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The Fuglede-Putnam theorem and quasinormality for class p-$wA(s,t)$ operators

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Abstract. In this work, we demonstrate that (i) if T is a class p-$wA(s,t)$ operator and $T(s,t)$ is quasinormal (resp., normal), then T is also quasinormal (resp., normal) (ii) If T and $T*$ are class p-$wA(s,t)$ operators, then T is normal; (iii) the normal portions of quasisimilar class p-$wA(s,t)$ operators are unitarily equivalent; and (iv) Fuglede-Putnam type theorem holds for a class p-$wA(s,t)$ operator T for $0 < s,t,s+t = 1$ and $0 < p \leq 1$ if T satisfies a kernel condition $\ker(T) \subset \ker(T^*)$.

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1. Introduction

On a complex Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators. Aluthge [2] investigated the p-hyponormal operator T, which is defined as $(T^*T)^p \geq (TT^*)^p$ with $0 \leq p \leq 1$ using the Furuta inequality [14]. When $p = 1$, T is said to be hyponormal. As a result, p-hyponormality is a broadening of hyponormality. Following [2], several authors are looking towards novel hyponormal operator generalizations.

It is known that p-hyponormal operators have many interesting properties as hyponormal operators, for example, Putnam's inequality, Fuglede-Putnam type theorem, Bishop's property (β), Weyl's theorem and polaroid. Let $T \in \mathcal{B}(\mathcal{H})$ and $|T| = (T^*T)^{\frac{1}{2}}$. By taking $U|T|x = Tx$ for $x \in \mathcal{H}$ and $Ux = 0$ for $x \in \ker|T|$, T has a unique polar decomposition $T = U|T|$ with condition $\ker U = \ker|T|$. We say that $T = U|T|$ is the polar decomposition of T. In [2], Aluthge extended the class of hyponormal operators by introducing p-hyponormal operators and obtained some properties with the help of the transformation

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$T(\frac{1}{2},\frac{1}{2}) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, which now known as the Aluthge transform. The introduction of these operators by Aluthge has inspired many researchers not only to expose some important properties of p-hyponormal operators but also to introduce the number of extensions ([1, 7, 8, 13]).

The Aluthge transform, and more broadly, the generalized Aluthge transform defined as $T(s,t) = |T|^sU|T|^t$ with $s,t > 0$, have proven to be useful tools in this attempt. The generalized Aluthge transform is used to analyze class p-$wA(s,t)$ operators in this article.

Definition 1. *Let $T = U|T|$ be the polar decomposition of an operator $T \in \mathcal{B}(\mathcal{H})$. Then the generalized Aluthge transform $T(s,t)$ of T is defined as follows:*

$$T(s,t) = |T|^sU|T|^t.$$

Moreover, for each nonnegative integer n, the n-th generalized Aluthge transform $\Delta^n(T(s,t))$ of $T(s,t)$ is defined as follows:

$$\Delta^n(T(s,t)) = \Delta(\Delta^{n-1}(T(s,t))), \Delta^0(T(s,t)) = T(s,t).$$

Definition 2. *Let $0 < s,t$, and $0 < p \leq 1$. An operator T is said to be a class*

(i) p-$wA(s,t)$ *if*

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$$

and

$$|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}}.$$

(ii) p-$A(s,t)$ *if* $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$.

(iii) p-A *if* $|T^2|^p \geq |T|^{2p}$.

(iv) (s,p)-w-*hyponormal if* $|T(s,s)|^p \geq |T|^{2sp} \geq |(T(s,s)^*|^p$.

It is known that p-hyponormal operators and log-hyponormal operators are class 1-$wA(s,t)$ for any $0 < s,t$. Class p-$wA(s,s)$ is called class (s,p)-w-hyponormal, class 1-$wA(1,1)$ is called class A and class 1-$wA(\frac{1}{2},\frac{1}{2})$ is called w-hyponormal [13, 15, 18, 19, 33]. Hence class p-$wA(s,t)$ operator is a generalization of class (s,p)-w-hyponormal, class A and w-hyponormal operators. C. Yang and J. Yuan [34–36] studied class $wF(p,r,q)$ operator T, i.e.,

$$\left(|T^*|^r|T|^{2p}|T^*|^r\right)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}$$

and

$$|T|^{2(p+r)(1-\frac{1}{q})} \geq \left(|T|^p|T^*|^{2r}|T|^p\right)^{1-\frac{1}{q}}$$

where $0 < p, 0 < r, 1 \leq q$. If we take small p_1 such that $0 < p_1 \leq \frac{p+r}{qr}$ and $p_1 \leq \frac{(p+r)(q-1)}{pq}$, then T is class p_1-$wA(p,r)$. Hence class p_1-$wA(p,r)$ is a generalization of class $wF(p,r,q)$. We will use this property frequently.

It is known that $T = U|T|$ is class p-$wA(s,t)$ if and only if

$$|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}, \quad |T|^{2sp} \geq |T(s,t)^*|^{\frac{2sp}{s+t}}$$

by [26]. Hence

$$|T(s,t)|^{\frac{2rp}{s+t}} \geq |T|^{2rp} \geq |T(s,t)^*|^{\frac{2rp}{s+t}}$$

and $T(s,t)$ is rp-hyponormal for all $r \in (0, \min\{s,t\}]$.

The following is a breakdown of the paper's structure: In section 2, we prove that if T is a class of p-$wA(s,t)$ operators and its Aluthge transform $T(s,t)$ is quasinormal (respectively, normal), then T is also quasinormal (resp., normal). The normal parts of quasisimilar class p-$wA(s,t)$ operators are unitarily equivalent in section 3. The major goal of Section 4 is to demonstrate that the Fuglede-Putnam theorem holds for a class p-$wA(s,t)$ operator T with $0 < s, t, s+t = 1$ and $0 < p \leq 1$ if T fulfills the kernel condition $\ker(T) \subset \ker(T^*)$.

2. Quasinormality

Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. T is said to be quasinormal if $|T|U = U|T|$, or equivalently, $TT^*T = T^*TT$. S. M. Patel, K. Tanahashi, A. Uchiyama and M. Yanagida [27] proved that if T is class $A(s,t)$ and $T(s,t)$ is quasinormal, then T is quasinormal and $T = T(s,t)$ if $s+t = 1$. The following is a generalization of this result.

Theorem 1. *Let T be a class p-$wA(s,t)$ operator with the polar decomposition $T = U|T|$. If $T(s,t) = |T|^sU|T|^t$ is quasinormal, then T is also quasinormal. Hence T coincides with its generalized Aluthge transform $T(s,t)$.*

Proof. Since T is a class p-$A(s,t)$ operator,

$$|T(s,t)|^{\frac{2rp}{s+t}} \geq |T|^{2rp} \geq |(T(s,t))^*|^{\frac{2rp}{s+t}} \tag{1}$$

for all $r \in (0, \min\{s,t\})$ by [19, Theorem 3] and Löwner-Heinz inequality. Then Douglas's theorem [11] implies

$$\overline{\operatorname{ran}(T(s,t))} = \overline{\operatorname{ran}((|T(s,t))^*|)} \subset \overline{\operatorname{ran}(|T|)} = \overline{\operatorname{ran}(|T(s,t)|)}$$

where $\overline{\mathcal{M}}$ denotes the norm closure of $\mathcal{M}$. Let $T(s,t) = W|T(s,t)|$ be the polar decomposition of $T(s,t)$. Then $E := W^*W = U^*U \geq WW^* =: F$. Put

$$|(T(s,t))^*|^{\frac{1}{s+t}} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, W = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}$$

on $\mathcal{H} = \overline{\operatorname{ran}(T(s,t))} \oplus \ker((T(s,t))^*)$.
Then X is injective and has a dense range. Since $T(s,t)$ is quasinormal, W commutes with $|T(s,t)|$ and

$$|T(s,t)|^{\frac{2rp}{s+t}} = W^*W|T(s,t)|^{\frac{2rp}{s+t}} = W^*|T(s,t)|^{\frac{2rp}{s+t}}W$$

$$\geq W^*|T|^{2rp}W \geq W^*|(T(s,t))^*|^{\frac{2rp}{s+t}}W = |T(s,t)|^{\frac{2rp}{s+t}}.$$

Hence

$$|T(s,t)|^{\frac{2rp}{s+t}} = W^*|T(s,t)|^{\frac{2rp}{s+t}}W = W^*|T|^{2rp}W,$$

and

$$|(T(s,t))^*|^{\frac{2rp}{s+t}} = W|T(s,t)|^{\frac{2rp}{s+t}}W^* = WW^*|T(s,t)|^{\frac{2rp}{s+t}}WW^* \quad (2)$$

$$= WW^*|T|^{2rp}WW^* = \begin{pmatrix} X^{2rp} & 0 \\ 0 & 0 \end{pmatrix}. \quad (3)$$

