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Approximation of the Quadratic and Cubic Functional Equations in RN–spaces

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Abstract. We prove a stability result for the quadratic and cubic functional equations in random normed (RN) spaces (in the sense of Sherstnev) under arbitrary t–norms.

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1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [37], concerning the stability of group homomorphisms, affirmatively answered

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for Banach spaces by Hyers [13]. Subsequently, the result of Hyers was generalized by Aoki [2], Bourgin [4], Gǎvruta [8] and Rassias [24] (see also [9] and [31]). The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is related to a symmetric bi–additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi–additive function B such that $f(x)=B(x,x)$ for all x (see [1, 16]). The bi–additive function B is given by

$$B(x,y)=\frac{1}{4}[f(x+y)-f(x-y)]. \tag{2}$$

Hyers–Ulam–Rassias stability problem for the quadratic functional equation (1) was proved by Skof for functions $f:A\to B$, where A is normed space and B is a Banach space (see [36]). Cholewa [5] noticed that the theorem of Skof is still true if relevant domain A is replaced by an abelian group. In the paper [7], Czerwik proved the Hyers–Ulam–Rassias stability of the functional equation (1). Grabiec [10] has generalized these result mentioned above. We only mention here the papers [14], [16], [23], [32], [33] [27–30] concerning the stability of the quadratic functional equations.

The following cubic functional equation, which is the oldest cubic functional equation, and was introduced by J. M. Rassias [25](in 2001):

$$f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y).$$

Jun and Kim [15] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{3}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (3) (in this case we have a much better possible upper bound for (1.3) than the Hyers–Ulam–Rassias stability). The function $f(x) = x^3$ satisfies the functional equation (3), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. There are many works in the very active area of the stability of functional equations. We only mention here the papers [26] and [14] concerning the stability of the cubic functional equation.

The generalized Hyers–Ulam–Rassias stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [17]– [22].

In the sequel we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [3, 6, 17, 19, 34, 35]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$, such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point–wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1. *([34]). A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous t–norm) if T satisfies the following conditions:*

(a) *T is commutative and associative;*

(b) *T is continuous;*

(c) *$T(a, 1) = a$ for all $a \in [0, 1]$;*

(d) $T(a,b) \leq T(c,d)$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a,b,c,d \in [0,1]$.

Typical examples of continuous t–norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm). Recall (see [11], [12]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^{n} x_i$ is defined recurrently by $T_{i=1}^{1} x_i = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$. $T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$. It is known([12]) that for the Lukasiewicz t–norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1-x_n) < \infty \tag{4}$$

for all $t \geq 0$.

Definition 2. *([35]). A* random normed space *(briefly, RN–space) is a triple* (X,μ,T)*, where* X *is a vector space,* T *is a continuous* t*-norm, and* μ *is a mapping from* X *into* D^+ *such that, the following conditions hold:*
$(RN1)$ $\mu_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$*;*
$(RN2)$ $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ *for all* $x \in X$, $\alpha \neq 0$ *and all* $t \geq 0$*;*
$(RN3)$ $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ *for all* $x,y \in X$ *and* $t,s \geq 0$.

Every normed space $(X, \|.\|)$ defines a random normed space (X,μ,T_M) where

$$\mu_x(t) = \frac{t}{t+\|x\|},$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 3. *Let* (X,μ,T) *be an RN–space.*
(1) *A sequence* $\{x_n\}$ *in* X *is said to be* convergent *to* x *in* X *if, for every* $\epsilon > 0$ *and* $\lambda > 0$*, there exists positive integer* N *such that* $\mu_{x_n - x}(\epsilon) > 1-\lambda$ *whenever* $n \geq N$.
(2) *A sequence* $\{x_n\}$ *in* X *is called* Cauchy sequence *if, for every* $\epsilon > 0$ *and* $\lambda > 0$*, there*

exists positive integer N such that $\mu_{x_n-x_m}(\epsilon) > 1-\lambda$ whenever $n \geq m \geq N$.

(3) *An RN–space (X,μ,T) is said to be* complete *if and only if every Cauchy sequence in X is convergent to a point in X.*

Theorem 1. *([34]). If (X,μ,T) is an RN–space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

The aim of this paper is to investigate the stability of the quadratic and cubic functional equations in random normed spaces (in the sense of Sherstnev), under arbitrary continuous t–norms.

