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A New Sushila Distribution: Properties and Applications

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Abstract. In this paper, we introduce a new continuous distribution mixing exponential and gamma distributions, called new Sushila distribution. We derive some properties of the distribution include: probability density function, cumulative distribution function, expected value, moments about the origin, coefficient of variation (C.V.), coefficient of skewness, coefficient of kurtosis, moment generating function, and reliability measures. The distribution includes, a special cases, the Sushila distribution as a particular case $p = \frac{1}{2}$ $(\theta = 1)$. The hazard rate function exhibits increasing. The parameter estimations as the moment estimation (ME), the maximum likelihood estimation (MLE), nonlinear least squares methods, and genetic algorithm (GA) are proposed. The application is presented to show that model to fit for waiting time and survival time data. Numerical results compare ME, MLE, weighted least squares (WLS), unweighted least squares (UWLS), and GA. The results conclude that GA method is better performance than the others for iterative methods. Although, ME is not the best estimate, ME is a fast estimate, because it is not an iterative method. Moreover, The proposed distribution has been compared with Lindley and Sushila distributions to a waiting time data set. The result shows that the proposed distribution is performing better than Lindley and Sushila distribution.

2020 Mathematics Subject Classifications: 60-08, 62-08, 49M99

Key Words and Phrases: Exponential distribution, Gamma distribution, Least square method, Maximum likelihood estimation, Method of moment, Genetic algorithm, Estimator

1. Introduction

Finite mixture models are a widespread tool to heterogeneous data analytic for across a broad number of fields including agriculture, botany, bioinformatics, cell biology, genetics, genomics, genealogy, paleontology, zoology, fisheries research, economics, machine learning, medicine, psychology, insurance, physics, engineering and chemistry, among many

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others. Because of their flexibility, finite mixture models can be used to cluster data, classify data, estimate densities, and increasingly.

An enormous number of research have presented some approaches into finite mixture models and many methods to fitting lifetime data for improve performance of parameters are used. The Bayesian approach is used into [17–20, 22, 23, 25]. The Maximum Likelihood Estimation approach is used into [7, 8, 10, 21, 23–25]. The Least Square Estimator and Weighted Least Square Estimator are used into [7, 10, 21]. The Lagrangian multiplier approach is used into [5] and the ARMA approach is used into [9]. There are focus on the parameter estimation which is important to bring into the next step on application.

A continuous random variable X which is called to have an exponential distribution with one parameter $\lambda > 0$, often called rate parameter, denoted by $X \sim \mathrm{Exp}(\lambda)$. The probability density function (PDF) of the distribution is given by

$$f(x;\lambda) = \lambda e^{-\lambda x}, \quad x \geq 0,$$

and the cumulative distribution function (CDF), has been obtained as

$$F(x) = 1 - \ e^{-\lambda x}, \quad x \geq 0.$$

A continuous random variable X is called to have a gamma distribution with two parameters $\alpha > 0$, often called shape parameter, and $\lambda > 0$, often called rate parameter, denoted by $X \sim \mathrm{Gamma}(\alpha, \lambda)$. The PDF of the distribution is given by

$$f(x;\alpha,\lambda) = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0,$$

where $\Gamma(\alpha)$ is gamma function. We can see that $\mathrm{Gamma}(1,\lambda) = \mathrm{Exp}(\lambda)$. Its CDF is given by

$$F(x) = \frac{1}{\Gamma(\alpha)}\gamma(\alpha, \lambda x), \quad x > 0.$$

where $\gamma(\alpha, \lambda x)$ is the lower incomplete gamma function.

There are studies that have used exponential, Lindley and gamma distributions to be used for modeling lifetime data. A two-parameter Lindley distribution is proposed by modifying the mixing weight of exponential and gamma [6, 11–16]. They proposed a two-parameter distribution by modifying the mixing weight of $\mathrm{Exp}(\frac{\theta}{\alpha})$ and $\mathrm{Gamma}(2, \frac{\theta}{\alpha})$.