Since $WW^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (1), (2) and (3) imply that $|T(s,t)|^{\frac{2rp}{s+t}}$ and $|T|^{2rp}$ are of the forms

$$|T(s,t)|^{\frac{2rp}{s+t}} = \begin{pmatrix} X^{2rp} & 0 \\ 0 & Y^{2rp} \end{pmatrix} \geq |T|^{2rp} = \begin{pmatrix} X^{2rp} & 0 \\ 0 & Z^{2rp} \end{pmatrix}, \quad (4)$$

where $\overline{\operatorname{ran}(Y)} = \overline{\operatorname{ran}(Z)} = \overline{\operatorname{ran}(|T|)} \ominus \overline{\operatorname{ran}(T(s,t))} = \ker((T(s,t))^*) \ominus \ker(T)$.
Since W commutes with $|T(s,t)|$,

$$\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}.$$

So $W_1X = XW_1$ and $W_2Y = XW_2$, and hence $\overline{\operatorname{ran}(W_1)}$ and $\overline{\operatorname{ran}(W_2)}$ are reducing subspaces of X . Since $W^*W|T(s,t)| = |T(s,t)|$, we have $W_1^*W_1 = 1$ and

$$X^k = W_1^*W_1X^k = W_1^*X^kW_1,$$
$$Y^k = W_2^*W_2Y^k = W_2^*X^kW_2,$$

for $k = 1, 2, \cdots$.

Put $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. Then $T(s,t) = |T|^sU|T|^t = W|T(s,t)|$ implies

$$\begin{pmatrix} X^s & 0 \\ 0 & Z^s \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} X^t & 0 \\ 0 & Z^t \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X^{s+t} & 0 \\ 0 & Y^{s+t} \end{pmatrix}.$$

Hence

$$X^sU_{11}X^t = W_1X^{s+t} = X^sW_1X^t,$$
$$X^sU_{12}Z^t = W_2Y^{s+t} = X^{s+t}W_2$$

and

$$X^s(U_{11} - W_1)X^t = 0,$$
$$X^s(U_{12}Z^t - X^tW_2) = 0.$$

Since X is injective and has a dense range, $U_{11} = W_1$ is isometry and $U_{12}Z^t = X^tW_2$. Then

$$U^*U = \begin{pmatrix} U_{11}^*U_{11} + U_{21}^*U_{21} & U_{11}^*U_{l2} + U_{21}^*U_{22} \\ U_{12}^*U_{11} + U_{22}^*U_{21} & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix}$$

on $\mathcal{H} = \overline{\operatorname{ran}(T(s,t))} \oplus \ker((T(s,t))^*)$ is the orthogonal projection onto $\overline{\operatorname{ran}(|T|)} \supset \overline{\operatorname{ran}(T(s,t))}$, we have $U_{21} = 0$ and

$$U^*U = \begin{pmatrix} 1 & 0 \\ 0 & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix}.$$

Since $U_{12}Z^t = X^tW_2$, we have

$$Z^{2t} \geq Z^tU_{12}^*U_{12}Z^t = W_2^*X^{2t}W_2 = Y^{2t},$$

and

$$Z^{2rp} \geq (Z^tU_{12}^*U_{12}Z^t)^{\frac{rp}{t}} = (W_2^*X^tW_2)^{\frac{rp}{t}} = Y^{2rp} \geq Z^{2rp}$$

by Löwner-Heinz inequality and (4). Hence

$$(Z^tU_{12}^*U_{12}Z^t)^{\frac{rp}{t}} = Z^{2rp} = Y^{2rp},$$

so $Z = Y$ and $|T(s,t)| = |T|^{s+t}$. Since

$$\begin{aligned} Z^{2t} &= Z^tU_{12}^*U_{12}Z^t \\ &\leq Z^tU_{12}^*U_{12}Z^t + Z^tU_{22}^*U_{22}Z^t \leq Z^{2t} \end{aligned}$$

$Z^tU_{22}^*U_{22}Z^t = 0$ and $U_{22}Z^t = 0$. This implies $\operatorname{ran}(U_{22}^*) \subset \ker(Z)$. Since $\operatorname{ran}(U_{12}^*U_{12} + U_{22}^*U_{22}) \subset \overline{\operatorname{ran}(Z)}$ and $U_{22}^*U_{22} \leq U_{12}^*U_{12} + U_{22}^*U_{22}$, we have $\operatorname{ran}(U_{22}^*) \subset \overline{\operatorname{ran}(Z)}$. Hence

$$U_{22} = 0, U = \begin{pmatrix} W_1 & U_{12} \\ 0 & 0 \end{pmatrix}$$

and

$$\operatorname{ran}(U) \subset \overline{\operatorname{ran}(T(s,t))} \subset \overline{\Re(|T|)} = \operatorname{ran}(E).$$

Since W commutes with $|T(s,t)| = |T|^{s+t}$, W commutes with $|T|$ and

$$\begin{aligned} |T|^s(W-U)|T|^t &= W|T|^s|T|^t - |T|^sU|T|^t \\ &= W|T(s,t)| - T(s,t) = 0. \end{aligned}$$

Hence $E(W-U)E = 0$ and

$$U = UE = EUE = EWE = WE = W.$$

Thus $U = W$ commutes with $|T|$ and T is quasinormal.

Corollary 1. *Let $T = U|T|$ be a class p-$wA(s,t)$ operator. If $T(s,t) = |T|^sU|T|^t$ is normal, then T is also normal.*

Proof. Since $T(s,t)$ is normal, T is quasinormal by Theorem 1. Hence $T(s,t) = |T|^s U|T|^t = U|T|^{s+t}$ and $(T(s,t))^* = |T|^{s+t}U^*$. Hence

$$|T|^{2(s+t)} = |T(s,t)|^2 = |(T(s,t))^*|^2 = |T^*|^{2(s+t)}.$$

This implies $|T| = |T^*|$ and T is normal.

Theorem 2. *[25] Let $s_1 > 0$, $s_2 > 0$, $t_1 > 0, t_2 > 0$ and $0 < p \leq 1$. If T belongs to class p_1-$wA(s_1,t_1)$ for $0 < p_1 \leq p$ and T^* belongs to class p_2-$wA(s_2,t_2)$ for $0 < p_2 \leq p$, then T is normal.*

To prove Theorem 2, we need the following results.

Lemma 1. *([21]) If T is class p-$wA(s,t)$ and $0 < s \leq s_1$, $0 < t \leq t_1$, $0 < p_1 \leq p < 1$, then T is class p_1-$wA(s_1,t_1)$.*

Theorem 3 (Furuta theorem [14])**.** *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{r+p}{q}}$ *and*

(ii) $A^{\frac{r+p}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Proposition 1. *([19]) Let $A \geq 0$ and $B \geq 0$. If*

$$B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq B^2 \qquad \text{and} \quad A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2, \tag{5}$$

then $A = B$.

Proof. [Proof of Theorem 2] Let $r = \max\{s_1, s_2, t_1, t_2\}$ and let $q = \min\{p_1, p_2\}$.

Firstly, if T belongs to class p_1-$wA(s_1,t_1)$, then T belongs to class q-$wA(r,r)$ by Lemma 1. Hence we have

$$(|T^*|^r|T|^{2r}|T^*|^r)^{\frac{q}{2}} \geq |T^*|^{2rq} \text{ and } |T|^{2rq} \geq (|T|^r|T^*|^{2r}|T|^r)^{\frac{q}{2}} \tag{6}$$

Secondly, if T^* belongs to class p_2-$wA(s_2,t_2)$, then T^* belongs to class q-$wA(r,r)$ by Lemma 1. Hence we have

$$(|T|^r|T^*|^{2r}|T|^r)^{\frac{q}{2}} \geq |T|^{2rq} \text{ and } |T^*|^{2rq} \geq (|T^*|^r|T|^{2r}|T^*|^r)^{\frac{q}{2}} \tag{7}$$

Therefore

$$|T^*|^r|T|^{2r}|T^*|^r = |T^*|^{4r} \text{ and } |T|^{4r} = |T|^r|T^*|^{2r}|T|^r$$

hold by (6) and (7), and then $|T| = |T^*|$ by Proposition 1.

The following result is very important in the sequal

Theorem 4. *[17, Jensen's Operator Inequality (JOI)] Suppose that f is a continuous function defined on an interval I. Then f is operator convex on an interval I containing 0 with $f(0) \leq 0$ if and only if $f(a^*xa) \leq a^*f(x)a$ for every self-adjoint x with spectrum in I and every contraction a.*

Theorem 5. *([11]) Let A and B be bounded linear operators on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) $ran(A) \subseteq ran(B)$;

(ii) $AA^* \leq \lambda^2 BB^*$ *for some* $\lambda \geq 0$; *and*

(i) there exists a bounded linear operator C on $\mathcal{H}$ so that $A = BC$.