2. Main Results

In this section we establish the stability of the quadratic and cubic functional equation

$$f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)=0 \tag{5}$$

in the setting of random normed spaces.

Theorem 2. *Let X be a real linear space, (Y,μ,T) be a complete RN-space and $f: X\to Y$ be a mapping with $f(0)=0$ for which there is $\rho: X\times X\to D^+$ ($\rho(x,y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(x+y)+f(x-y)-2f(x)-2f(y)}(t) \geq \rho_{x,y}(t) \tag{6}$$

for all $x,y\in X$ and all $t>0$. If

$$\lim_{n\to\infty} T_{i=1}^{\infty}(\rho_{2^{n+i-1}x,2^{n+i-1}x}(2^{2n+2i}t)) = 1 \tag{7}$$

and

$$\lim_{n\to\infty}\rho_{2^n x,2^n y}(2^{2n}t) = 1 \tag{8}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{Q(x)-f(x)}(t) \geq T_{i=1}^{\infty}(\rho_{2^{i-1}x,2^{i-1}x}(2^{2i}t)). \tag{9}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (6), we get

$$\mu_{\frac{f(2x)}{2^2}-f(x)}(t) \geq \rho_{x,x}(2^2 t). \tag{10}$$

Therefore,

$$\mu_{\frac{f(2^{k+1}x)}{2^{2(k+1)}}-\frac{f(2^k x)}{2^{2k}}}(t) \geq \rho_{2^k x,2^k x}(2^{2(k+1)}t), \tag{11}$$

for all $k \in \mathbb{N}$ and all $t > 0$. By the triangle inequality it follows that

$$\begin{aligned}\mu_{\frac{f(2^n x)}{2^{2n}}-f(x)}(t) \geq T_{k=0}^{n-1}(\mu_{\frac{f(2^{k+1}x)}{2^{2(k+1)}}-\frac{f(2^k x)}{2^{2k}}}(t)) &\geq T_{k=0}^{n-1}(\rho_{2^k x,2^k x}(2^{2(k+1)}t)) \\ &= T_{i=1}^{n}(\rho_{2^{i-1}x,2^{i-1}x}(2^{2i}t)) \end{aligned} \tag{12}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$, we replace x with $2^m x$ in (12) to find that

$$\begin{aligned}\mu_{\frac{f(2^{n+m}x)}{2^{2(n+m)}}-\frac{f(2^m x)}{2^{2m}}}(t) &= \mu_{\frac{f(2^{n+m}x)}{2^{2n}}-f(2^m x)}(2^{2m}t) \\ &\geq T_{i=1}^{n}(\rho_{2^{i+m-1}x,2^{i+m-1}x}(2^{2i+2m}t)). \end{aligned} \tag{13}$$

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$ is a Cauchy sequence. Therefore, we may define $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$ for all $x \in X$. Now, we show that Q is a quadratic function. Replacing x, y with $2^n x$ and $2^n y$, respectively, in (6), it follows that

$$\mu_{\frac{f(2^n x+2^n y)}{2^{2n}}+\frac{f(2^n x-2^n y)}{2^{2n}}-2\frac{f(2^n x)}{2^{2n}}-2\frac{f(2^n y)}{2^{2n}}}(t) \geq \rho_{2^n x,2^n y}(2^{2n}t). \tag{14}$$

Taking the limit as $n \to \infty$, we find that Q satisfies (5) for all $x, y \in X$.
To prove (9), take the limit as $n \to \infty$ in (12). Finally, to prove the uniqueness of

the quadratic function Q subject to (9), let us assume that there exists a quadratic function Q' which satisfies (9) Since $Q(2^n x) = 2^{2n} Q(x)$ and $Q'(2^n x) = 2^{2n} Q'(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (9) it follows that

$$\begin{aligned}
&\mu_{Q(x)-Q'(x)}(2t) = \mu_{Q(2^n x)-Q'(2^n x)}(2^{2n+1} t) \\
&\geq T(\mu_{Q(2^n x)-f(2^n x)}(2^{2n} t), \mu_{f(2^n x)-Q'(2^n x)}(2^{2n} t)) \\
&\geq T(T_{i=1}^{\infty}(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(2^{2n+2i} t)), T_{i=1}^{\infty}(\rho_{2^{n+i-1}x, 2^{n+i-1}}(2^{2n+2i} t))) \qquad (15)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in (15), we find that $Q = Q'$.

