasi hyponormal operator is $M-$paranormal operator.

2. Every $M-$paranormal operator is $M-$quasi paranormal operator.

Proof. Let be $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}, \|x\| = 1$. We may assume that $Tx \neq 0$.

1. Since T is a $M-$quasi hyponormal operator we have

$$\|T^*Tx\| \leq M\|T^2x\|,$$

$\forall x \in \mathcal{H}, \|x\| = 1$ and for a fixed real positive number M.

We know that for any bounded linear operator T on $\mathcal{H}$ and $\forall x \in \mathcal{H}, \|x\| = 1$ it is valid:

$$\|Tx\|^2 \leq \|T^*Tx\|$$

Therefore

$$M\|T^2x\| \geq \|T^*Tx\| \geq \|Tx\|^2$$

$\forall x \in \mathcal{H}, \|x\| = 1$. This prove that operator T is also $M-$paranormal.

2. Now let's suppose that T is a $M-$paranormal operator. Then we have:

$$M\|T^3x\| = M\Big\|T^2\frac{Tx}{\|Tx\|}\Big\| \cdot \|Tx\| \geq$$
$$\Big\|T\frac{Tx}{\|Tx\|}\Big\|^2 \cdot \|Tx\| =$$
$$\frac{\|T^2x\|^2}{\|Tx\|}$$

Therefore,

$$M\|T^3x\| \cdot \|Tx\| \geq \|T^2x\|^2.$$

Which prove that operator T is $M-$quasi paranormal operator.

From this proposition we have the inclusion:
$M-$quasi hyponormal $\subseteq$ $M-$paranormal $\subseteq$ $M-$quasi paranormal
In the following proposition we give the necessary and sufficient conditions under which an operator T is a $M-$quasi paranormal operator.

Proposition 2. *An operator $T \in \mathcal{L}(\mathcal{H})$ is $M-$quasi paranormal operator if and only if*

$$M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T \geq 0$$

$\forall k > 0$.

Proof. Since T is a $M-$quasi paranormal operator, for a fixed real positive number M, then

$$\|T^2x\|^2 \leq M\|T^3x\| \cdot \|Tx\|,$$

$\forall x \in \mathcal{H}$. Then,

$$\|T^2x\|^2 - M\|T^3x\| \cdot \|Tx\| \leq 0,$$

i.e.,

$$4(\|T^2x\|^2)^2 - 4M^2\|T^3x\|^2 \cdot \|Tx\|^2 \leq 0.$$

By elementary properties of real quadratic forms, this gives

$$k^2\|Tx\|^2 - 2k\|T^2x\|^2 + M^2\|T^3x\|^2 \geq 0, \forall x \in H, \forall k > 0;$$
$$k^2\langle T^*Tx|x\rangle - 2k\langle T^{*2}T^2x|x\rangle + M^2\langle T^{*3}T^3x|x\rangle \geq 0, \forall x \in H, \forall k > 0;$$
$$\langle (M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T)x|x\rangle \geq 0, \forall x \in H, \forall k > 0.$$

Hence,

$$M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T \geq 0, \forall x \in H, \forall k > 0.$$

The reverse implication follows by retracing the steps back.

In following we give an example of $M-$quasi paranormal operator.

Example 1. *Let* $T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(l_2 \oplus l_2)$. *Then* T *is* $M-$*quasi paranormal operator* $\forall M \geq 1$.

By simple calculation we have that:

$$T^* = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$
$$T^{*2} = T^{*3} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$
$$T^2 = T^3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$
$$T^*T = T^{*2}T^2 = T^{*3}T^3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$
$$M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T$$
$$= \begin{pmatrix} 2M^2 - 4k + 2k^2 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2[(1-k)^2 + M^2 - 1] & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore T *is* $M-$*quasi paranormal operator* $\forall M \geq 1, \forall k > 0$.

The next proposition give the necessary and sufficient conditions for a weighted shift operator T with decreasing weighted sequence (α_n) to be a $M-$quasi paranormal operator.

Proposition 3. *Let* T *be a weighted shift operator with decreasing weighted sequence* (α_n). *Then* T *is a* $M-$*quasi paranormal operator if and only if*

$$|\alpha_{n+1}| \leq M|\alpha_{n+2}|$$

$\forall n \in N$.