Lemma 2. *Let A, B and C be positive operators. Then the following assertions hold for each $p \geq 0$, $r \in [0,1]$ and $0 < q \leq 1$:*

(i) If $(B^{r/2}A^pB^{r/2})^{\frac{rq}{p+r}} \geq B^{rq}$ *and* $B \geq C$, *then* $(C^{r/2}A^pC^{r/2})^{\frac{rq}{p+r}} \geq C^{rq}$.

(ii) If $A \geq B$, $B^{rq} \geq (B^{r/2}C^pB^{r/2})^{\frac{rq}{p+r}}$ *and the condition*

$$\begin{aligned} \text{if } \lim_{n\to\infty} B^{1/2}x_n &= 0 \text{ and } \lim_{n\to\infty} A^{1/2}x_n \text{ exists,} \\ \text{then } \lim_{n\to\infty} A^{1/2}x_n &= 0 \text{ for any sequence of vectors} \{x_n\} \end{aligned} \tag{8}$$

hold, then $A^{rq} \geq (A^{r/2}C^pA^{r/2})^{\frac{rq}{p+r}}$.

Lemma 2 can be obtained as an application of the following results.

Theorem 6. *([11]) Let A and B be bounded linear operators on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) $ran(A) \subseteq ran(B)$;

(ii) $AA^* \leq \lambda^2 BB^*$ *for some* $\lambda \geq 0$; *and*

(iii) there exists a bounded linear operator C on $\mathcal{H}$ so that $A = BC$.

Moreover, if (i), (ii) and (iii) are valid, then there exists a unique operator C so that

(a) $\|C\|^2 = \inf\{\mu : AA^* \leq \mu BB^*\}$;

(b) $\ker(A) = \ker(C)$; *and*

(c) $ran(C) \subseteq \overline{ran(B^*)}$.

Theorem 7. *([16]) Let X and A be bounded linear operator on a Hilbert space $\mathcal{H}$. We suppose that $A \geq 0$ and $\|X\| \leq 1$. If f is an operator monotone function defined on $[0, \infty)$, then*

$$X^*f(A)X \leq f(X^*AX).$$

We remark that the condition (c) of Theorem 6 is equivalent to (c'): $\overline{ran(C)} \subseteq \overline{ran(B^*)}$. Here we consider when the equality of (c') holds.

Lemma 3. *([33]) Let A and B be operators which satisfy (i), (ii) and (iii) of Theorem 6 and C be the operator which is given in (iii) and determined uniquely by (a), (b) and (c) of Theorem 6. Then the following assertions are mutually equivalent:*

(i) $\overline{ran(C)} = \overline{ran(B^*)}$.

(ii) If $\lim_{n\to\infty} A^*x_n = 0$ *and* $\lim_{n\to\infty} B^*x_n$ *exists, then* $\lim_{n\to\infty} B^*x_n = 0$ *for any sequence of vectors* $\{x_n\}$.

We also prepare the following lemma in order to give a proof of Lemma 2.

Lemma 4. *([33]) Let S be a positive operator and $0 < q \leq 1$. If* $\lim_{n\to\infty} Sx_n = 0$ *and* $\lim_{n\to\infty} S^q x_n$ *exists, then* $\lim_{n\to\infty} S^q x_n = 0$ *for any sequence of vectors* $\{x_n\}$.

Proof. [Proof of Lemma 2] (i) The hypothesis $B \geq C$ ensures then $B^t \geq C^t$ for each $t \in (0,1]$ by Löwner-Heinz theorem. By Theorem 6, there exists an operator X with $\|X\| \leq 1$ such that

$$B^{\frac{t}{2}}X = X^*B^{\frac{t}{2}} = C^{\frac{t}{2}}. \tag{9}$$

Then we have

$$\begin{aligned}
(C^{r/2}A^pC^{r/2})^{\frac{rq}{p+r}} &= (X^*B^{r/2}A^pB^{r/2}X)^{\frac{rq}{p+r}} \\
&\geq X^*(B^{r/2}A^pB^{r/2})^{\frac{rq}{p+r}}X \text{ (by Theorem 7)} \\
&\geq X^*B^{rq}X \text{ (by the hypothesis)} \\
&= X^*(B^r)^qX \geq (X^*B^{\frac{r}{2}}B^{\frac{r}{2}}X)^q \text{ (by Theorem 4)} \\
&= (C^{\frac{r}{2}}C^{\frac{r}{2}})^q = C^{rq} \quad \text{(by Equation (9))}.
\end{aligned}$$

(ii) The hypothesis $A \geq B$ ensures $A^s \geq B^s$ for $s \in (0,1]$ by Löwner-Heinz theorem. By Theorem 6, there exists an operator X with $\|X\| \leq 1$ such that

$$A^{s/2}X = X^*A^{s/2} = B^{s/2}. \tag{10}$$

Then we have

$$\begin{aligned}
X^*(A^{r/2}C^pA^{r/2})^{\frac{rq}{p+r}}X &\leq (X^*A^{r/2}C^pA^{r/2}X)^{\frac{rq}{p+r}} \text{ (by Theorem 7)} \\
&= (B^{r/2}C^pB^{r/2})^{\frac{rq}{p+r}} \\
&\leq B^{rq} \text{ (by the hypothesis)} \\
&= (B^r)^q = (X^*A^{\frac{r}{2}}A^{\frac{r}{2}}X)^q \leq X^*A^{rq}X \text{ (by Theorem 4)}
\end{aligned}$$

so that $A^{rq} \geq (A^{r/2}C^pA^{r/2})^{\frac{rq}{p+r}}$ holds on $\overline{ran(X)}$. On the other hand, the hypothesis (8) implies the following (11)

$$\text{If } \lim_{n\to\infty} B^{r/2}x_n = 0 \text{ and } \lim_{n\to\infty} A^{r/2}x_n \text{ exists,}$$

$$\text{then } \lim_{n\to\infty} A^{r/2}x_n = 0 \text{ for any sequence of vectors } \{x_n\}. \tag{11}$$

since $\lim_{n\to\infty} B^{r/2}x_n = 0$ and $\lim_{n\to\infty} A^{r/2}x_n$ exists, then $\lim_{n\to\infty} B^{1/2}x_n = B^{(1-r)/2}(\lim_{n\to\infty} B^{r/2}x_n) = 0$ and $\lim_{n\to\infty} A^{1/2}x_n = A^{(1-r)/2}(\lim_{n\to\infty} A^{r/2}x_n)$ exists, so that $\lim_{n\to\infty} A^{1/2}x_n = 0$ by (8), hence $\lim_{n\to\infty} A^{r/2}x_n = 0$ by Lemma 4. (11) ensures $\overline{ran(X)} = \overline{ran(A^{r/2})}$ by Lemma 3, hence we have

$$\begin{aligned} \ker((A^{r/2}C^pA^{r/2})^{\frac{rq}{p+r}}) &= \ker(A^{r/2}C^pA^{r/2}) \\ &\supseteq \ker(A^{r/2}) = \ker(A^r) = \ker(A^{qr}) = \ker(X^*), \end{aligned}$$

so that $A^{qr} = (A^{r/2}C^pA^{r/2})^{\frac{rq}{p+r}} = 0$ holds on $\ker(X^*)$. Consequently the proof is complete since $\mathcal{H} = \overline{ran(X)} \oplus \ker(X^*)$.

Lemma 5. *([26]) Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ be the polar decomposition of T. Then T is class p-$wA(s,t)$ if and only if $|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$ and $|T|^{2sp} \geq |(T(s,t))^*|^{\frac{2sp}{s+t}}$.*

Lemma 6. *Let $0 < s,t,s+t \leq 1$ and $0 < p \leq 1$. Let $T \in \mathcal{B}(\mathcal{H})$ be class p-$wA(s,t)$ and let $\mathcal{M}$ an invariant subspace of T. Then the restriction $T|_{\mathcal{M}}$ is also class p-$wA(s,t)$.*

Proof. Let $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and P the orthogonal projection onto $\mathcal{M}$. Let $T_0 := TP = PTP = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$|T_0|^{2t} = (P|T|^2P)^t \geq P|T|^{2t}P \text{ for each } 0 < t \leq 1$$

by Hansen's inequality, and

$$|T^*|^2 = TT^* \geq TPT^* = |T_0^*|^2.$$

Hence

$$\begin{aligned} T \text{ is class } p\text{-}A(s,t) &\Longleftrightarrow |T^*|^{2tp} \leq (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \\ &\Longrightarrow |T_0^*|^{2tp} \leq (|T_0^*|^t|T|^{2s}|T_0^*|^t)^{\frac{tp}{s+t}} \text{ (by Lemma 2)} \\ &\Longrightarrow |T_0^*|^{2tp} \leq (|T_0^*|^t|T_0|^{2s}|T_0^*|^t)^{\frac{tp}{s+t}} \text{ (since } |T_0^*|^t = |T_0^*|^tP = P|T_0^*|^t \text{ for every } 0 < t \leq 1). \end{aligned}$$

Now

$$|T_0| = P|\widetilde{T}|P \geq P|T|P \geq P|(\widetilde{T})^*|P = |T_0^*|.$$

Then by Theorem 3 it follows that

$$|T_0|^{2sp} \geq (|T_0|^s|T_0^*|^{2t}||T_0|^s|)^{\frac{ps}{s+t}}.$$

Therefore, $T|_{\mathcal{M}}$ is class p-$A(s,t)$ operator.