Theorem 3. *Let X be a real linear space, (Y, μ, T) be a complete RN-space and $f : X \to Y$ be a mapping which there is $\tau : X \times X \to D^+$ ($\tau(x, y)$ is denoted by $\tau_{x,y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)}(t) \geq \tau_{x,y}(t) \qquad (16)$$

for all $x, y \in X$ and all $t > 0$. If

$$\lim_{n\to\infty} T_{i=1}^{\infty}(\tau_{2^{n+i-1}x,0}(2^{3n+2i} t)) = 1 \qquad (17)$$

and

$$\lim_{n\to\infty} \tau_{2^n x, 2^n y}(2^{3n} t) = 1 \qquad (18)$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{C(x)-f(x)}(t) \geq T_{i=1}^{\infty}(\tau_{2^{i-1}x,0}(2^{2i} t)). \qquad (19)$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = 0$ in (16), we get

$$\mu_{\frac{f(2x)}{2^3} - f(x)}(t) \geq \tau_{x,0}(2^4 t) \geq \tau_{x,0}(2^3 t). \qquad (20)$$

Therefore,

$$\mu_{\frac{f(2^{k+1}x)}{2^{3(k+1)}} - \frac{f(2^k x)}{2^{3k}}}(t) \geq \tau_{2^k x,0}(2^{3(k+1)} t), \qquad (21)$$

for every $k \in \mathbb{N}$ and $t > 0$. Thus we have

$$\mu_{\frac{f(2^{k+1}x)}{2^{3(k+1)}} - \frac{f(2^k x)}{2^{3k}}}\left(\frac{t}{2^{k+1}}\right) \geq \tau_{2^k x,0}(2^{2(k+1)}t), \tag{22}$$

for every $k \in \mathbb{N}$ and $t > 0$. As $1 > \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n}$, by the triangle inequality it follows

$$\begin{aligned}\mu_{\frac{f(2^n x)}{2^{3n}} - f(x)}(t) &\geq T_{k=0}^{n-1}\left(\mu_{\frac{f(2^{k+1}x)}{2^{3(k+1)}} - \frac{f(2^k x)}{2^{3k}}}\left(\frac{t}{2^{k+1}}\right)\right) \geq T_{k=0}^{n-1}(\tau_{2^k x,0}(2^{2(k+1)}t)) \\ &= T_{i=1}^{n}(\tau_{2^{i-1}x,0}(2^{2i}t)) \end{aligned} \tag{23}$$

for all $x \in x$ and all $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^n x)}{2^{3n}}\}$, we replace x with $2^m x$ in (23) to find that

$$\begin{aligned}\mu_{\frac{f(2^{n+m}x)}{2^{3(n+m)}} - \frac{f(2^m x)}{2^{3m}}}(t) &= \mu_{\frac{f(2^{n+m}x)}{2^{3n}} - f(2^m x)}(2^{3m}t) \\ &\geq T_{i=1}^{n}(\tau_{2^{i+m-1}x,0}(2^{2i+3m}t)). \end{aligned} \tag{24}$$

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{\frac{f(2^n x)}{2^{3n}}\}$ is a Cauchy sequence. Therefore, we may define $C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}}$ for all $x \in X$. Now, we show that C is a cubic function. Replacing x, y with $2^n x$ and $2^n y$ respectively in (16), it follows that

$$\mu_{\frac{f(2^{n+1}x+2^n y)}{2^{3n}} + \frac{f(2^{n+1}x-2^n y)}{2^{3n}} - 2\frac{f(2^n x+2^n y)}{2^{3n}} - 2\frac{f(2^n x-2^n y)}{2^{3n}} - 12\frac{f(2^n x)}{2^{3n}}}(t) \geq \tau_{2^n x,2^n y}(2^{3n}t). \tag{25}$$

Taking the limit as $n \to \infty$, we find that C satisfies (25) for all $x, y \in X$.