A continuous random variable X is called to have Sushila distribution (SD) [16] with two parameters $\theta > 0$ and $\alpha > 0$, denoted by $X \sim \mathrm{SD}(\alpha, \theta)$. The PDF of the distribution is given by

$$f(x;\alpha,\theta) = \frac{\theta^2}{\alpha(\theta+1)}(1 + \frac{x}{\alpha})e^{-\frac{\theta}{\alpha}x}, \quad x > 0.$$

Its CDF is given by

$$F(x) = 1 - \frac{\alpha(\theta+1) + \theta x}{\alpha(\theta+1)}e^{-\frac{\theta}{\alpha}x}, \quad x > 0.$$

The PDF of Sushila distribution can be shown as a mixture of Exp($\frac{\theta}{\alpha}$) and Gamma($2, \frac{\theta}{\alpha}$) as follows:

$$f(x, \alpha, \theta) = pf_1(x) + (1-p)f_2(x),$$

where $p = \dfrac{\theta}{\theta+1}$, $f_1(x) = \dfrac{\theta}{\alpha}e^{-\frac{\theta}{\alpha}x}$, and $f_2(x) = \dfrac{\theta^2}{\alpha^2}xe^{-\frac{\theta}{\alpha}x}$.

In this paper, we focus on the Sushila distribution. The distribution is mixed by exponential and gamma distributions. Sushila distribution therefore pays attention to the weight value for exponential distribution. Notice that the gamma distribution has two parameters that provide a flexible model to fit the reliability data. For this reason, we pay attention to the weight value for gamma distribution. The main aim of this paper is to study a two-parameter ($\theta > 0$ and $\alpha > 0$) continuous distribution is introduced as follows:

$$f(x, \alpha, \theta) = (1-p)f_1(x) + pf_2(x).$$

Its CDF, expected value, first four moments, and some of the related measures have been proposed.

The paper is organized as follows. We define new Sushila distribution and its properties in Section 2. The moments, generating function, and related measures are derived in Section 3. The reliability measures, like survival function, hazard function, and mean residual life function for new Sushila distribution in Section 4. Five methods of estimation are discussed in Section 5. The numerical simulation and comparison are given to the proposed methods in Section 6. Conclusion is discussed in Section 7.

2. Definition and Properties of the new Sushila Distribution

In this section, we introduce a continuous distribution with two parameters α and θ, called new Sushila distribution. The PDF of new Sushila distribution can be shown as a mixture of Exp ($\frac{\theta}{\alpha}$) and Gamma($2, \frac{\theta}{\alpha}$) distributions as follows:

$$f(x; \alpha, \theta) = (1-p)f_1(x) + pf_2(x), \tag{1}$$

where $p = \dfrac{\theta}{\theta+1}$, $f_1(x) = \dfrac{\theta}{\alpha}e^{-\frac{\theta}{\alpha}x}$, and $f_2(x) = \left(\dfrac{\theta}{\alpha}\right)^2 xe^{-\frac{\theta}{\alpha}x}$.

Definition 1. *A continuous random variable X is said to be a new Sushila distribution random variable with two parameters α and θ, its PDF is given by*

$$f(x; \alpha, \theta) = \frac{\theta(\alpha + \theta^2 x)e^{-\frac{\theta}{\alpha}x}}{\alpha^2(\theta+1)}, \quad \textit{for all} \;\; x > 0, \alpha > 0, \theta > 0. \tag{2}$$

Probability plots of new Sushila distribution are given in Figure 1 for particular values of α and θ. From Figure 1, we can see that new Sushila distribution is a light-tailed distribution and the tail approaches to zero at a faster rate for lower values of α. It is