The following example shows that there exists a class p-$wA(s,t)$ operator T such that $T|_{\mathcal{M}}$ is quasinormal but $\mathcal{M}$ does not reduce T.

Example 1. *Let T be a bilateral shift on $\ell^2(\mathbb{Z})$ defined by $Te_n = e_{n+1}$ and $\mathcal{M} = \bigvee_{n\geq 0} \mathbb{C}e_n$. Then T is unitary and $T|_{\mathcal{M}}$ is isometry. However, $\mathcal{M}$ does not reduce T.*

Lemma 7. *Let $0 < s, t, s+t = 1$ and $0 < p \leq 1$. Let $T \in \mathcal{B}(\mathcal{H})$ be class p-$wA(s,t)$ operator, let $\mathcal{M}$ be an invariant subspace for T and a reducing subspace for $T(s,t)$ such that $T(s,t)|_{\mathcal{M}}$ the restriction of $T(s,t)$ to $\mathcal{M}$ is an injective normal operator, then $T|_{\mathcal{M}} = T(s,t)|_{\mathcal{M}}$ and $\mathcal{M}$ reduces T.*

Proof. Let

$$T(s,t) = \begin{pmatrix} T_0 & 0 \\ 0 & A \end{pmatrix}, \; T = \begin{pmatrix} S & B \\ 0 & D \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Since T is class p-$wA(s,t)$ we have $|T(s,t)|^{2rp} \geq |T|^{2rp} \geq |(T(s,t))^*|^{2rp}$ for $r \in \min\{s,t\}$. Let P be the orthogonal projection onto $\mathcal{M}$. Then

$$|T_0| = P|T(s,t)|P \geq P|T|P \geq P|(T(s,t))^*|P = |T_0^*|.$$

By Löwner-Heinz theorem we get

$$|T_0|^{2rp} = P|T(s,t)|^{2rp}P \geq P|T|^{2rp}P \geq P|(T(s,t))^*|^{2rp}P = |T_0^*|^{2rp}.$$

Since $|T|^sT = T(s,t)|T|^s$ and $P|T|^sP = |T_0|^s$, we deduce that

$$|T_0|^sS = T_0|T_0|^s.$$

We have T_0 is an injective normal operator, then $S = T|_{\mathcal{M}} = T_0 = T(s,t)|_{\mathcal{M}}$, consequently

$$T = \begin{pmatrix} T_0 & B \\ 0 & D \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Hence

$$T^*T = \begin{pmatrix} T_0^*T_0 & T_0^*B \\ B^*T_0 & B^*B + D^*D \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

So we can write

$$|T|^{rp} = \begin{pmatrix} |T_0|^{rp} & X \\ X^* & Y \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Since

$$P|T|^{pr}|T|^{pr}P = |T_0|^{2rp},$$

then $|T_0|^{2rp} = |T_0|^{2rp} + XX^*$, and thus $X = 0$.

It follows that $|T|^{rp} = |T_0|^{rp} \oplus Y^2$ implying $|T|^{2rp} = |T_0|^{2rp} \oplus Y^4$. Consequently we get $B^*B = 0$ it follows that $B = 0$ and hence $\mathcal{M}$ reduces T.

The next lemma is a simple consequence of the preceding one.

Lemma 8. *Let* $0 < s, t, s + t = 1$ *and* $0 < p \leq 1$. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a class* p*-*$wA(s, t)$ *operator with* $\ker(T) \subset \ker(T^*)$. *Then* $T = T_1 \oplus T_2$ *on* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ *where* T_1 *is normal,* $\ker(T_2) = \{0\}$ *and* T_2 *is pure class* p*-*$wA(s, t)$ *i.e.,* T_2 *has no non-zero invariant subspace* $\mathcal{M}$ *such that* $T_2|_{\mathcal{M}}$ *is normal.*

Lemma 9. *Let* $0 < s, t, s+t = 1$ *and* $0 < p \leq 1$. *Let* $T = U|T| \in \mathcal{B}(\mathcal{H})$ *be class* p*-*$wA(s, t)$ *and* $\ker(T) \subset \ker(T^*)$. *Suppose* $T(s, t) = |T|^s U|T|^t$ *be of the form* $N \oplus T'$ *on* $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, *where* N *is a normal operator on* $\mathcal{M}$. *Then* $T = N \oplus T_1$ *and* $U = U_{11} \oplus U_{22}$ *where* T_1 *is class* p*-*$wA(s, t)$ *with* $\ker(T_1) \subset \ker(T_1^*)$ *and* $N = U_{11}|N|$ *is the polar decomposition of* N.

Proof. Since

$$|T(s,t)|^{2rp} \geq |T|^{2rp} \geq |(T(s,t))^*|^{2rp}$$

for $r \in \min\{s, t\}$, we have

$$|N|^{2rp} \oplus |T'|^{2rp} \geq |T|^{2rp} \geq |N|^{2rp} \oplus |T'^*|^{2rp}$$

by assumption. This implies that $|T|$ is of the form $|N| \oplus L$ for some positive operator L. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ be 2×2 matrix representation of U with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Then the definition $T(s, t)$ means

$$\begin{pmatrix} N & 0 \\ 0 & T' \end{pmatrix} = \begin{pmatrix} |N|^s & 0 \\ 0 & L^s \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |N|^t & 0 \\ 0 & L^t \end{pmatrix}$$

Hence, we have

$$N = |N|^s U_{11} |N|^t, \ |N|^s U_{12} L^t = 0 \text{ and } L^s U_{21} |N|^t = 0.$$

Since $\ker(T) \subset \ker(T^*)$,

$$\overline{\text{ran}(U)} = \overline{\text{ran}(T)} = \ker(T^*)^{\perp} \subset \ker(T)^{\perp} = \overline{\text{ran}(|T|)}.$$

Let $Nx = 0$ for $x \in \mathcal{M}$. Then $x \in \ker(|T|) = \ker(U)$, and

$$Ux = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} = 0.$$

Hence

$$\ker(N) \subset \ker(U_{11}) \cap \ker(U_{21}).$$

Let $x \in \mathcal{M}$. Then

$$U \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} \in \overline{\text{ran}(|T|)} = \overline{\text{ran}(|N| \oplus L)}.$$

Hence

$$\text{ran}(U_{11}) \subset \text{ran}(|N|), \ \text{ran}(U_{21}) \subset \overline{\text{ran}(L)}.$$

Similarly

$$\operatorname{ran}(U_{12}) \subset \operatorname{ran}(|N|), \ \operatorname{ran}(U_{22}) \subset \overline{\operatorname{ran}(L)}.$$

Let $Lx = 0$ for $x \in \mathcal{M}^{\perp}$. Then $x \in \ker(|T|) = \ker(U)$ and

$$U \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} U_{12}x \\ U_{22}x \end{pmatrix} = 0$$

Hence

$$\ker(L) \subset \ker(U_{12}) \cap \ker(U_{22}).$$

Let $N = V|N|$ be the polar decomposition of N. Then

$$(V|N|^s - |N|^s U_{11})|N|^t = 0.$$

Hence $V|N|^s - |N|^s U_{11} = 0$ on $\overline{\operatorname{ran}(|N|)}$. Since $\ker(N) \subset \ker(U_{11})$, this implies $0 = V|N|^s - |N|^s U_{11} = |N|^s(V - U_{11})$. Hence

$$\operatorname{ran}(V - U_{11}) \subset \ker(|N|) \cap \overline{\operatorname{ran}(|N|)} = \{0\}.$$

Hence $V = U_{11}$ and $N = U_{11}|N|$ is the polar decomposition of N. Since $|N|^s U_{12} L^t = 0$,

$$\operatorname{ran}(U_{11}L^t) \subset \ker(|N|) \cap \overline{\operatorname{ran}(|N|)} = \{0\}.$$

Hence $U_{12}L^t$ and $U_{12} = 0$. Similarly we have $U_{21} = 0$ by $L^s U_{21}|N|^t = 0$. Hence $U = U_{11} \oplus U_{22}$. So we obtain

$$T = U|T| = U_{11}|N| \oplus U_{22}L = N \oplus T_1,$$

where $T_1 = U_{22}L$.