To prove (19), take the limit as $n \to \infty$ in (23). Finally, to prove the uniqueness of the cubic function C subject to (19), let us assume that there exists a cubic function C' which satisfies (19). Since $C(2^n x) = 2^{3n}C(x)$ and $C'(2^n x) = 2^{3n}C'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (19) it follows that

$$\begin{aligned}&\mu_{C(x)-C'(x)}(2t) = \mu_{C(2^n x)-C'(2^n x)}(2^{3n+1}t) \\ &\geq T(\mu_{C(2^n x)-f(2^n x)}(2^{3n}t), \mu_{f(2^n x)-C'(2^n x)}(2^{3n}t)) \\ &\geq T(T_{i=1}^{\infty}(\tau_{2^{n+i-1}x,0}(2^{3n+2i}t)), T_{i=1}^{\infty}(\tau_{2^{n+i-1}x,0}(2^{3n+2i}t))), \end{aligned} \tag{26}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in (26), we find that $C = C'$.

Example 1. *Let $(A, \|.\|)$ be a Banach algebra and*

$$\mu_x(t) = \begin{cases} \max\{1 - \frac{\|x\|}{t}, 0\}, & if \quad t > 0, \\ 0, & if \quad t \leq 0, \end{cases}$$

for every $x, y \in A$. Let

$$\rho_{x,y}(t) = \max\{1 - \frac{32\|x\| + 32\|y\|}{t}, 0\} \quad (t > 0)$$

and $\rho_{x,y}(t) = 0$ if $t \leq 0$. We note that $\rho_{x,y}$ is a distribution function and

$$\lim_{n\to\infty} \rho_{2^n x, 2^n y}(2^{2n} t) = 1$$

for all $x, y \in A$ and all $t > 0$. It is straightforward to show that (A, μ, T_L) is an RN-space. Indeed,

$$(\forall t > 0\ ;\ \ \mu_x(t) = 1) \Longrightarrow$$
$$(\forall t > 0\ ;\ \ \frac{\|x\|}{t} = 0) \Longrightarrow x = 0$$

and

$$\mu_{\lambda x}(t) = 1 - \frac{\|\lambda x\|}{t} = 1 - \frac{|\lambda|\|x\|}{t} = 1 - \frac{\|x\|}{\frac{t}{\lambda}} = \mu_x(\frac{t}{\lambda}) \tag{27}$$

for all $x \in A$ and all $t > 0$. Also, for every $x, y \in A$ and $t, s > 0$ we have

$$\begin{aligned} \mu_{x+y}(t+s) &= \max\{1 - \frac{\|x+y\|}{t+s}, 0\} = \max\{1 - \|\frac{x+y}{t+s}\|, 0\} \\ &= \max\{1 - \|\frac{x}{t+s} + \frac{y}{t+s}\|, 0\} \\ &\geq \max\{1 - \|\frac{x}{t} + \frac{y}{s}\|, 0\} \\ &\geq \max\{1 - \|\frac{x}{t}\| - \|\frac{y}{s}\|, 0\} \\ &= T_L(\mu_x(t), \mu_y(s)). \end{aligned}$$

It is also easy to see that (A, μ, T_L) is complete, for

$$\mu_{x-y}(t) \geq 1 - \frac{\|x-y\|}{t}\ ; \quad (x, y \in A, t > 0) \tag{28}$$

and $(A, \|.\|)$ *is complete. Define* $f : A \to A$, $f(x) = \|x\|x_0$, *where* x_0 *is a unit vector in* A. *A simple computation shows that*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|$$
$$= \|x+y\| + \|x-y\| - 2\|x\| - 2\|y\|$$
$$\leq 32\|x\| + 32\|y\|$$