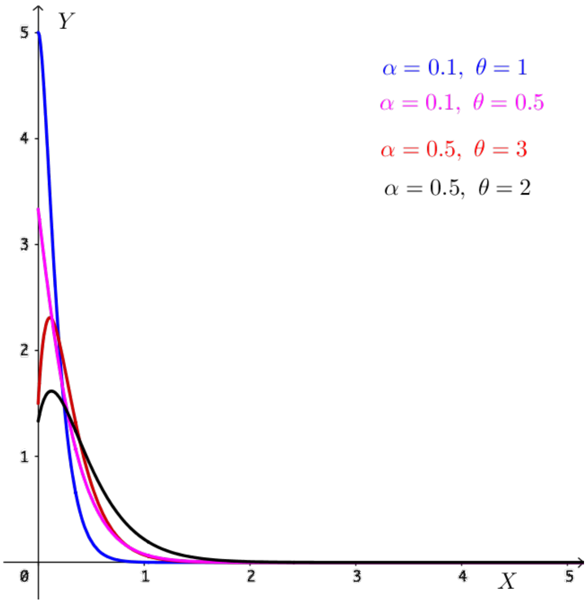

Figure 1: Plots of the PDF of new Sushila distribution for various parameter values.

evident from the density plots of new Sushila distribution that the distribution is most likely to appropriately model those data sets where the observations of lower magnitude appear more frequently than that of the higher magnitude values. Such data sets are waiting time, claim and lifetime.

The first derivative of Equation (2) with respect to x is obtained as

$$\frac{d}{dx}f(x;\alpha,\theta) = \frac{\theta^2(\alpha\theta - \alpha - \theta^2 x)e^{-\frac{\theta}{\alpha}x}}{\alpha^3(\theta+1)}. \tag{3}$$

Now, based on Equation (3), we obtain,
1. $f'(x) = 0$ gives $x = \dfrac{\alpha(\theta-1)}{\theta^2}$ as the critical point. For $\theta > 1, x_0 = \dfrac{\alpha(\theta-1)}{\theta^2}$ is the unique critical point at which $f(x)$ is maximum.
2. for $\theta < 1, f'(x) \leq 0$ i.e. $f(x)$ is decreasing in x.
Therefore, the mode of the distribution (2) is given by

$$\text{Mode} = \begin{cases} \dfrac{\alpha(\theta-1)}{\theta^2} & \text{if } \theta > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we obtain the cumulative distribution function of new Sushila distribution as follows.

Theorem 1. *The cumulative distribution function (CDF) of new Sushila distribution is given by*

$$F(x) = 1 - \frac{(\alpha\theta + \alpha + \theta^2 x)e^{-\frac{\theta}{\alpha}x}}{\alpha(\theta+1)}, \quad x > 0, \theta > 0, \alpha > 0. \tag{4}$$

Proof. For all $x > 0, \alpha > 0$ and $\theta > 0$,
consider

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= 1 - P(X \geq x) \\ &= 1 - \int_x^\infty \frac{\theta(\alpha + \theta^2 t)e^{-\frac{\theta}{\alpha}t}}{\alpha^2(\theta+1)}\, dt \\ &= 1 - \frac{(\alpha\theta + \alpha + \theta^2 x)e^{-\frac{\theta}{\alpha}x}}{\alpha(\theta+1)}. \end{aligned}$$

The CDF plots of new Sushila distribution are given in Figure 2 for particular values of α and θ.

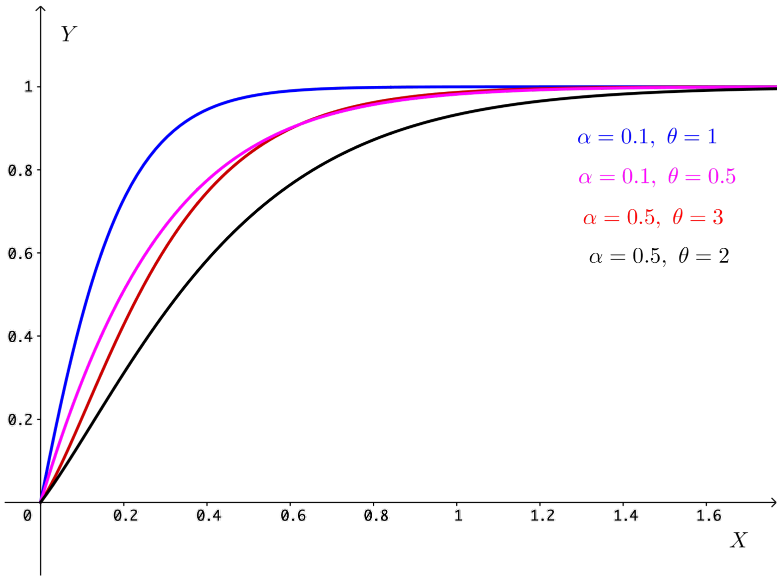

Figure 2: Plots of the CDF of new Sushila distribution for various parameter values.

