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On the existence, uniqueness and application of the Finite difference method for solving Cauchy-Dirichlet problem

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Abstract. In this paper we treat the existence, the uniqueness and the numerical resolution of the problem at the elliptic limits case of the Cauchy-Dirichlet problem of the type the stationary convection-diffusion equation. By applying the Lax-Milgram theorem, we proved the existence and the uniqueness of the problem, then we solved the problem numerically by the finite difference method. In addition, we solved the problem analytically using the method of variation of constants. Finally, we performed a numerical simulation of said problem to approach the exact solution by the numerical solution using the software *Scilab*

2020 Mathematics Subject Classifications: 65M06, 65M12, 65K05

Key Words and Phrases: Cauchy-Dirichlet Problem, convection-diffusion equation, Finite difference method, numerical simulation

1. Introduction

Partial differential equations (PDEs) are used in several fields, including engineering, mechanics, physics, aeronautics, petroleum industry, but also in economics, chemistry, biology; in medicine ... they are defined in an open Ω of space $\mathbb{R}^n$, $n \geq 1$ (PFEs integration domain) of border $\partial\Omega$. They can be listed by type. Thus, the equations of the elliptical type describe the phenomena of stationary diffusion, the equations of the parabolic type describe the phenomena of diffusion and the equations of the hyperbolic type describe the phenomena of transport at finite speed. As a general rule, it is difficult to find a unique solution to an EDP without boundary conditions (additional information on the border $\partial\Omega$). Thus, we speak of Dirichlet's boundary condition, of Cauchy condition (that is to say of Dirichlet type on a part of the edge $\partial\Omega$ and of Cauchy type on the other part).

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Furthermore, a boundary problem is broken down into a PDEs (or ODE) inside the Ω domain and the boundary conditions of the problem. We speak in this case of the Dirichlet problem, Cauchy problem, Cauchy-Dirichlet problem (problem whose boundary conditions are a combination of a Cauchy condition and a Dirichlet condition). In addition, it is necessary, to approach the theoretical (mathematical) study of these type of problems, to have a good knowledge of vector and functional analysis (normed vector spaces, Lebesgue spaces), Hilbert analysis, matrix numerical analysis and also some knowledge of distributions, Sobolev spaces , traces, Green's formulas, the use of the Lax-Milgram theorem and some fundamental inequalities (Cauchy-Schwartz, Hölder, Poincaré, etc.) which emanate from the analysis and which are generally used in the framework of functional analysis of PDE [2–6, 8]. The PDE numerical resolution represents a field of research in mathematical sciences. There are several numerical methods of resolution available for each type of boundary problem, classified by categories [1, 12–18]. Thus, we speak of finite difference method, finite element method, finite volume method, etc. The manipulation of these resolution methods and the implementation of numerical simulations depend on the type of problem studied. However, incontestably few numerical methods have been used to solve the Cauchy-Dirichlet problem. The object of our work is to propose a theoretical (mathematical) and numerical study of the solution of the Cauchy-Dirichlet problem of the type of stationary convection-diffusion equation by the method of finite differences and to implement numerical simulations using the *Scilab* software, via the analytical and numerical solutions obtained by solving the problem.

2. Cauchy- Dirichlet Problem

In mathematics, a Cauchy–Dirichlet problem is the problem of finding a function that solves a specified partial differential equation (PDE) inside a given region that takes on prescribed values at the boundary of the region. The problem finds its applications in several fields of engineering sciences (for example in aeronautics) and is often presented in the form: let Ω be a bounded non-empty domain of class C^1 of $\mathbb{R}^n$ $(n \geq 1)$, f and c are two functions defined respectively on Ω and on $\partial\Omega = \Gamma$ (the border of Ω).
The Cauchy-Dirichlet problem [2, 9, 10] is an elliptical boundary problem of solution $u = u(x,t)$ which presents itself in the following way:

$$\begin{cases} -\nabla(\mu\nabla u) + c(x)\nabla u = f(x), x \in]0;1[\\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u_0(x) = \sin(19\pi x) \end{cases} \tag{1}$$

where $-\nabla(\mu\nabla u)$ and $c(x)\nabla u$ represent respectively , the diffusion and convection terms, $f(x) = 5\cos(\pi x) + (x-5)$, $c(x) = \sin(2x)$, $\mu > 0$ fixed positive real.