Proof. Since T is a weighted shift, its adjoint T^* is also a weighted shift and defined by $T(e_n) = |\alpha_n|e_{n+1}$ we have:

$$\begin{aligned}
&T^*(e_n) = |\alpha_{n-1}|e_{n-1},\\
&(T^*T)(e_n) = |\alpha_n|^2 e_n,\\
&(T^{*2}T^2)(e_n) = |\alpha_n|^2|\alpha_{n+1}|^2 e_n,\\
&(T^{*3}T^3)(e_n) = |\alpha_n|^2|\alpha_{n+1}|^2|\alpha_{n+2}|^2 e_n.
\end{aligned}$$

Now, since T is a $M-$quasi paranormal operator then,

$$\begin{aligned}
&M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T \geq 0, \forall k > 0\\
\Leftrightarrow &M^2|\alpha_n|^2|\alpha_{n+1}|^2|\alpha_{n+2}|^2 - 2k|\alpha_n|^2|\alpha_{n+1}|^2 + k^2|\alpha_n|^2 \geq 0, \forall k > 0\\
\Leftrightarrow &M^2|\alpha_{n+1}|^2|\alpha_{n+2}|^2 - 2k|\alpha_{n+1}|^2 + k^2 \geq 0, \forall k > 0.
\end{aligned}$$

By elementary properties of real quadratic forms, this gives

$$\begin{aligned}
&4|\alpha_{n+1}|^4 - 4M^2|\alpha_{n+1}|^2|\alpha_{n+2}|^2 \leq 0\\
&|\alpha_{n+1}| \leq M|\alpha_{n+2}|
\end{aligned}$$

Example 2. *A weighted shift operator T with decreasing weighted sequence $\alpha_n = 2^n, n \in N$ is a $M-$quasi paranormal operator for every fixed real number $M \geq \frac{1}{2}$ (it is clear from Proposition 3).*

Proposition 4. *Let T be a non singular weighted shift operator with decreasing weighted sequence (α_n). Then T^{-1} is a $M-$quasi paranormal operator if and only if*

$$|\alpha_{n-3}| \leq M|\alpha_{n-2}|$$

$\forall n \geq 3$.

In the next propositions we will prove some properties of $M-$quasi paranormal operators.

Proposition 5. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $M-$quasi paranormal operator.*
a) If T double commutes with an isometric operator S, then TS is a $M-$quasi paranormal operator.
b) If S is unitarily equivalent to operator T, then S is a $M-$quasi paranormal operator.
c) If A is a closed T invariant subset of $\mathcal{H}$, then, the restriction $T_{|A}$ is a $M-$quasi paranormal operator.

Proof. Let be $T \in \mathcal{L}(\mathcal{H})$ a $M-$quasi paranormal operator.
a) Let be S an isometric operator and let be $B = TS$. Since operator T double commutes with operator S we have $TS = ST, S^*T = TS^*$ and $S^*S = I$. Now,

$$\begin{aligned} &M^2B^{*3}B^3 - 2kB^{*2}B^2 + k^2B^*B \\ &= M^2(TS)^{*3}(TS)^3 - 2k(TS)^{*2}(TS)^2 + k^2(TS)^*(TS) \\ &= M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T \geq 0, \forall k > 0 \end{aligned}$$

so TS is a $M-$quasi paranormal operator.

b) Since operator S is unitarly equivalent to operator T, then there exists an unitary operator U such that $S = U^*TU$. Since T is a $M-$quasi paranormal operator then

$$M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T \geq 0, \forall k > 0.$$

Hence,

$$\begin{aligned} &M^2S^{*3}S^3 - 2kS^{*2}S^2 + k^2S^*S \\ &= M^2(U^*TU)^{*3}(U^*TU)^3 - 2k(U^*TU)^{*2}(U^*TU)^2 + k^2(U^*TU)^*(U^*TU) \\ &= U^*(M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T)U \geq 0, \forall k > 0 \end{aligned}$$

so S is a $M-$quasi paranormal operator.

c)

$$\begin{aligned} &\|(T\mid_A)^2u\|^2 \\ &= \|T^2u\|^2 \leq M(\|T^3u\| \cdot \|Tu\|) \\ &= M(\|(T\mid_A)^3u\| \cdot \|Tu\|). \end{aligned}$$

This implies that $T\mid_A$ is a $M-$quasi paranormal operator.