3. Moments, Generating function and Related Measures

In this section, we derive the r^{th} moment about origin, moment generating function, the coefficient of variation, coefficient of skewness, and coefficient of kurtosis.

Theorem 2. *The r^{th} moment about the origin of new Sushila distribution is defined by,*

$$\mu_r' = \frac{r!\alpha^r[(r+1)\theta + 1]}{\theta^r(\theta+1)}, \quad r = 1, 2, 3, \ldots.$$

Proof. By definition, for $r = 1, 2, \ldots$, the moment about origin, μ_r' is obtained as,

$$\mu_r' = E[X^r] = \int_0^\infty x^r \frac{\theta(\alpha + \theta^2 x)e^{-\frac{\theta}{\alpha}x}}{\alpha^2(\theta+1)}\, dx$$

$$
\begin{aligned}
&= \frac{\theta}{\alpha^2(\theta+1)}\left[\int_0^\infty \alpha x^r e^{-\frac{\theta}{\alpha}x}\,dx + \theta^3\int_0^\infty x^{r+1}e^{-\frac{\theta}{\alpha}x}\,dx\right]\\
&= \frac{\theta}{\alpha^2(\theta+1)}\left[\alpha\int_0^\infty \left(\frac{x\theta}{\alpha}\right)^r\left(\frac{\alpha}{\theta}\right)^r e^{-\frac{\theta}{\alpha}x}\,dx + \theta^3\int_0^\infty \left(\frac{x\theta}{\alpha}\right)^{r+1}\left(\frac{\alpha}{\theta}\right) e^{-\frac{\theta}{\alpha}x}\,dx\right]\\
&= \frac{1}{\alpha(\theta+1)}\left(\frac{\alpha^{r+1}}{\theta^r}\Gamma(r+1) + \frac{\alpha^{r+1}}{\theta^{r-2}}\Gamma(r+2)\right)\\
&= \frac{\alpha^{r+1}}{\alpha(\theta+1)}\left(\frac{r!}{\theta^r} + \frac{1}{\theta^{r-2}}(r+1)!\right)\\
&= \frac{r!\alpha^r[(r+1)\theta+1]}{\theta^r(\theta+1)}.
\end{aligned}
$$

The first four moments about origin of new Sushila distribution can be obtained as:

$$\mu_1' = \frac{\alpha(2\theta+1)}{\theta(\theta+1)},\ \mu_2' = \frac{2\alpha^2(3\theta+1)}{\theta^2(\theta+1)},\ \mu_3' = \frac{6\alpha^3(4\theta+1)}{\theta^3(\theta+1)},\ \mu_4' = \frac{24\alpha^4(5\theta+1)}{\theta^4(\theta+1)}.$$

In particular, the 1^{st} moment about origin take the form,

$$\mu = E[X] = \frac{\alpha(2\theta+1)}{\theta(\theta+1)}.$$

Using the relationship between moments about mean and the moments about the origin, we have the moment about mean as the following theorem.

Theorem 3. *The first four moments about mean of new Sushila distribution is defined by,*

$$
\begin{aligned}
\mu_2 &= \frac{\alpha^2(2\theta^2+4\theta+1)}{\theta^2(\theta+1)^2},\\
\mu_3 &= \frac{2\alpha^3(2\theta^3+6\theta^2+6\theta+1)}{\theta^3(\theta+1)^3},\\
\mu_4 &= \frac{3\alpha^4(8\theta^4+32\theta^3+44\theta^2+24\theta+3)}{\theta^4(\theta+1)^4}.
\end{aligned}
$$