Variational formulation of the boundary value problems

Let us transform the problem (1) into a variational problem As $f \in L^2(\Omega)$ then

$$-\mu\Delta u \in L^2(\Omega)$$

$$u \in H^1(\Omega)$$

$u \in H^1(\Omega)$ and $u = 0$ on $\partial\Omega$, then $u \in H_0^1(\Omega)$.

Now, let's give the variational formulation of the problem (1)

Let $v \in H_0^1(\Omega)$

Multiply the first equation of the system (1) by v and integrate member to member on Ω

$$\int_\Omega -\mu\Delta u.v\,d\Omega + \int_\Omega c\nabla u.v\,d\Omega = \int_\Omega f.v\,d\Omega \tag{2}$$

Let's use Green's formula [8, 11], then

$$\mu\int_\Omega \nabla u.\nabla v\,d\Omega + \int_\Omega c\nabla u.v\,d\Omega = \int_\Omega f.v\,d\Omega \tag{3}$$

We thus obtain the following variational formulation

$$\begin{cases} u \in H_0^1(\Omega) \\ a(u,v) = L(v) \quad \forall v \in H_0^1(\Omega) \end{cases} \tag{4}$$

where $\quad a(u,v) = \mu\int_\Omega \nabla u.\nabla v\,d\Omega + \int_\Omega c\nabla u.v\,d\Omega \quad$ and $\quad L(v) = \int_\Omega f.v\,d\Omega$

3. Existence and uniqueness of the solution

Let us prove the existence and the uniqueness of the solution $u \in H_0^1(\Omega)$ by Lax-Milgram theorem [8, 9, 11].
let $u \in V = H_0^1(\Omega)$

- **Let us show that the bilinear form a is continuous**
  we have

$$\mid a(u,v) \mid = \mid \mu\int_\Omega \nabla u.\nabla v\,d\Omega + \int_\Omega c\nabla u.v\,d\Omega \mid \tag{5}$$

$$\mid a(u,v) \mid \leq \mid \mu \mid \int_\Omega \mid \nabla u.\nabla v \mid\,d\Omega + \int_\Omega \mid c\nabla u.v \mid\,d\Omega \tag{6}$$

Since

$$c \in L^\infty(\Omega), \quad \text{then} \quad \mid c \mid \leq M$$

thus

$$| a(u,v) | \leq | \mu | \int_{\Omega} | \nabla u.\nabla v | \; d\Omega + M \int_{\Omega} | \nabla u.v | \; d\Omega \tag{7}$$

According to the Cauchy-Schwarz inequality [8, 10, 11]

$$| a(u,v) | \leq | \mu | (\int_{\Omega} | \nabla u |^2 \; d\Omega)^{\frac{1}{2}} (\int_{\Omega} | \nabla v |^2 \; d\Omega)^{\frac{1}{2}} + M (\int_{\Omega} | \nabla u |^2 \; d\Omega)^{\frac{1}{2}} (\int_{\Omega} | v |^2 \; d\Omega)^{\frac{1}{2}} \tag{8}$$

$$\leq | \mu | \| \nabla u \|_{L^2(\Omega)} . \| \nabla v \|_{L^2(\Omega)} + M \| \nabla u \|_{L^2(\Omega)} . \| v \|_{L^2(\Omega)} \tag{9}$$

$$\leq | \mu | \| u \|_{H_0^1(\Omega)} . \| v \|_{H_0^1(\Omega)} + Mc \| u \|_{H_0^1(\Omega)} . \| v \|_{H_0^1\Omega)} \tag{10}$$

$$\leq | \mu | \| u \|_{H_0^1(\Omega)} . \| v \|_{H_0^1(\Omega)} + M' \| u \|_{H_0^1(\Omega)} . \| v \|_{H_0^1\Omega)} \tag{11}$$

$$\leq (| \mu | + M') \| u \|_{H_0^1(\Omega)} . \| v \|_{H_0^1(\Omega)} \tag{12}$$

$$| a(u,v) | \leq C \| u \|_{H_0^1(\Omega)} . \| v \|_{H_0^1(\Omega)}, \quad \text{where} \quad C = | \mu | + M', \text{whith} \quad M' = \text{constant} \tag{13}$$

therefore a is continue.