for all $x, y \in A$, *hence*

$$\mu_{f(x+y)+f(x-y)-2f(x)-2f(y)}(t) \geq \rho_{x,y}(t), \tag{29}$$

for all $x, y \in A$ *and all* $t > 0$. *Fix* $x \in A$ *and* $t > 0$, *then*

$$\begin{aligned}(T_L)_{i=1}^{\infty}(\rho_{2^{n+i-1}x,2^{n+i-1}x}(2^{2n+2i}t)) &= \max\{\sum_{i=1}^{\infty}(\rho_{2^{n+i-1}x,2^{n+i-1}x}(2^{2n+2i}t) - 1) + 1, 0\}\\ &= \max\{1 - \frac{32\|x\|}{2^n t}, 0\},\end{aligned}$$

hence $\lim_{n\to\infty}(T_L)_{i=1}^{\infty}(\rho_{2^{n+i-1}x,2^{n+i-1}x}(2^{2n+2i}t)) = 1$. *Hence, all the conditions of Theorem 3 hold. Since*

$$\begin{aligned}(T_L)_{i=1}^{\infty}(\rho_{2^{i-1}x,2^{i-1}x}(2^{2i}t)) &= \max\{\sum_{i=1}^{\infty}(\rho_{2^{i-1}x,2^{i-1}x}(2^{2i}t) - 1) + 1, 0\}\\ &= \max\{1 - \frac{32\|x\|}{t}, 0\},\end{aligned}$$

we obtain that there exists a unique quadratic mapping $Q : A \longrightarrow A$ *such that*

$$\mu_{Q(x)-f(x)}(t) \geq \max\{1 - \frac{32\|x\|}{t}, 0\} \tag{30}$$

for all $x \in A$ *and all* $t > 0$.

Let

$$\tau_{x,y}(t) = \max\{1 - \frac{64\|x\| + 64\|y\|}{t}, 0\} \quad (t > 0) \tag{31}$$

and $\tau_{x,y}(t) = 0$ *if* $t \leq 0$. *We note that* $\tau_{x,y}$ *is a distribution function and*

$$\lim_{n\to\infty} \tau_{2^n x, 2^n y}(2^{3n}t) = 1 \tag{32}$$

for all $x, y \in A$ and all $t > 0$. It is obviously that (A, μ, T_L) is an RN–space. It is also easy to see that (A, μ, T_L) is complete, for

$$\mu_{x-y}(t) \geq 1 - \frac{\|x-y\|}{t} \quad (x, y \in A, t > 0) \tag{33}$$

and $(A, \|.\|)$ is complete. Define $g : A \to A$, $g(x) = x^3 + \|x\|x_0$, where x_0 is a unit vector in A. A simple computation shows that

$$\|g(2x+y) + g(2x-y) - 2g(x+y) - 2g(x-y) - 12g(x)\|$$
$$\leq 64\|x\| + 64\|y\|$$

for all $x, y \in A$, hence

$$\mu_{g(2x+y)+g(2x-y)-2g(x-y)-2g(x-y)-12g(x)}(t) \geq \tau_{x,y}(t),$$

for all $x, y \in A$ and all $t > 0$. Fix $x \in A$ and $t > 0$, then

$$\begin{aligned}(T_L)_{i=1}^{\infty}(\tau_{2^{n+i-1}x,0}(2^{3n+2i}t)) &= \max\{\sum_{i=1}^{\infty}(\tau_{2^{n+i-1}x,0}(2^{3n+2i}t) - 1) + 1, 0\} \\ &= \max\{1 - \frac{32\|x\|}{2^{2n}t}, 0\},\end{aligned}$$

hence $\lim_{n\to\infty}(T_L)_{i=1}^{\infty}(\tau_{2^{n+i-1}x,0}(2^{3n+2i}t)) = 1$.
Thus, all the conditions of Theorem 3 hold. Since

$$\begin{aligned}(T_L)_{i=1}^{\infty}(\tau_{2^{i-1}x,0}(2^{2i}t)) &= \max\{\sum_{i=1}^{\infty}(\tau_{2^{i-1}x,0}(2^{2i}t) - 1) + 1, 0\} \\ &= \max\{1 - \frac{32\|x\|}{t}, 0\},\end{aligned}$$

we obtain that there exists a unique cubic mapping $C : A \longrightarrow A$ such that

$$\mu_{c(x)-g(x)}(t) \geq \max\{1 - \frac{32\|x\|}{t}, 0\} \tag{34}$$

for all $x \in A$ and all $t > 0$.

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