Proof. For $x > 0, \alpha > 0$ and $\theta > 0$.
Consider

$$
\begin{aligned}
\mu_2 &= E[(X-\mu)^2]\\
&= E[X^2] - \mu^2\\
&= \frac{2\alpha^2(3\theta+1)}{\theta^2(\theta+1)} - \left(\frac{\alpha(2\theta+1)}{\theta(\theta+1)}\right)^2
\end{aligned}
$$

$$= \frac{\alpha^2(2\theta^2+4\theta+1)}{\theta^2(\theta+1)^2},$$

$$\begin{aligned}\mu_3 &= E[X^3] - 3\mu E[X^2] + 2\mu^3\\ &= \frac{2\alpha^2(3\theta+1)}{\theta^2(\theta+1)} - 3\frac{\alpha(2\theta+1)}{\theta(\theta+1)}\left[\frac{2\alpha^2(3\theta+1)}{\theta^2(\theta+1)}\right] + 2\left[\frac{\alpha(2\theta+1)}{\theta(\theta+1)}\right]^3\\ &= \frac{2\alpha^3(2\theta^3+6\theta^2+6\theta+1)}{\theta^3(\theta+1)^3},\end{aligned}$$

$$\begin{aligned}\mu_4 &= E[(X-\mu)^4]\\ &= E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4\\ &= \frac{24\alpha^4(5\theta+1)}{\theta^4(\theta+1)} - 4\left(\frac{\alpha(2\theta+1)}{\theta(\theta+1)}\right)\left(\frac{6\alpha^3(4\theta+1)}{\theta^3(\theta+1)}\right)\\ &+ \quad 6\left(\frac{\alpha(2\theta+1)}{\theta(\theta+1)}\right)^2\left(\frac{2\alpha^2(3\theta+1)}{\theta^2(\theta+1)}\right) - 3\left(\frac{\alpha(2\theta+1)}{\theta(\theta+1)}\right)^4\\ &= \frac{3\alpha^4(8\theta^4+32\theta^3+44\theta^2+24\theta+3)}{\theta^4(\theta+1)^4}.\end{aligned}$$

In particular, the 2^{nd} moment about the mean is variance that is,

$$\sigma^2 = \mu_2 = \frac{\alpha^2(2\theta^2+4\theta+1)}{\theta^2(\theta+1)^2}.$$

The coefficient of variation (C.V.), coefficient of skewness ($\sqrt{\beta_1}$), and coefficient of kurtosis (β_2) of new Sushila are given by

$$\begin{aligned}\text{C.V.} &= \frac{\sigma}{\mu_1'} = \frac{\sqrt{2\theta^2+4\theta+1}}{2\theta+1},\\ \sqrt{\beta_1} &= \frac{\mu_3}{\mu_2^{3/2}} = \frac{4\theta[\theta(\theta+3)+3]+2}{\theta^6(\theta+1)^6[2\theta(\theta+2)+1]^{3/2}},\\ \beta_2 &= \frac{3(8\theta^4+32\theta^3+44\theta^2+24\theta+3)}{(2\theta^2+4\theta+1)^2}.\end{aligned}$$

Now, we derive the moment generating function (MGF) of new Sushila distribution as the following.

Theorem 4. *The moment generating function $M_X(t)$ of new Sushila distribution is defined by*

$$M_X(t) = E[e^{tX}] = \frac{\theta(\alpha t+\theta^2-\theta)}{(\theta+1)(\theta-\alpha t)^2}, \quad \frac{\theta}{\alpha} > t.$$

Proof.

$$
\begin{aligned}
M_X(t) = E[e^{tX}] &= \frac{\theta}{\alpha^2(\theta+1)} \int_0^{\infty} (\alpha + \theta^2 x) e^{(t-\frac{\theta}{\alpha})x}\, dx \\
&= \frac{\theta}{\alpha^2(\theta+1)} \lim_{b\to\infty} \left[\frac{\alpha(\alpha^2 t + \alpha\theta + \alpha\theta(\theta(xt-1)-1)) + \theta^3(-x)}{(\theta-\alpha t)^2} e^{(t-\frac{\theta}{\alpha})x} \right]_{x=0}^{b} \\
&= \frac{\theta(\alpha t + \theta^2 - \theta)}{(\theta+1)(\theta-\alpha t)^2}, \quad \frac{\theta}{\alpha} > t.
\end{aligned}
$$

4. Reliability Measures

In this section, we derive expressions for the reliability measures, like survival function, hazard function, and mean residual life function for new Sushila distribution.