3. Quasisimilarity

An operator $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is called quasiaffinity if X is both injective and has a dense range. For $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, if there exist quasiaffinities $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $TX = XS$ and $YT = SY$, then we say that T and S are quasisimilar. The operator $T \in \mathcal{B}(\mathcal{H})$ is said to be pure if there exists no non-trivial reducing subspace $\mathcal{M}$ of $\mathcal{H}$ such that the restriction of T to $\mathcal{M}$ is normal and is completely hyponormal if it is pure. Recall that every operator $T \in \mathcal{B}(\mathcal{H})$ has a direct sum decomposition $T = T_1 \oplus T_2$, where T_1 and T_2 are normal and pure parts, respectively. Of course in the sum decomposition, either T_1 or T_2 may be absent. The following lemma is due to Williams [32, Lemma 1.1].

Lemma 10. *Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be normal operators. It there exist injective operators $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $TX = XS$ and $YT = SY$, then T and S are unitarily equivalent.*

Corollary 2. *Let $T \in \mathcal{B}(\mathcal{H})$ be class p-$wA(s,t)$operator for $0 < s,t,s+t = 1$ and $0 < p \leq 1$. Then $T = T_1 \oplus T_2$ on the space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where T_1 is normal and T_2 is pure and class p-$wA(s,t)$, i.e., T_2 has no invariant subspace $\mathcal{M}$ such that $T_2|_{\mathcal{M}}$ is normal.*

The next result was proved for dominant operators in [28, Theorem 1], for p-hyponormal operators in [20] and for w-hyponormal operators in [22, Lemma 2.12].

Proposition 2. *Let $T \in \mathcal{B}(\mathcal{H})$ be class p-$wA(s,t)$operator for $0 < s,t,s+t = 1$ and $0 < p \leq 1$ such that $\ker(T) \subset \ker(T^*)$ and let $S \in \mathcal{B}(\mathcal{K})$ be a normal operator. If there exists a quasiaffinity $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ with dense range such that $TX = XS$, then T is normal.*

To prove Proposition 2, we need the following lemmas.

Lemma 11. *[9] If N is a normal operator on $\mathcal{H}$, then we have*

$$\bigcap_{\lambda \in \mathbb{C}} (N-\lambda)\mathcal{H} = \{0\}.$$

Lemma 12. *([10]) Let $T \in \mathcal{B}(\mathcal{H})$, $D \in \mathcal{B}(\mathcal{H})$ with $0 \leq D \leq M(T-\lambda)(T-\lambda)^*$ for all $\lambda \in \mathbb{C}$, where M is a positive real number. Then for every $x \in D^{\frac{1}{2}}\mathcal{H}$ there exists a bounded function $f : \mathbb{C} \longrightarrow \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$.*

Proof. [Proof of Proposition 2] $\ker(T) \subset \ker(T^*)$ implies $\ker(T)$ reduces T. Also $\ker(S)$ reduces S since S is normal. Using the orthogonal decompositions $\mathcal{H} = \overline{\text{ran}(|T|)} \oplus \ker(T)$ and $\mathcal{H} = \overline{\text{ran}(S)} \oplus \ker(S)$, we can represent T and S as follows: $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, where T_1 is an injective class p-$wA(s,t)$ operator on $\overline{\text{ran}(|T|)}$ and S_1 is injective normal on $\overline{\text{ran}(S)}$. The assumption $TX = XS$ asserts that X maps $\text{ran}(S)$ to $\text{ran}(T) \subset \overline{\text{ran}(|T|)}$ and $\ker(S)$ to $\ker(T)$, hence X is the form: $X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$, where $X_1 \in \mathcal{B}(\overline{\text{ran}(S)}, \overline{\text{ran}(|T|)})$, $X_2 \in \mathcal{B}(\ker(S), \ker(T))$. Since $TX = XS$, we have that $T_1X_1 = X_1S_1$. Since X is injective with dense range, X_1 is also injective with dense range. Put $W_1 = |T_1|^sX_1$, then W_1 is also injective with dense range and satisfies $T(s,t)W_1 = W_1S$. Put $W_n = |\Delta^n(T(s,t))|^sW_{n-1}$, then W_n is also injective with dense range and satisfies $\Delta^n(T(s,t))W_n = W_nS$. From [26, Corollary 2.7] and [6], if there exists an integer m such that $\Delta^m(T(s,t))$ is a hyponormal operator, then $\Delta^n(T(s,t))$ is a hyponormal operator for $n \geq m$. It follows from Lemma 12 that there exists a bounded function $f : \mathbb{C} \longrightarrow \mathcal{H}$ such that $(\Delta^n(T_1(s,t))^* - \lambda)f(\lambda) \equiv x$, for every $x \in (\Delta^n(T_1(s,t))^*\Delta^n(T_1(s,t) - \Delta^n(T_1(s,t)(\Delta^n(T_1(s,t))^*)^{\frac{1}{2}}\mathcal{H}$. Hence

$$\begin{aligned} W_n^*x &= W_n^*(\Delta^n(T_1(s,t))^* - \lambda)f(\lambda) \\ &= (S_1^* - \lambda)W_n^*f(\lambda) \in \text{ran}(S_1^* - \lambda) \text{ for all } \lambda \in \mathbb{C}. \end{aligned}$$

By Lemma 11, we have $W_n^* x = 0$, and hence $x = 0$ because W_n^* is injective. This implies that $\Delta^n(T_1(s,t)$ is normal. By Corollary 1, T_1 is normal and therefore $T = T_1 \oplus 0$ is also normal.

Theorem 8. *Let T and S^* be class p-$wA(s,t)$ operators with $0 < s,t,s+t = 1$ and $0 < p \leq 1$ such that $\ker(T) \subset \ker(T^*)$ and $\ker(S^*) \subset \ker(S)$. If there exist a quasiaffinity X such that $TX = XS$, then T and S are unitarily equivalent normal operators.*

Proof. First decompose T and S^* into their normal and pure parts by $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $S^* = S_1^* \oplus S_2^*$ on $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where T_1, S_1 are normal and T_2, S_2^* are pure. Let $X = [X_{ij}]_{i,j=1}^2$. Then $TX = XS$ implies that $T_2 X_{21} = X_{21} S_1$ and $T_2 X_{22} = X_{22} S_2$. Let $T_2 = U_2|T_2|$, $S_2^* = V_2^*|S_2^*|$ be the polar decompositions of T_2 and S_2^*, respectively and

$$T_2(s,t) = |T_2|^s U_2 |T_2|^t,\ S_2^*(s,t) = |S_2^*|^s V_2^* |S_2^*|^t,\ W = |T_2|^s X_{22} |S_2^*|^s.$$

Then

$$\begin{aligned} T_2(s,t)W &= |T_2|^s T_2 X_{22} |S_2^*|^s \\ &= |T_2|^s X_{22} S_2 |S_2^*|^s \\ &= W(S_2^*(s,t))^*. \end{aligned}$$

Since $\overline{\operatorname{ran}(W)}$ reduces $T_2(s,t)$ and $\ker(W)^\perp$ reduces $S_2^*(s,t)$ and $T_2(s,t)|_{\overline{\operatorname{ran}(W)}}$ and $S_2^*(s,t)|_{\ker(W)^\perp}$ are unitarily equivalent normal operators, and since T_2, S_2^* are injective class p-$wA(s,t)$ operators, we have $T_2|_{\overline{\operatorname{ran}(W)}} = T_2(s,t)|_{\overline{\operatorname{ran}(W)}}$ and $S_2^*|_{\ker(W)^\perp} = S_2^*(s,t)|_{\ker(W)^\perp}$ by Lemma 9. Since T_2, S_2^* are pure, it implies $W = |T_2|^s X_{22} |S_2^*|^s = 0$. Hence $X_{22} = 0$. Similarly $X_{12} = 0, X_{21} = 0$. Hence $X = X_{11}$ and S, T are unitarily equivalent normal operators.

The following lemma is due to Williams [32, Lemma 1.1]

Lemma 13. *Let $N_1 \in \mathcal{B}(\mathcal{H})$ and $N_2 \in \mathcal{B}(\mathcal{K})$ be normal. If $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H},\mathcal{K})$ are injective such that $N_1 X = X N_2$ and $Y N_1 = N_2 Y$, then N_1 and N_2 are unitarily equivalent.*

Stampfli and Wadhwa [28] proved that the normal parts of quasisimilar dominant operators are unitarily equivalent. This result was generalized to classes of p-hyponormal operators in [12]. We prove that theses results hold for class p-$wA(s,t)$ operators.