- **Let us show that the bilinear form a is coercive**
  let $v = u$, then

$$a(u,u) = \mu \int_{\Omega} (\nabla u)^2 \, d\Omega + \int_{\Omega} c \nabla u.u \, d\Omega \tag{14}$$

Assuming that, $c \geq c_0 > 0$, therefore

$$a(u,u) \geq \mu \int_{\Omega} (\nabla u)^2 \, d\Omega + c_0 \int_{\Omega} \nabla u.u \, d\Omega \tag{15}$$

$$a(u,u) \geq \mu \int_{\Omega} (\nabla u)^2 \, d\Omega \tag{16}$$

$$a(u,u) \geq \mu \| u \|^2_{H_0^1(\Omega)} \tag{17}$$

$$a(u,u) \geq \alpha \| u \|^2_{H_0^1(\Omega)}, \text{with} \mu = \alpha \tag{18}$$

$$\text{therefore} \quad a \quad \text{is coercive.} \tag{19}$$

$$\tag{20}$$

- **Let us show that the linear form L is continuous on** $V = H_0^1(\Omega)$

$$\text{we have,} \quad L(v) = \int_{\Omega} f.v \, d\Omega \tag{21}$$

$$= | \int_{\Omega} f.v \, d\Omega | \tag{22}$$

$$\leq \int_{\Omega} \mid f.v \mid \ d\Omega \tag{23}$$

According to the Cauchy-Schwartz inequality

$$\mid L(v) \mid \leq (\int_{\Omega} \mid f \mid^2 \ d\Omega)^{\frac{1}{2}} (\int_{\Omega} \mid v \mid^2 \ d\Omega)^{\frac{1}{2}} \tag{24}$$

$$\mid L(v) \mid \leq \parallel f \parallel_{L^2(\Omega)} . \parallel v \parallel_{L^2(\Omega)} \tag{25}$$

$$\tag{26}$$

Since $f \in L^2(\Omega)$ then $\parallel f \parallel_{L^2(\Omega)} \leq M_1$ and $\parallel v \parallel_{L^2(\Omega)} \leq C \parallel v \parallel_{H_0^1(\Omega)}$

therefore $\mid L(v) \mid \leq M_1 C \parallel v \parallel_{H_0^1(\Omega)}$ hence $\mid L(v) \mid \leq \beta \parallel v \parallel_{H_0^1(\Omega)}$ with$\beta = M_1 C$

Thus L is continue.

All the hypotheses of the Lax-Milgram theorem being satisfied, we deduce that the variational problem admits a unique solution $u \in H_0^1(\Omega)$.

4. Numerical resolution of the problem

In this section, we will try to solve the Cauchy-Dirichlet problem by a numerical method, in particular the finite difference method. To do this, let's use the Cauchy-Dirichlet problem in dimension one, which is presented in the following way:

$$\begin{cases} -\mu u''(x) + c(x)u'(x) = f(x), x \in]0;1[\\ u(0,t) = u(1,t) \end{cases} \tag{27}$$

One describes the method in three parts: choice of the mesh, choice of the numerical scheme and discretization of the problem.

First step: Choice of the discretization.

We consider a subdivision, $0 = x_0 < x_1 < x_2 < ... < x_N < x_{N+1} = 1$,
of the interval $[0, 1]$, where $N \in \mathbb{N}$.

For $i = 0, ..., N$, we put, $\Delta x_i = x_{i+1} - x_i$, with $\Delta x = \max\limits_{1 \leqslant i \leqslant N} \Delta x_i$, the mesh step.
the first step of discretization consists in approximating the functions u, c and f at the nodes x_i, that is to say:

$$u(x_i) \simeq u_i, c(x_i) \simeq c_i, f(x_i) \simeq f_i$$

the problem (27) become

$$\begin{cases} -\mu.u''_i + c_i.u'_i = f_i, \forall i = 1, ..., N \\ u(0) = u_{N+1} \end{cases} \tag{28}$$

Second step: Construction of the numerical scheme
Assuming that, $u \in C^2([0;1])$,
then u admits a Taylor expansion in the neighborhood of x_i in the form:

$$u(x_i+1) - u(x_i+\Delta x) = u(x_i) - \frac{\Delta x}{1!}u'(x_i) + \frac{\Delta x^2}{2!}u''(x_i) + 0(\Delta x^3) \tag{29}$$

$$u(x_i-1) - u(x_i-\Delta x) = u(x_i) - \frac{\Delta x}{1!}u'(x_i) + \frac{\Delta x^2}{2!}u''(x_i) + 0(\Delta x^3) \tag{30}$$