Theorem 5. *For $x > 0, \theta > 0, \alpha > 0$. The survival function $S(x)$, the hazard rate function $h(x)$, and mean residual life function $m(x)$ of new Sushila distribution is defined by,*

$$
S(x) = \frac{(\alpha\theta + \alpha + \theta^2 x)e^{-\frac{\theta}{\alpha}x}}{\alpha(\theta+1)}, \qquad h(x) = \frac{\theta(\alpha+\theta^2 x)}{\alpha(\alpha\theta+\alpha+\theta^2 x)},
$$

and

$$
m(x) = \frac{\alpha^2(\theta+1)(2\alpha\theta+\alpha+\theta^2 x)}{\alpha\theta+\alpha+\theta^2 x}.
$$

Proof. By definition of the survival function, we have

$$
\begin{aligned}
S(x) &= 1 - F(x) \\
&= \frac{(\alpha\theta + \alpha + \theta^2 x)e^{-\frac{\theta}{\alpha}x}}{\alpha(\theta+1)}, \quad x > 0, \theta > 0, \alpha > 0.
\end{aligned}
$$

By definition of the hazard rate function, we have

$$
\begin{aligned}
h(x) &= \lim_{\Delta x\to 0} \frac{P(X < x + \Delta x | X > x)}{\Delta x} \\
&= \frac{f(x)}{1-F(x)} \\
&= \frac{\theta(\alpha+\theta^2 x)}{\alpha(\alpha\theta+\alpha+\theta^2 x)}.
\end{aligned}
$$

By definition of the mean residual life function, we have

$$
m(x) = E[X - x | X > x]
$$

$$= \frac{1}{1-F(x)}\int_x^\infty (1-F(t))\,dt$$
$$= \frac{\alpha^2(\theta+1)(2\alpha\theta+\alpha+\theta^2 x)}{\alpha\theta+\alpha+\theta^2 x}.$$

It can be easily verified that $h(0) = \dfrac{\theta\alpha}{\theta+\alpha^2} = f(0)$ and $m(0) = \mu_1'$. The derivative $h'(x) > 0$ for all x, so $h(x)$ is an increasing function of x, α and θ, whereas $m(x)$ is a decreasing function of x, α and θ.

5. Estimation of Parameters

In this section, we proposed five methods for estimating parameters. The first two methods are the moment estimation (ME), the maximum likelihood estimates (MLE). These methods are widely used for estimation because they are simple. The next two methods are the weighted and unweighted least squares methods via the CDF. These methods are more complicated than the first two methods, but the least squares are better than in some situations. The last method is the genetic algorithm which is a heuristic search that mimics the process of natural evolution. This method is an efficient and effective technique.

5.1. The Moment Estimation (ME)

The new Sushila distribution has two parameters to be estimated. Using the first moment about origin, we have

$$E[X] = \frac{\alpha(2\theta+1)}{\theta(\theta+1)},$$
$$E[X] = \bar{X}.$$

We obtain,

$$\bar{X} = \frac{\alpha(2\theta+1)}{\theta(\theta+1)}.$$

So

$$\hat{\alpha} = \frac{\theta(\theta+1)\bar{X}}{(2\theta+1)}.$$

Using the second moment about mean, we have

$$\mu_2 = E[X^2] - \mu^2 = \frac{\alpha^2(2\theta^2+4\theta+1)}{\theta^2(\theta+1)^2},$$

and

$$E[X^2] - \mu^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{X}^2,$$

so

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{X}^2 = \frac{\alpha^2(2\theta^2+4\theta+1)}{\theta^2(\theta+1)^2},$$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{X}^2 = \left(\frac{\theta(\theta+1)\bar{X}}{(2\theta+1)}\right)^2 \frac{(2\theta^2+4\theta+1)}{\theta^2(\theta+1)^2}.$$