Theorem 9. *Suppose that $0 < s,t,s+t = 1$ and $) < p \leq 1$. For each $i = 1,2$, let $T_i \in \mathcal{B}(\mathcal{H}_i)$ be class p-$wA(s,t)$ operators such that $\ker(T_j) \subset \ker(T_j^*)$ and let $T_i = N_i \oplus V_i$ on $\mathcal{H}_i = \mathcal{H}_{i1} \oplus \mathcal{H}_{i2}$, where N_i and V_i are the normal and pure parts, respectively of T_i. If T_1 and T_2 are quasisimilar, then N_1 and N_2 are unitarily equivalent and there exist $X_* \in \mathcal{B}(\mathcal{H}_{22},\mathcal{H}_{12})$ and $Y_* \in \mathcal{B}(\mathcal{H}_{12},\mathcal{H}_{22})$ having dense range such that $V_1 X_* = X_* V_2$ and $Y_* V_1 = V_2 Y_*$.*

Proof. By hypothesis there exist quasiaffinities $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_1X = XT_2$ and $YT_1 = T_2Y$. Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$$

with respect to $\mathcal{H}_2 = \mathcal{H}_{21} \oplus \mathcal{H}_{22}$ and $\mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12}$, respectively. A simple matrix calculation shows that

$$V_1X_3 = X_3N_2 \text{ and } V_2Y_3 = Y_3N_1.$$

We claim that $X_3 = Y_3 = 0$. Let $\mathcal{M} = \overline{\operatorname{ran}(X_3)}$. Then $\mathcal{M}$ is a non-trivial invariant subspace of V_1. Since $V_1^*X_3 = X_3N_2^*$ by Proposition 2, $\mathcal{M}$ is an invariant subspace of V_1^*. Hence $\mathcal{M}$ reduces V_1, $\sigma(V_1|_{\mathcal{M}}) \subset \sigma(V_1)$ and $V_1|_{\mathcal{M}}$ is invertible. Let $V_1' = V_1|_{\mathcal{M}}$ and define an operator $X_3' : \mathcal{H}_{12} \longrightarrow \mathcal{M}$ by $X_3'x = X_3x$ for each $x \in \mathcal{H}_{12}$. Then V_1' is class p-$wA(s,t)$ by Lemma 6, so that X_3' has dense range and satisfies $V_1'X_3' = X_3'N_2$. Hence V_1' is normal by Propsition 2. Since V_1 is pure, this implies that $\mathcal{M} = \{0\}$ and $X_3 = 0$. Similarly, we have $Y_3 = 0$. Hence X_1 and Y_1 are injective.

SInce $N_1X_1 = X_1N_2$ and $Y_1N_1 = N_2Y_1$, N_1 and N_2 are unitarily equivalent, by Lemma 13. Also, X_4 and Y_4 have dense ranges. Hence $V_1X_4 = X_4V_2$ and $Y_4V_1 = V_2Y_4$, so the proof is complete.

Corollary 3. *Let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ be quasisimilar class p-$wA(s,t)$ operators for $0 < s, t, s+t = 1$ and $0 < p \leq 1$. If T_1 is pure, then T_2 is also pure.*

Corollary 4. *Let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ be class p-$wA(s,t)$ operators for $0 < s, t, s+t = 1$ and $0 < p \leq 1$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ be normal. If T_1 and T_2 are quasisimilar, then T_1 and T_2 are unitarily equivalent normal operators.*

4. The Fuglede-Putnam Theorem

We offer various results related to the Fuglede-Putnam theorem in this section. If $T^*X = XS^*$ whenever $TX = XS$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, a pair (T, S) is said to have the Fuglede-Putnam property. In operator theory, the Fuglede-Putnam theorem is well-known. It claims that the pair (T, S) possesses the Fuglede-Putnam property for any normal operators T and S. There are several generalizations of this theorem, the majority of which loosen the normality of T and S; see, for example, [22–24, 27, 28], and some references therein and for more details (see [3],[5],[4]). The Fuglede-Putnam theorem is the subject of the next lemma, which we will require in the future.

Lemma 14. *([29]) Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$. Then the following assertions equivalent.*

(i) The pair (T, S) has the Fuglede-Putnam property.

(ii) If $TX = XS$, then $\overline{\operatorname{ran}(X)}$ reduces T, $\ker(X)^\perp$ reduces S, and $T|_{\overline{\operatorname{ran}(X)}}$, $S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators.

Remark 1. *A necessary condition for the pair (T, T^*) to satisfy Fuglede-Putnam's theorem is $\ker(T) \subset \ker(T^*)$. Since for a class p-$wA(s,t)$ operator this is not always true, class p-$wA(s,t)$ operator do not Fuglede-Putnam's theorem. For example, if P is the orthogonal projection onto $\ker(T)$, with T is class p-$wA(s,t)$, then $TP = PT^*$ but $T^*P \neq PT$. The following result (Corollary 6) prove that if T^*, S are p-class $A(s,t)$ operators for $0 < s,t,s+t = 1$ and $0 < p \leq 1$ such that $\ker(T^*)$ reduces T^* and $\ker(S)$ reduces S, then the pair (T,S) satisfy Fuglede-Putnam's theorem.*

Theorem 10. *Let $T \in \mathcal{B}(\mathcal{H})$ be class p-$wA(s,t)$ operator for $0 < s,t,s+t = 1$ and $0 < p \leq 1$ and $\ker(T) \subset \ker(T^*)$. If L is self-adjoint and $TL = LT^*$, then $T^*L = LT$.*

Proof. Since $\ker(T) \subset \ker(T^*)$ and $TL = LT^*$, $\ker(T)$ reduces T and L. Hence

$$T = T_1 \oplus 0,\ L = L_1 \oplus L_2 \text{ on } \mathcal{H} = \overline{\operatorname{ran}(T^*)} \oplus \ker(T),$$

$T_1L_1 = L_1T^*$ and $\{0\} = \ker(T_1) \subset \ker(T_1^*)$. Since $\overline{\operatorname{ran}(L_1)}$ is invariant under T_1 and reduces L_1,

$$T = \begin{pmatrix} T_{11} & S \\ 0 & T_{22} \end{pmatrix},\ L_1 = L_{11} \oplus 0 \text{ on } \mathcal{H} = \overline{\operatorname{ran}(T^*)} = \overline{\operatorname{ran}(L_1)} \oplus \ker(L_1).$$

T_{11} is an injective class p-$wA(s,t)$ operator by Lemma 6 and L_{11} is an injective self-adjoint operator (hence it has dense range) such that $T_{11}L_{11} = L_{11}T_{11}^*$. Let $T_{11} = V_{11}|T_{11}|$ be the polar decomposition of T_{11} and $T_{11}(s,t) = |T_{11}|^sV_{11}|T_{11}|^t$, $W = |T_{11}|^sL_{11}|T_{11}|^s$. Then

$$\begin{aligned} T_{11}(s,t)W &= |T_{11}|^sV_{11}|T_{11}|^t|T_{11}|^sL_{11}|T_{11}|^s \\ &= |T_{11}|^sT_{11}L_{11}|T_{11}|^s \\ &= |T_{11}|^sL_{11}T_{11}^*|T_{11}|^s \\ &= |T_{11}|^sL_{11}|T_{11}|^s|T_{11}|^tV_{11}^*|T_{11}|^s \\ &= W(T_{11}(s,t))^*. \end{aligned}$$

Since $T_{11}(s,t)$ is $\min\{sp, tp\}$-hyponormal and $\operatorname{ran}(W)$ is dense (because $\ker(W) = \{0\}$), $T_{11}(s,t)$ is normal by [12, Theorem 7]. Hence T_{11} is normal and $T_{11} = T_{11}(s,t)$ by Corollary 1. Then $\overline{\operatorname{ran}(L_1)}$ reduces T_1 by Lemma 7 and $T_{11}^*L_{11} = L_{11}T_{11}$ by Lemma 14. Hence

$$\begin{aligned} T &= T_{11} \oplus T_{22} \oplus 0, \\ L &= L_{11} \oplus 0 \oplus L_2 \end{aligned}$$

and

$$T^*L = T_{11}^*L_{11} \oplus 0 \oplus 0 = L_{11}T_{11} \oplus 0 \oplus 0 = LT.$$

Example 2. *Let $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathbb{C}^2$ and define an operator R on $\mathcal{H}$ by*

$$R(\cdots \oplus x_{-2} \oplus x_{-1} \oplus x_0^{(0)} \oplus x_1 \oplus \cdots) = \cdots \oplus Ax_{-2} \oplus Ax_{-1}^{(0)} \oplus Bx_0 \oplus Bx_1 \oplus \cdots,$$

where

$$A=\frac{1}{4}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

Then R is a class p-$wA(s,t)$. Moreover, $\operatorname{ran}(E)=\ker(R)$, E is not a self-adjoint and $\ker(R)\neq\ker(R^*)$, where E is the Riesz idempotent with respect to 0, see [31, Example 13]. Let $T=R$ and $L=P$ be the orthogonal projection onto $\ker(T)$. Then T is a class p-$wA(s,t)$ operator and $TL=0=LT^*$, but $T^*L\neq LT$. Hence the kernel condition $\ker(T)\subset\ker(T^*)$ is necessary for Theorem 10.