Going to the approximations, we have:

$$u_{i+1} = u_i + \frac{\Delta x}{1!}u'_i + \frac{\Delta x^2}{2!}u''_i + 0(\Delta x^3) \tag{31}$$

$$u_{i-1} = u_i - \frac{\Delta x}{1!}u'_i + \frac{\Delta x^2}{2!}u''_i + 0(\Delta x^3) \tag{32}$$

Adding the two equalities, to obtain the following expression

$$u''_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \tag{33}$$

The first derivative has been approximated using the forward finite difference method of order 1, that is to say

$$(\frac{\partial u}{\partial x})_i \simeq \frac{u_{i+1} - u_i}{\Delta x}, \tag{34}$$

with $u_0 = u_{N+1} = 0$.
we have

$$-\mu u''(x) + c(x)u'(x) = f(x), \tag{35}$$

that is to say

$$-\mu(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}) + c_i(\frac{u_{i+1} - u_i}{\Delta x}) = f_i \tag{36}$$

therefore,

$$\frac{-\mu}{\Delta x^2}(u_{i+1} - 2u_i - u_{i-1}) + \frac{c_i}{\Delta x}(u_{i+1} - u_i) = f_i \tag{37}$$

Multiplying the equation (37) by $\frac{\Delta x^2}{\mu}$, to obtain

$$(\lambda c_i - 1)u_{i+1} + (2 - \lambda c_i)u_i - u_{i-1} = \lambda \Delta x f_i, \quad \text{with} \quad \lambda = \frac{\Delta x}{\mu} \tag{38}$$

for $i = 1$, $(\lambda c_1 - 1)u_2 + (2 - \lambda c_1)u_1 - u_0 = \lambda \Delta x f_1$
for $i = 2$, $(\lambda c_2 - 1)u_3 + (2 - \lambda c_2)u_2 - u_1 = \lambda \Delta x f_2$
for $i = 3$, $(\lambda c_3 - 1)u_4 + (2 - \lambda c_3)u_3 - u_2 = \lambda \Delta x f_3$
$\vdots$
for $i = N$, $(\lambda c_N - 1)u_{N+1} + (2 - \lambda c_N)u_N - u_{N-1} = \lambda \Delta x f_N$

third step: writing Matrix
Taking into account the boundary conditions, we obtain the following linear system

$$\begin{cases} (2-\lambda c_1)u_1 + (\lambda c_1 - 1)u_2 = \lambda \Delta x f_1 \\ -u_1 + (2-\lambda c_2)u_2 + (\lambda c_3 - 1)u_3 = \lambda \Delta x f_2 \\ -u_2 + (2-\lambda c_3)u_3 + (\lambda c_3 - 1)u_4 = \lambda \Delta x f_3 \\ \vdots \\ -u_{N-1} + (2-\lambda c_N)u_N = \lambda \Delta x f_N \end{cases} \tag{39}$$

hence the matrix below:

$$\begin{pmatrix} 2-\lambda c_1 & \lambda c_1 - 1 & 0 & 0 & ... & 0 \\ -1 & 2-\lambda c_2 & \lambda c_2 - 1 & 0 & ... & 0 \\ 0 & -1 & 2-\lambda c_3 & \lambda c_3 - 1 & ... & \vdots \\ \vdots & \vdots & \vdots & \vdots & ... & \lambda c_{N-1} \\ 0 & 0 & ... & -1 & & 2-\lambda c_N \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{pmatrix} = \lambda \Delta x \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix} \tag{40}$$