Solving this equation for θ, we get

$$\hat{\theta} = \frac{\left(2\bar{X}^2 - \frac{1}{n}\sum_{i=1}^{n} x_i^2\right) \pm \sqrt{\bar{X}^2\left(\bar{X}^2 - \frac{1}{2n}\sum_{i=1}^{n} x_i^2\right)}}{\frac{2}{n}\sum_{i=1}^{n} x_i^2 - 3\bar{X}^2}, \quad \frac{2}{n}\sum_{i=1}^{n} x_i^2 - 3\bar{X}^2 \neq 0,$$

where $\hat{\alpha}, \hat{\theta}$ are the estimators of the parameter θ and α, respectively. This method will be referred to as Method 1.

5.2. The Maximum Likelihood Estimates (MLE)

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from new Sushila distribution and let f_x be the observed frequency in the sample corresponding to $X = x$ $(x = 1, 2, ..., k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function, L of new Sushila distribution is given by

$$L = \left(\frac{\theta}{\alpha^2(\theta+1)}\right)^n \prod_{x=1}^{k} \left(\alpha + \theta^2 x\right)^{f_x} e^{-\frac{\theta}{\alpha}(n\bar{x})},$$

and so the log likelihood function is obtained as

$$\log L = n\log\theta - 2n\log\alpha - n\log(\theta+1) + \sum_{x=1}^{k} f_x \log(\alpha+\theta^2 x) - \frac{n\theta\bar{x}}{\alpha}. \tag{5}$$

By differentiating Equation (5) with respect to θ and α, is obtained by:

$$\frac{\partial \log L}{\partial\theta} = 0, \quad \frac{\partial \log L}{\partial\alpha} = 0. \tag{6}$$

The equation (6) do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \dfrac{\partial^2 \log L}{\partial \theta^2} & \dfrac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \\ \dfrac{\partial^2 \log L}{\partial \alpha \partial \theta} & \dfrac{\partial^2 \log L}{\partial \alpha^2} \end{bmatrix} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \log L}{\partial \theta} \\ \\ \dfrac{\partial \log L}{\partial \alpha} \end{bmatrix},$$

where θ_0 and α_0 are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained. This method will be referred to as Method 2.

5.3. The Unweighted Least Squares (UWLS) Method via the CDF

Let $X_1, X_2, \ldots, X_n$ be n independent random variables having the new Sushila distribution with parameters α and θ. Without loss of generality, suppose that $X_1 < X_2 < \cdots < X_n$ are the order statistics and by taking the natural logarithm on the both sides of Equation (4), we obtain that

$$\begin{aligned} \log(F(x)) &= \log\left(1 - \frac{(\alpha\theta + \alpha + \theta^2 x)e^{-\frac{\theta x}{\alpha}}}{\alpha(\theta+1)}\right), \quad x, \alpha, \theta > 0 \\ &= \log\left((\theta+1) - (\theta + 1 + \frac{\theta^2 x}{\alpha})e^{-\frac{\theta x}{\alpha}}\right) - \log(\theta+1). \end{aligned}$$

Let $0 < x_1 < x_2 < \cdots < x_n$ be n orders observations, then

$$\log(F(x_i)) = \log\left((\theta+1) - (\theta + 1 + \frac{\theta^2 x_i}{\alpha})e^{-\frac{\theta x_i}{\alpha}}\right) - \log(\theta+1), \quad x, \alpha, \theta > 0.$$

Let the empirical distribution function of X be denoted by $F_n(x)$, By reference [4], the estimator of $F(x_i)$ can be considered

$$F_n(x_i) = \frac{i - d}{n - 2d + 1}, \quad i = 1, 2, \ldots, n, \tag{7}$$

for some real number d, $0 \leq d \leq 1$. So, we choose four popular expressions (see also [1]) that are often used as estimators of $F(x_i)$

$$u_{ik} = \begin{cases} \frac{i}{n+1}, & k = 1 \\ \\ \frac{i-0.3}{n+0.4}, & k = 2 \\ \\ \frac{i-0.375}{n+0.25}, & k = 3 \\ \\ \frac{i-r}{n}, & k = 4, \end{cases}$$