Corollary 5. *Let $T\in\mathcal{B}(\mathcal{H})$ be a class p-$wA(s,t)$ operator for $0<s,t,s+t=1$ and $0<p\leq 1$ and $\ker(T)\subset\ker(T^*)$. If $TX=XT^*$ for some $X\in\mathcal{B}(\mathcal{H})$ then $T^*X=XT$.*

Proof. Let $X=L+iJ$ be the Cartesian decomposition of X. Then we have $TL=LT^*$ and $TJ=JT^*$ by the assumption. By Theorem 10, we have $T^*L=LT$ and $T^*J=JT$. This implies that $T^*X=XT$.

If we use the 2×2 matrix trick, we easily deduce the following result.

Corollary 6. *Suppose that $0<s,t,s+t=1$ and $0<p\leq 1$. Let $T^*\in\mathcal{B}(\mathcal{H})$ be a class p-$wA(s,t)$ operator and $S\in\mathcal{B}(\mathcal{K})$ be a class p-$wA(s,t)$ operator with $\ker(T^*)\subset\ker(T)$ and $\ker(S)\subset\ker(S^*)$. If $X\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and $XT=SX$, then $XT^*=S^*X$.*

Proof. Put $A=\begin{pmatrix}T^* & 0\\ 0 & S\end{pmatrix}$ and $B=\begin{pmatrix}0 & 0\\ X & 0\end{pmatrix}$ on $\mathcal{H}\oplus\mathcal{K}$. Then A is a class p-$wA(s,t)$ operator on $\mathcal{H}\oplus\mathcal{K}$ that satisfies $BA^*=AB$ and $\ker(A)\subset\ker(A^*)$. Hence we have $BA=A^*B$, by Corollary 5, and so $XT^*=S^*X$.

Example 3. *Let $S=T^*=R$ as in Example 2 and $X=P$ be the orthogonal projection onto $\ker(S)$. Then $SX=0=XT$, but $S^*X\neq XT^*$. Hence the kernel condition is necessary for Corollary 6.*

As an application of Corollary 6, we establish the following result.

Corollary 7. *Suppose that $0<s,t,s+t=1$. Let $T\in\mathcal{B}(\mathcal{H})$ and $S^*\in\mathcal{B}(\mathcal{K})$ be class p-$wA(s,t)$ and $\ker(T)\subset\ker(T^*)$, $\ker(S^*)\subset\ker(S)$. Let $TX=XS$ for some operator $X\in\mathcal{B}(\mathcal{K},\mathcal{H})$. Then $\overline{\operatorname{ran}(X)}$ reduces T, $\ker(S)^\perp$ reduces S and $T|_{\overline{\operatorname{ran}(X)}}$, $S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators.*

Proof. By Corollary 6, $T^*X=XS^*$. Therefore $T^*TX=XS^*S$ and so $|T|X=X|S|$. Let $T=U|T|,S=V|S|$ be the polar decomposition. Then $UX|S|=U|T|X=TX=XS=XV|S|$. Let $x\in\ker(|S|)$. Then $Vx=0$ and $TXx=XSx=0$. Hence $Xx\in\ker(T)=\ker(U)$ and $UXx=0$. Hence $UX=XV$. Since $\ker(U)=\ker(T)\subset\ker(T^*)=\ker(U^*)$, $UU^*\leq U^*U$. Hence $U^*UU=U^*UUU^*U=UU^*U=U$. This implies U and V^* are quasinormal. Hence $U^*X=XV^*$, $\overline{\operatorname{ran}(X)}$ reduces U, $|T|$, $\ker(X)^\perp$ reduces V, $|S|$. We may assume $t<s$. Then T,S^* are class p-$wA(s,s)$ operators with reducing kernels.

Let $T(s,s) = |T|^s U|T|^s, S(s,s) = |S|^s V|S|^s$. Then $T(s,s)$, $S^*(s,s) = |S^*|^s V^*|S^*|^s = VS(s,s)^* V^*$ are $\frac{p}{2}$-hyponormal. Also, since

$$|S(s,s)^*| - |S(s,s)| = V^*(|S^*(s,s)| - |S^*(s,s)^*|)V \geq 0,$$

$S(s,s)^*$ is $\frac{p}{2}$-hyponormal, too. Then

$$\begin{aligned} T(s,s)X &= |T|^s U|T|^s X = |T|^s UX|S|^s \\ &= |T|^s XV|S|^s = XS(s,s), \end{aligned}$$

hence $T(s,s)^*X = XS(s,s)^*$, $\overline{\text{ran}(X)}$ reduces $T(s,s)$, $\ker(X)^\perp$ reduces $S(s,s)$ and

$$T|_{\overline{\text{ran}(X)}}(s,s) = T(s,s)|_{\overline{\text{ran}(X)}} \simeq S(s,s)|_{\ker(X)^\perp} = S|_{\ker(X)^\perp}(s,s)$$

are unitarily equivalent normal operators. Hence $T|_{\overline{\text{ran}(X)}}$, $S|_{\ker(X)^\perp}$ are normal by Corollary 1, and that they are unitarily equivalent follows from the fact that if $N = U|N|$ and $M = W|M|$ are normal operators, then for a unitary operator V, $N = V^*MV$ if and only if $U = V^*WV$ and $|N|^s = V^*|M|^s V$ for any $s > 0$.

Theorem 11. *Suppose that $0 < s, t, s+t = 1$. Let $T \in \mathcal{B}(\mathcal{H})$ be class p-$wA(s,t)$ and N a normal operator. Let $TX = XN$. Then the following assertions hold.*

(i) If the range $\text{ran}(X)$ *is dense, then T is normal.*

(ii) If $\ker(X^*) \subset \ker(T^*)$, *then T is quasinormal.*

Proof. Let $Z = |T|^s X$. Then

$$\begin{aligned} T(s,t)Z &= |T|^s U|T|^t|T|^s X = |T|^s TX \\ &= |T|^s XN = ZN. \end{aligned}$$

Since $T(s,t)$ is $\min\{sp, tp\}$-hyponormal, we have

$$T(s,t)^*Z = ZN^*$$

by [30]. Hence

$$\begin{aligned} (T(s,t)^*T(s,t) - T(s,t)T(s,t)^*)|T|^s X &= T(s,t)^*T(s,t)Z - T(s,t)T(s,t)^*Z \\ &= T(s,t)^*ZN - T(s,t)ZN^* = ZN^*N - ZNN^* = 0. \end{aligned}$$

(i) If $\overline{\text{ran}(X)}$ is dense, then

$$(T(s,t)^*T(s,t) - T(s,t)T(s,t)^*)|T|^s = 0.$$

Since

$$\ker(|T|^s) \subset \ker(T(s,t)) \cap \ker(T(s,t)^*),$$

this implies $T(s,t)$ is normal. Hence T is normal by Corollary 1.
(ii) Let $X^*|T|^s x = 0$. Then $|T|^s x \in \ker(X^*) \subset \ker(T^*) = \ker(U^*)$ and $T(s,t)^* x = |T|^t U^* |T|^s x = 0$. Hence $\ker(X^*|T|^s) \subset \ker(T(s,t)^*)$ and $\overline{\operatorname{ran}(T(s,t))} \subset \overline{\operatorname{ran}(|T|^s X)}$. Hence

$$(T(s,t)^* T(s,t) - T(s,t)T(s,t)^*)T(s,t) = 0$$

by (i). This implies $T(s,t)$ is quasinormal, and T is quasinormal by Theorem 1.