Proposition 6. *If $T \in \mathcal{L}(\mathcal{H})$ is a invertible $M-$quasi paranormal operator then T^{-1} is also $M-$quasi paranormal operator.*

Proof. Since T is a $M-$quasi paranormal operator, for a fixed real positive number M, then

$$\|T^2x\|^2 \leq M\|T^3x\| \cdot \|Tx\|,$$

$\forall x \in \mathcal{H}$. Then,

$$\frac{\|T^2x\|}{\|T^3x\|} \leq \frac{M\|Tx\|}{\|T^2x\|}$$

$\forall x \in \mathcal{H}$. Now replacing x by $T^{-4}x$, we have

$$\begin{aligned}
&\frac{\|T^2T^{-4}x\|}{\|T^3T^{-4}x\|} \leq \frac{M\|TT^{-4}x\|}{\|T^2T^{-4}x\|}\\
&\frac{\|T^{-2}x\|}{\|T^{-1}x\|} \leq \frac{M\|T^{-3}x\|}{\|T^{-2}x\|}\\
&\|T^{-2}x\|^2 \leq M\|T^{-3}x\| \cdot \|T^{-1}x\|
\end{aligned}$$

$\forall x \in \mathcal{H}$. This shows that T^{-1} is a $M-$quasi paranormal operator.

Proposition 7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $M-$quasi paranormal operator. If T^k has dense range, then T is a $M-$paranormal operator.*

Proof. Since T^k has dense range, $\overline{T^k(\mathcal{H})} = \mathcal{H}$. Let $y \in \mathcal{H}$. Then there exists a sequence $\{x_n\}_{n=1}^{+\infty}$ in $\mathcal{H}$ such that $T^k(x_n) \to y$, $n \to +\infty$. Since T is a $M-$quasi paranormal operator, then

$$\begin{aligned}
&\left\langle (M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T)x_n, x_n \right\rangle \geq 0, \forall k > 0;\\
&\left\langle (T^*(M^2T^{*2}T^2 - 2kT^*T + k^2)T)x_n, x_n \right\rangle \geq 0, \forall k > 0;\\
&\left\langle (M^2T^{*2}T^2 - 2kT^*T + k^2)Tx_n, Tx_n \right\rangle \geq 0, \forall k > 0.
\end{aligned}$$

By the continuity of the inner product, we have

$$\langle (M^2T^{*2}T^2 - 2kT^*T + k^2)y, y\rangle \geq 0, \forall y \in \mathcal{H}, \forall k > 0.$$

Therefore T is a $M-$paranormal operator.

In following we give the inclusion of approximate point spectrum of this class of operators.

Proposition 8. *Let $T \in L(H)$ be a regular $M-$quasi paranormal operator. Then the approximate point spectrum of operator T lies in the disc*

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{1}{\sqrt{M}\|T^{-2}\| \cdot \sqrt{\|T^2\|}} \leq |\lambda| \leq \|T\|\}.$$

Proof. Let T be a regular $M-$quasi paranormal operator, $\forall x \in \mathcal{H}, \|x\| = 1$ we have:

$$\begin{aligned}
\|x\|^2 &= \|T^{-2} \cdot T^2x\|^2\\
&\leq \|T^{-2}\|^2 \cdot \|T^2x\|^2\\
&\leq \|T^{-2}\|^2 \cdot M \cdot \|T^3x\| \cdot \|Tx\|
\end{aligned}$$

$$\leq M \cdot \|T^{-2}\|^2 \cdot \|T^2\| \cdot \|Tx\| \cdot \|Tx\|$$
$$= M \cdot \|T^{-2}\|^2 \cdot \|T^2\| \cdot \|Tx\|^2.$$

So,

$$1 \leq M \cdot \|T^{-2}\|^2 \cdot \|T^2\| \cdot \|Tx\|^2,$$

where we have

$$\|Tx\| \geq \frac{1}{\sqrt{M}\|T^{-2}\| \cdot \sqrt{\|T^2\|}}.$$

Now, assume that $\lambda \in \sigma_a(T)$, then $\exists (x_n), \|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0, n \to +\infty$.

From the last inequation we have:

$$\|Tx_n - \lambda x_n\| \geq \|Tx_n\| - |\lambda| \cdot \|x_n\| \geq \frac{1}{\sqrt{M}\|T^{-2}\| \cdot \sqrt{\|T^2\|}} - |\lambda|.$$

Now, when $n \to +\infty$ we have

$$|\lambda| \geq \frac{1}{\sqrt{M}\|T^{-2}\| \cdot \sqrt{\|T^2\|}}.$$

So, we have

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{1}{\sqrt{M}\|T^{-2}\| \cdot \sqrt{\|T^2\|}} \leq |\lambda| \leq \|T\|\}.$$

Therefore the proof is completed.

Now we will give some results for the matrix representation of $M-$quasi paranormal operators.

Proposition 9. *Let $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be the operator defined as*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

If A is a $M-$paranormal operator, then T is a $M-$quasi paranormal operator.

