Second Duals of Measure Algebras

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Abstract. In this paper we show that \(M(G)^{**}\) determines \(G\) when \(G\) is a compact topological group. It is a new proof for theorem of Gharamani and Mcclure.

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The second dual space \(\mathcal{A}^{**}\) of a Banach algebra \(\mathcal{A}\) admits the Banach algebra product known as first (left) Arens product. This product extends the product of \(\mathcal{A}\) as canonically embedded in \(\mathcal{A}^{**}\). We briefly recall the definition of this product. For \(m, n \in \mathcal{A}^{**}\), their first (left) Arens product indicated by \(mn\) is given by

\[
\langle mn, f \rangle = \langle m, nf \rangle \quad (f \in \mathcal{A}^*)
\]

where \(nf \in \mathcal{A}^*\) is defined by

\[
\langle nf, a \rangle = \langle n, fa \rangle \quad (a \in \mathcal{A}).
\]

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(See [1] and [2]). Wendel in [6] proved that for locally compact groups $G_1$ and $G_2$, the group algebras $L^1(G_1)$ and $L^1(G_2)$ are isometrically isomorphic if and only if $G_1$ and $G_2$ are isomorphic in the category of topological groups. Johnson in [5] proved that the algebra $M(G)$ determines $G$ when $G$ is a locally compact group. In [3] Ghahramani and Lau have proved that $L^1(G)^{**}$ determines $G$ when $G$ is a locally compact group. Ghahramani and Mcclure in [4] proved that the algebra $(M(G))^{**}$ determines $G$ when $G$ is a compact topological group. In this paper we define some new ideals in Banach algebras and we apply this ideals to consider a new proof to show that $(M(G))^{**}$ determines $G$ when $G$ is compact. Let $\mathcal{A}$ be a Banach algebra. We consider

$$Z_l(\mathcal{A}) := \{ a \in \mathcal{A} : \mathcal{A}^{**} \cdot \hat{1} \subseteq \mathcal{A} \}.$$ 

It is easy to show that $Z_l(\mathcal{A})$ is a two sided ideal of $\mathcal{A}$ so it is a left ideal of $\mathcal{A}^{**}$. Also $Z_l(\mathcal{A})$ is the union of all two sided ideals of $\mathcal{A}$ which are left ideals of $\mathcal{A}^{**}$. First we prove the following lemma.

**Lemma 1.** Let $\theta : \mathcal{A} \to \mathcal{B}$ be an isometrically isomorphism between Banach algebras. Then $\theta(Z_l(\mathcal{A})) = Z_l(\mathcal{B})$.

**Proof.** Let $\theta : \mathcal{A} \to \mathcal{B}$ be an isometrically isomorphism between Banach algebras. Then $\theta^{**}$ is an isometrically isomorphism between Banach algebras $\mathcal{A}^{**}$ and $\mathcal{B}^{**}$. Let $a \in Z_l(\mathcal{A})$ and $b^{**} \in \mathcal{B}^{**}$. Then there exists $a^{**} \in \mathcal{A}^{**}$ such that $b^{**} = \theta^{**}(a^{**})$. Thus

$$b^{**} \theta(a) = \theta^{**}(a^{**}) \theta(a) = \theta^{**}(a^{**} \hat{1}) = \theta(a^{**} \hat{1}) \in \theta(\mathcal{A}) = \mathcal{B}.$$ 

Then $\theta(Z_l(\mathcal{A})) \subset Z_l(\mathcal{B})$. ■

**Theorem 1.** Let $G$ be a compact group. Then $Z_l((M(G))^{**}) = \pi^{**}(L^1(G))^{**}$.

**Proof.** Let $(e_a)$ be a bounded approximate identity of $L^1(G)$ with bound 1, and with cluster point $E \in L^1(G)^{**}$. We denote $\pi : L^1(G) \to M(G)$ the inclusion map,
then the map
\[ m \mapsto (\pi''(E))\hat{m} : M(G) \rightarrow \pi''(L^1(G)^{**}) \]
is isometric embedding. We denote this map with \( \Gamma \). Since the restriction of \( \Gamma_E \) to \( L^1(G) \) is identity map, then \( \Gamma_E(m) \in \pi(L^1(G)) \) if and only if \( m \in L^1(G) \). It is easy to show that \( \Gamma_E'' \) is isometrically embedding from \( (M(G))^{**} \) into \( \pi'''((L^1(G))^{****}) \).

The restriction of \( \Gamma_E'' \) to \( \pi''(L^1(G)^{**}) \) is identity map, then for every \( m'' \in (M(G))^{**} \), \( \Gamma_E''(m'') \in \pi''(L^1(G)^{**}) \) if and only if \( m'' \in \pi''(L^1(G)^{**}) \). Let now \( m'' \in Z_i((M(G))^{**}) \), then \( (M(G))^{****}m'' \subseteq (M(G))^{**} \). Thus
\[ \pi'''(L^1(G))^{****}m'' \subseteq (M(G))^{**}. \] (1)

On the other hand, we have direct sum decompositions
\[ (L^1(G))^{****} = \widehat{L^1(G)^{**}} \oplus \widehat{(L^1(G)^*)} \] (2)
and
\[ (M(G))^{****} = \hat{M(G)^{**}} \oplus \hat{(M(G)^*)}. \] (3)

So we have
\[ \pi'''((L^1(G)^*)^\perp) \subseteq (M(G)^*)^\perp. \] (4)

Since \( \pi'''(L^1(G)^{****}) \) is an ideal of \( M(G)^{****} \), then by (2) and (4), we have
\[ \pi'''(L^1(G)^{****}m'') \subseteq [(M(G)^*)^\perp \cap \pi'''(L^1(G)^{****})] = \pi'(L^1(G)^{**}). \]
Therefore \( \Gamma_E''(m'') \in \pi'(L^1(G)^{**}) \) and \( m'' \in \pi''(L^1(G)^{**}) \), hence, \( Z_i(M(G)^{**}) \subseteq \pi''(L^1(G)^{**}) \). On the other hand since \( G \) is compact then \( \pi''(L^1(G)^{**}) \) is a two sided ideal of \( \pi'''(L^1(G)^{****}) \), so \( Z_i(\pi''(L^1(G)^{**})) = \pi''(L^1(G)^{**}) \) and \( Z_i(\pi''(L^1(G)^{**})) \) is a two sided ideal of \( M(G)^{****} \). Hence, \( \pi''(L^1(G)^{**}) \subseteq Z_i(M(G)^{**}) \).

We now apply above theorem to show that \( M(G)^{**} \) determines \( G \) when \( G \) is a compact topological group. It is a new proof for the main result of [4]. By Lemma 1 and Theorem 1 we have the following.
Corollary 1 (Theorem 7 of 4). If $G_1$ and $G_2$ are compact groups, and if $\theta$ is an isometric isomorphism from $M(G_1)^{**}$ onto $M(G_2)^{**}$, then $\theta(L^1(G_1)^{**}) = L^1(G_2)^{**}$.

Since $L^1(G)^{**}$ determines $G$ [3], then we have

Corollary 2. If $G$ is a compact group, then $M(G)^{**}$ determines $G$.

References