Theorem 12. *Suppose that $0 < s,t,s+t = 1$ and $0 < q \leq 1$. Let $T \in \mathcal{B}(\mathcal{H})$ be such that T^* is p-hyponormal or* log*-hyponormal. Let $S \in \mathcal{B}(\mathcal{K})$ be class q-$wA(s,t)$ with* $\ker(S) \subset \ker(S^*)$. *If $XT = SX$, for some $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then $XT^* = S^*X$.*

Proof. Let T^* be a p-hyponormal operator for $p \geq \frac{1}{2}$ and let $T = U|T|$ be the polar decomposition of T. Then the generalized Aluthge transform $T^*(s,t)$ of T^* is hyponormal and satisfies

$$|T^*(s,t)|^2 \geq |T|^2 \geq |(T^*(s,t))^*|^2, \tag{12}$$

$$X'T(s,t) = SX' \tag{13}$$

where $X' = XU|T|^t$. Using the decompositions $\mathcal{H} = \ker(X')^\perp \oplus \ker(X')$ and $\mathcal{K} = \overline{\operatorname{ran}(X')} \oplus \operatorname{ran}(X')^\perp$, we see that $T(s,t)$, S and X' are of the form

$$T^*(s,t) = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix},\ S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix},\ X' = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where T_1^* is hyponormal, S_1 is class q-$wA(s,t)$ with $\ker(S_1) \subset \ker(S_1^*)$ and X_1 is a one-one operator with dense range. Since $X'T(s,t) = SX'$, we have

$$X_1 T_1 = S_1 X_1. \tag{14}$$

Hence T_1 and S_1 are normal by Corollary 6, so that $T_2 = 0$, by Lemma 12 of [30] and $S_2 = 0$ by Lemma 7. Then $|T| = |T_1| \oplus P$, for some positive operator P, by (12) and $U = \begin{pmatrix} U_1 & U_2 \\ 0 & U_3 \end{pmatrix}$ by Lemma 13 of [30]. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ be a 2×2 matrix representation of X with respect to the decomposition $\mathcal{H} = \ker(X')^\perp \oplus \ker(X')$ and $\mathcal{K} = \overline{\operatorname{ran}(X')} \oplus \operatorname{ran}(X')^\perp$. Then $X' = XU|T|^t$ implies that $X_1 = X_{11}U_1|T_1|^t$ and hence $\ker(T_1) \subset \ker(X_1) = \{0\}$. This shows that T_1 is one-one and hence it has dense range, so that $U_2 = 0$ and $T = T_1 \oplus T_4$ for some hyponormal operator T_4^* by [30, Lemma 13]. Since

$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = X' = XU|T|^t = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} U_1|T_1|^t & 0 \\ 0 & U_3|T_4|^t \end{pmatrix}$$

we deduce the following assertions.

$$X_{12}U_2|T_4|^t = 0;\ \text{hence } X_{12}T_3 = 0 \text{ because } T_4 = U_3|T_4|.$$

$$X_{21}U_1|T_1|^t;\ \text{hence } X_{12}=0 \text{ because } U_1|T_1|^{\frac{1}{2}} \text{ has dense range.}$$

$$X_{22}U_3|T_4|^t=0;\ \text{hence } X_{22}T_3=0.$$

The assumption $XT=SX$ tell us that,

$$\begin{aligned} X_{11}T_1 &= S_1X_{11} \\ X_{12}T_4 &= S_1X_{12}=0, \\ X_{22}T_4 &= S_3X_{22}=0. \end{aligned}$$

Since T_1 and S_1 are normal, we have $X_{11}T_1^*=S_1^*X_{11}$, by Fuglede-Putnam theorem. The p-hyponormality of T_4^* shows that $\overline{\text{ran}(T_4^*)}\subset\overline{\text{ran}(T_4)}$. Also, we have $\ker(S_3)\subset\ker(S_3^*)$. Hence, we also have $X_{12}T_4^*=S_1^*X_{12}=0$ and $X_{22}T_4^*S_3^*X_{22}=0$. This implies that $XT^*=X_{11}T_1^*\oplus 0=S_1^*X_{11}\oplus 0=S^*X$.

Next, we prove the case where T^* is p-hyponormal for $0<p\leq\frac{1}{2}$. Let X' be as above. Then $T^*(s,t)$ is $(p+\frac{1}{2})$-hyponormal and satisfies $X'T(s,t)=SX'$. Use the same argument as above. We obtain $T(s,t)=T_1\oplus T_3$ on $\mathcal{H}=\ker(X')^\perp\oplus\ker(X')$ and $S=S_1\oplus S_3$, where T_1 is an injective normal operator and S_1 is also normal. Hence we have $T=T_1\oplus T_4$ for some p-hyponormal T_4^* , by Lemma 13 of [30]. Again using the same argument as above, we obtain $X_{21}=0, X_{11}T_1^*=S_1^*X_{11}, X_{12}T_4^*=S_1^*X_{12}=0$ and $X_{22}T_4^*=S_3^*X_{22}=0$. Hence we have $XT^*=S^*X$.

Finally, we assume that T^* is log-hyponormal. Let $T(s,t)$ and X' be as above. Then $X'T(s,t)=SX'$ and $T^*(s,t)$ is semi-hyponormal and satisfies

$$|T^*(s,t)|\geq|T^*|\geq|(T^*(s,t)^*|.$$

By the same argument as above, we have $T(s,t)=T_1\oplus T_3$ on $\mathcal{H}=\ker(X')^\perp\oplus\ker(X')$ and $S=S_1\oplus S_3$ on $\mathcal{K}=\overline{\text{ran}(X')}\oplus\text{ran}(X')^\perp$, where T_1 is an injective normal operator, S_1 is normal, T_3^* is invertible semi-hyponormal and S_3 is class q-$wA(s,t)$ with $\ker(S_3)\subset\ker(S_3^*)$. By Lemma 13 of [30], we have that T is of the form $T=T_1\oplus T_4$, for some log-hyponormal T_4^*. Let $X=\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{pmatrix}$. Then $X'=XU|T|^t$ implies that $X_{12}=0, X_{21}=0$ and $X_{22}=0$. The assumption $XT=SX$ implies that $X_{11}T_1=S_1X_{11}$, hence $X_{11}T_1^*\oplus 0=S_1^*X_{11}$ by Fuglede-Putnam theorem. Thus we have $XT^*=X_{11}T_1^*\oplus 0=S_1^*X_{11}\oplus 0=S^*X$. Therefore, the proof of the theorem is achieved.

Example 4. *Let R be an operator such that* $\ker(R)$ *does not reduce R and let P be the orthogonal projection onto* $\ker(R)$. *Then P does not commute with T; otherwise* $\text{ran}(R)=\ker(R)$ *reduce T. Hence $PR\neq 0=RP$. It is easy to see that $RP=PR^*=0$ but $R^*P\neq PR(\neq 0)$ because* $\text{ran}(R^*P)\subset\text{ran}(R^*)\subset\ker(R^\perp)=I-P$. *If we put $T=R$, then the assertion of Theorem 10 does not hold for such T. Also, if we put $T=R^*$, $S=I-P$ and $X=P$, then $XT=PR^*=0=(I-P)P=SX$. However, $XT^*=PR\neq 0=(I-P)P=S^*X$. Hence the assertion of Theorem 12 does not hold for such T.*

Theorem 13. *Let $T \in \mathcal{B}(\mathcal{H})$ be such that T^* is an injective class p-$wA(s,t)$ for $0 < s, t, s+t =$ and $0 < p \leq 1$. Let $S \in \mathcal{B}(\mathcal{K})$ be dominant. If $XT = SX$, for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $XT^* = S^*X$.*

Proof. Assume that T^* is an injective p-w-hyponormal and let $T = U|T|$ be the polar decomposition of T. Let $T(s,t)$ be the aluthge transform of T and $X' = XU|T|^t$. Then $X'T(s,t) = SX'$ and $T^*(s,t)$ is rp-hyponormal and satisfies

$$|T^*(s,t)|^{2rp} \geq |T^*|^{2rp} \geq |(T^*(s,t))^*|^{2rp}$$

for $r \in \min\{s,t\}$. By the same argument in the proof of Theorem 12, we conclude that $T^*(s,t) = T_1 \oplus T_3$ on $\mathcal{H} = \ker(X')^{\perp} \oplus \ker(X')$ and $S = S_1 \oplus S_3$, where T_1 is an injective normal operator and S_1 is also normal, T_3^* is invertible class p-$wA(s,t)$ and S_3 is dominant. Hence by Lemma 7, we have that T is of the form $T = T_1 \oplus T_4$ for some class p-$wA(s,t)$ T_4^*. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Then $X' = XU|T|^t$ implies that $X_{12} = 0$, $X_{21} = 0$ and $X_{22} = 0$. The assumption $XT = SX$ implies that $X_{11}T_1 = S_1X_{11}$, hence $X_{11}T_1^* = S_1^*X_{11}$ by Fuglede-Putnam theorem. Thus we have $XT^* = X_{11}T_1^* \oplus 0 = S_1^*X_{11} \oplus 0 = S^*X$. Therefore, the proof of the theorem is achieved.

Example 5. *Let $T^* = R$ as in Example 2. Let $X = P$ be the orthogonal projection onto* $\ker(T^*)$ *and $S = I - P$. Then $SX = 0 = XT^*$, but $0 = S^*X \neq XT^*$. Hence the injectivity condition is necessary for Theorem 13.*

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