Thus, the problem (40) boils down to the following linear system:

$$A_\lambda U = B \tag{41}$$

It remains to verify that if A_λ is symmetric positive definite to prove the existence and the uniqueness of the solution u of the system (41). That is to say:

$$\begin{cases} A_\lambda^t = A_\lambda & (1) \\ \forall v \in \mathbb{R}^{\mathbb{N}}, & \\ A_\lambda v.v \ > 0 & (2) \\ A_\lambda v.v = 0 \Longrightarrow v = 0 & (3) \end{cases}$$

let's suppose that $\lambda c_i - 1 = -1$, that is to say $\lambda c_i = 0$ for all $i \in \{1, 2, 3, ..., N\}$, $c = 0$ and $\lambda \neq 0$.
The relation (1) is trivial because the matrix A_λ is tridiagonal with the values of the overdiagonal which are equal to the values of the subdiagonal.
Admitting $\lambda c_i = 0$ et $v_0 = v_{N+1} = 0$, to get:

$$A_\lambda v = \begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & 0 \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & ... & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{N-1} \\ v_N \end{pmatrix}$$

$$< A_\lambda v, v >= \begin{pmatrix} 2v_1 - v_2 - v_1 + 2v_2 - v_3 \\ -v_2 + 2v_3 - v_4 \\ \vdots \\ -v_{N-2} + 2v_{N-1} - v_N \\ -v_{N-1} + 2v_N \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{N-1} \\ v_N \end{pmatrix};$$

$$< A_\lambda v, v >= (2v_1-v_2)v_1+(-v_1+2v_2-v_3)v_2+(-v_2+2v_3-v_4)v_3+(-v_{N-2}+2v_{N-1}-v_N)v_{N-1}+(-v_{N-1}+2v_N)v_N$$

$$< A_\lambda v, v >= \sum_{i=1}^{N}[-v_{i-1} + 2v_i - v_{i+1}]v_i$$

$$< A_\lambda v, v >= \sum_{i=1}^{N}(-v_{i-1}v_i) + \sum_{i=1}^{N}(2v_i^2) + \sum_{i=1}^{N}(-v_{i+1}v_i)$$

or

$$\sum_{i=1}^{N}(-v_{i+1}v_i) = \sum_{i=2}^{N+1}(-v_{i-1}v_i)$$

$$\sum_{i=2}^{N+1}(-v_iv_{i-1}) + \sum_{i=1}^{N}(-v_{i-1}v_i) = \sum_{i=1}^{N}(-v_{i-1}v_i)$$

and

$$\sum_{i=1}^{N}(2v_i^2) = \sum_{i=1}^{N}(-v_i^2 + v_{i-1}^2) + v_N^2$$

thus

$$< A_\lambda v, v >= \sum_{i=1}^{N}(-2v_{i-1}v_i) + \sum_{i=1}^{N}(v_i^2 + v_{i-1}^2) + v_N^2$$

$$< A_\lambda v, v >= \sum_{i=1}^{N}(v_i^2 - 2v_iv_{i-1} + v_{i-1}^2) + v_N^2$$

$$< A_\lambda v, v >= [\sum_{i=1}^{N}(v_i - v_{i-1})^2 + v_N^2] \geq 0$$

the relation (3) gives:

$$< A_\lambda v, v >= 0 \Longrightarrow \begin{cases} v_i - v_{i-1} = 0 \\ v_i = 0 \\ \vdots \\ v_1 = 0 \end{cases}$$

$$\Longrightarrow \; v = 0$$

(1),(2) et (3) being verified, then the problem (41) admits a unique solution.

5. Analytical resolution of the problem

To solve the problem analytically, we are going to use the method of variation of constants which is done in three steps:

First step: Resolution of the associated homogeneous equation

Consider: $f(x) = 5\cos(\pi x) + (x-5)$, $h(x) = \sin(19\pi x)$ et $c(x) = c_0$.

Thus, we obtain:

$$\begin{cases} -\mu u''(x) + c_0 u'(x) = 5\cos(\pi x) + (x-5) \\ u(0) = 0, u(1) = 0 \end{cases}$$

Solving the associated homogeneous equation

$$-\mu u''(x) + c_0 u'(x) = 0$$

to get the solution:

$$u(x) = c_1 + c_2 e^{\beta x}$$

with $\beta = \frac{c_o}{\mu}$, $\quad c_1, c_2 \in \mathbb{R}$

Second step: Variation of constants

Let us determine $c_1(x)$ and $c_2(x)$ such that:

$$u(x) = c_1(x) + c_2(x)e^{\beta x}, \tag{42}$$

Let be the solution of the equation with second member. For that, it is a question of solving the following system:

$$\begin{cases} c_1'(x) + c_2'(x)e^{\beta x} = 0 \\ \beta c_2'(x)e^{\beta x} = \frac{f(x)}{a} = -\frac{5\cos(\pi x)+(x-5)}{\mu} \end{cases} \tag{43}$$