Proof. A simple calculation shows that:

$$T^* = \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix},$$
$$T^{*2} = \begin{pmatrix} A^{*2} & 0 \\ B^*A^* & 0 \end{pmatrix},$$
$$T^2 = \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix},$$
$$T^{*3} = \begin{pmatrix} A^{*3} & 0 \\ B^*A^{*2} & 0 \end{pmatrix},$$

$$T^3 = \begin{pmatrix} A^3 & A^2B \\ 0 & 0 \end{pmatrix},$$

$$T^{*3}T^3 = \begin{pmatrix} A^{*3}A^3 & A^{*3}A^2B \\ B^*A^{*2}A^3 & B^*A^{*2}A^2B \end{pmatrix}.$$

$$M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T$$

$$= \begin{pmatrix} A^*(M^2A^{*2}A^2 - 2kA^*A + k^2)A & A^*(M^2A^{*2}A^2 - 2kA^*A + k^2)B \\ B^*(M^2A^{*2}A^2 - 2kA^*A + k^2)A & B^*(M^2A^{*2}A^2 - 2kA^*A + k^2)B \end{pmatrix}, \forall k > 0.$$

Let $u = x \oplus y \in \mathcal{H} \oplus \mathcal{H}$. Then,

$$\begin{aligned}
&\langle (M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*Tu, u\rangle \\
&= \langle A^*(M^2A^{*2}A^2 - 2kA^*A + k^2)Ax, x\rangle + \langle A^*(M^2A^{*2}A^2 - 2kA^*A + k^2)By, x\rangle \\
&+ \langle B^*(M^2A^{*2}A^2 - 2kA^*A + k^2)Ax, y\rangle + \langle B^*(M^2A^{*2}A^2 - 2kA^*A + k^2)By, y\rangle \\
&= \langle (M^2A^{*2}A^2 - 2kA^*A + k^2)Ax, Ax\rangle + \langle (M^2A^{*2}A^2 - 2kA^*A + k^2)By, Ax\rangle \\
&+ \langle (M^2A^{*2}A^2 - 2kA^*A + k^2)Ax, By\rangle + \langle (M^2A^{*2}A^2 - 2kA^*A + k^2)By, By\rangle \\
&= \langle (M^2A^{*2}A^2 - 2kA^*A + k^2)(Ax + By), (Ax + By)\rangle \geq 0, \forall k > 0
\end{aligned}$$

because A is a $M-$paranormal operator this prove that T is a $M-$quasi paranormal operator.

Proposition 10. *Let T be a $M-$quasi paranormal operator, the range of T not to be dense, and*

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad on \ \ \mathcal{H} = \overline{T(\mathcal{H})} \oplus \ker T^*.$$

Then, A is a $M-$paranormal operator on $\overline{T(\mathcal{H})}$, $C = O$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof.

Suppose that T is a $M-$quasi paranormal operator. Since that T does not have dense range, we can represent T as the upper triangular matrix:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \ \ \mathcal{H} = \overline{T(\mathcal{H})} \oplus \ker T^*.$$

Since T is a $M-$quasi paranormal operator, we have

$$M^2T^{*3}T^3 - 2kT^{*2}T^2 + k^2T^*T \geq 0, \forall k > 0$$

$$\Rightarrow T^*(M^2T^{*2}T^2 - 2kT^*T + k^2)T \geq 0, \forall k > 0.$$

Therefore, after some calculation similar as in Proposition 9 we get:

$$\langle (M^2T^{*2}T^2 - 2kT^*T + k^2)x, x\rangle = \langle (M^2A^{*2}A^2 - 2kA^*A + k^2)y, y\rangle \geq 0,$$

$\forall y \in \overline{T(\mathcal{H})}, \forall k > 0$.

Hence

$$M^2 A^{*2} A^2 - 2kA^*A + k^2 \geq 0, \forall k > 0.$$

This shows that A is a $M-$paranormal operator, on $\overline{T(\mathcal{H})}$.

Let P be the orthogonal projection of H onto $\overline{T(\mathcal{H})}$. For any

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} = \overline{T(\mathcal{H})} \oplus \ker T^*.$$

Then

$$\langle Cx_2, x_2 \rangle = \langle T(I-P)x, (I-P)x \rangle = \langle (I-P)x, T^*(I-P)x \rangle = 0.$$

Thus $T^* = 0$.

Since $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$ by [7, Corollary 7]. Since $\sigma(A) \cap \sigma(C)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$ and $C^k = 0$.

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