we obtain

$$c_1(x) = \frac{5}{\mu\beta\pi}\sin(\pi x) + \frac{1}{\mu\beta}(\frac{x^2}{2} - 5x) + \alpha_1 \quad \text{with} \quad \alpha_1 \in \mathbb{R} \tag{44}$$

$$c_2(x) = \frac{1}{c_o}\left[\frac{-5}{\pi^2+\beta^2}(\pi\sin\pi x - \beta\cos\pi x) + \frac{1}{\beta^2} + \frac{1}{\beta}(x-5)\right]e^{-\beta x} + \alpha_2 \quad \text{with} \quad \alpha_2 \in \mathbb{R} \tag{45}$$

Thus, the relation (42) becomes:

$$u(x) = \frac{5}{c_o\pi}\sin(\pi x)+\frac{1}{c_o}(\frac{x^2}{2}-5x)+\frac{1}{c_o}\left[\frac{-5}{\pi^2+\beta^2}(\pi\sin\pi x-\beta\cos\pi x)+\frac{1}{\beta^2}+\frac{1}{\beta}(x-5)\right]+\alpha_1+\alpha_2 e^{-\beta x} \tag{46}$$

Third step: General solution of the complete equation

Let's find α_1 and α_2

Using the boundary conditions, we have:

$$\begin{cases} \frac{5\beta}{c_o(\pi^2+\beta^2)}+\frac{1}{c_o\beta^2}-\frac{5}{c_o\beta}+\alpha_1+\alpha_2=0 \\ \frac{-9}{2c_0}-\frac{5\beta}{c_0(\pi^2+\beta^2)}+\frac{1}{c_0\beta^2}-\frac{4}{c_0\beta}+\alpha_1+\alpha_2 e^{\beta}=0 \end{cases} \tag{47}$$

Solving the equation (47), to obtain

$$\alpha_1 = \frac{1}{c_o(e^\beta-1)}\left[-\frac{9}{2}+\frac{1-4\beta+(5\beta-1)e^\beta}{\beta^2}-\frac{5\beta(1+e^\beta)}{\beta^2+\pi^2}\right] \tag{48}$$

$$\alpha_2 = \frac{1}{c_o(e^\beta-1)}\left[\frac{9}{2}+\frac{10\beta}{\beta^2+\pi^2}-\frac{-1}{\beta}\right] \tag{49}$$

Thus , replacing (49) and (48) in (46),

$$u(x) = \frac{5}{c_o\pi}\sin(\pi x)+\frac{1}{c_o}(\frac{x^2}{2}-5x)+\frac{1}{c_o}\left[\frac{-5}{\pi^2+\beta^2}(\pi\sin\pi x-\beta\cos\pi x)+\frac{1}{\beta^2}+\frac{1}{\beta}(x-5)\right]+$$
$$+\frac{1}{c_o(e^\beta-1)}\left[\frac{9}{2}(e^\beta-1)+\frac{1-4\beta+(5\beta-1)e^\beta}{\beta^2}+\frac{5\beta(e^\beta-1)}{\beta^2+\pi^2}-\frac{e^\beta}{\beta}\right]. \tag{50}$$

6. Numerical simulation

The goal here is to represent on the same graph the solutions (50) and (41) exact and numeric, respectively, taking into account the number of points N and the step h of the finite difference method with the objective of making the two solutions converge (exact and numerical). This simulation will be implemented in *Scilab*. The figure 1 illustrates the exact solution (50) for $\mu = 1$ and $c_0 = 3$.

By fixing the number of points $N = 5$ in the finite difference method (41), we sought to

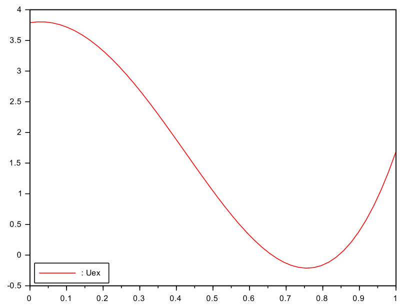
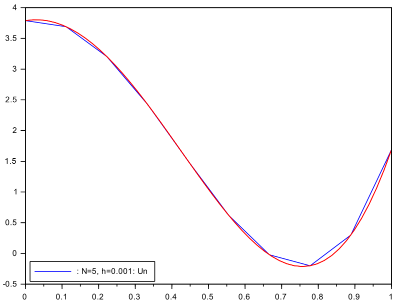


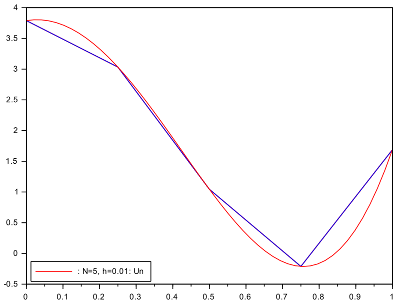
Figure 1: Representation of the exact solution.

vary the step h of the method to verify the numerical convergence of (41) on (50). (cf fig 2).

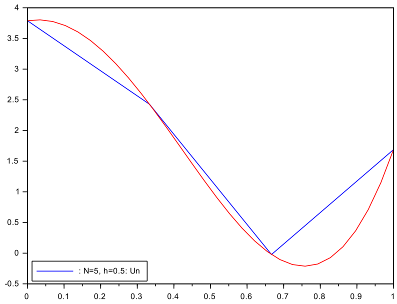
- Taking for step $h = 0.001$, on can notice that both exact and numerical solutions converge almost everywhere numerically (figure 2 a).
- Taking for step $h = 0.01$, we note that at the beginning the two solutions (exact and numerical) tend to move away and at a moment converge and move away very quickly after (figure ref fig2 b), which is explained by a weak numerical convergence.
- Taking for step $h = 0.5$, one can notice that both exact and numerical solutions respectively tend not to converge except on almost negligible points, which is explained by a numerical divergence (figure 2 c).



(a)



(b)

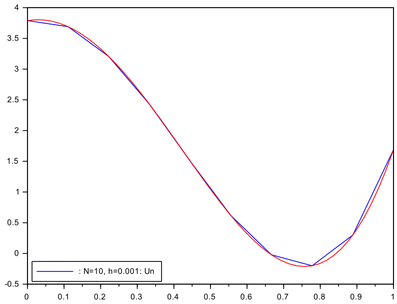


(c)

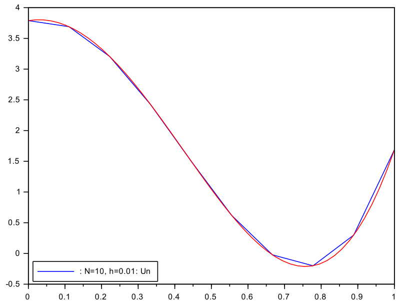
Figure 2: Representation of the exact and numerical solution for $N = 5$.

In the figure 3, let's fix $N = 10$.

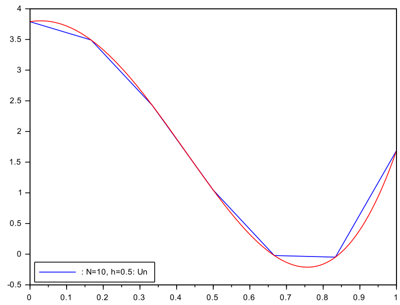
- By taking $h = 0.001$ and $h = 0.01$, we note that both exact and numerical solutions respectively (Figure 3 d and Figure 3 e) converge very fast numerically.
- By taking $h = 0.5$, one can see that the exact and numerical solutions respectively converge numerically almost everywhere.(Figure 3f)



(d)



(e)



(f)

Figure 3: Representation of the exact and numerical solution for $N = 10$

The finite difference method being linked to a multiplicative term of the form $\frac{1}{h^2}$, ensuring its convergence towards the exact solution requires a very large number of points N and a very good choice of the step h of the method.

Conclusion

The resolution of the problem of Cauchy-Dirichlet remains a challenge to be taken up of which several authors introduced the concept of very weak solution to a problem of Cauchy for the elliptic equations. The Cauchy-Dirichlet problem is regularized by a non-local boundary value problem whose solution is understood in this very weak sense [7]. In our this work we have numerically solved the Cauchy-Dirichlet problem by the finite difference method. In addition to prove the existence and the uniqueness of the solution to the linear system obtained we set $\lambda c_i - 1 = -1$, $i = 1, 2, ..., N$ so that the matrix is symmetrical. We then solved the problem analytically using the method of the variation of the constants. Finally, we have implemented numerical simulations in order to make the numerical solution converge towards the exact solution, using the *Scilab* software.

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