$i = 1, 2, 3, \ldots, n$ and for some real number $r \in (0, 1)$.
We estimate α and θ by the unweighted least squares method, by using minimizing function

$$E_k(\alpha, \theta) = \sum_{i=1}^{n} \left(\log(u_{ik}) - \log\left((\theta + 1) - (\theta + 1 + \frac{\theta^2 x_i}{\alpha})e^{-\frac{\theta x_i}{\alpha}} \right) + \log(\theta + 1) \right)^2, \quad (8)$$

$k = 1, 2, 3, 4$. By solving

$$\frac{\partial}{\partial \alpha} E_k(\alpha, \theta) = 0$$
$$\frac{\partial}{\partial \theta} E_k(\alpha, \theta) = 0,$$

$k = 1, 2, 3, 4$.
We denoted by

$$\begin{aligned} A(x_i, \alpha, \theta) &= x_i(\alpha + \theta^2 x_i) \log\left(-e^{-\frac{\theta x_i}{\alpha}} (\frac{\theta^2 x_i}{\alpha} + \theta + 1) + \theta + 1 \right), \\ B_k(x_i, \alpha, \theta) &= x_i(\alpha + \theta^2 x_i) \log(u_{ik}), \\ C(x_i, \alpha, \theta) &= \alpha(\theta + 1)(e^{\frac{\theta x_i}{\alpha}} - 1) - \theta^2 x_i, \\ D_k(x_i, \alpha, \theta) &= \frac{x_i[\log(u_{ik}) + \log(\theta + 1) - \log(-e^{-\frac{\theta x_i}{\alpha}} (\frac{\theta^2 x_i}{\alpha} + \theta + 1) + \theta + 1)]}{(\theta + 1)[\alpha(\theta + 1)(e^{\frac{\theta x_i}{\alpha}} - 1) - \theta^2 x_i}, \end{aligned}$$

where $\hat{\alpha}$ is the estimator of the parameter α with this method. Using some algebraic manipulations, $\hat{\theta}$ satisfies the following equation:

$$\hat{\theta} = \exp\left(\frac{\sum_{i=1}^{n} \frac{A(x_i, \alpha, \theta)}{C_k(x_i, \alpha, \theta)} - \sum_{i=1}^{n} \frac{B_k(x_i, \alpha, \theta)}{C(x_i, \alpha, \theta)}}{\sum_{i=1}^{n} \frac{x_i(\alpha + \theta^2 x_i)}{C(x_i, \alpha, \theta)}} \right) - 1,$$

and

$$\hat{\alpha} = \sum_{i=1}^{n} \frac{(\theta + 1)\theta^2 x_i D_k(x_i, \alpha, \theta)}{D_k(x_i, \alpha, \theta)},$$

for $k = 1, 2, 3, 4$. These four UWLS methods via u_{ik}, $k = 1, 2, 3, 4$, will be referred to as Methods 3 through 6, respectively.

5.4. The Weighted Least Squares (WLS) Method via the CDF

Using the same weight for all datum can be erroneous [2], therefore, we weigh the values in Equation (8) via a weighting factor. To find one of these factors we may use a variance stabilizing transformation. This approach, proposed by Bickel and Doksum [3] relates the variance of $\log(u_{ik})$, denoted by $Var(\log(u_{ik}))$, to the uncertainty of u_{ik} and that we denote by $Var(u_{ik})$. We obtain that

$$Var(\log(u_{ik})) = \left(\frac{\partial \log(u_{ik})}{\partial u_{ik}}\right)^2 Var(u_{ik}),$$

which gives

$$Var(\log(u_{ik})) = \left(\frac{1}{\partial u_{ik}}\right)^2 Var(u_{ik}).$$

Therefore, we use w_{ik} as a weighting factor, denoted by $w_{ik} = u_{ik}^2$ and minimizing function

$$E_k(\alpha,\theta) = \sum_{i=1}^{n} w_{ik}\left(\log(u_{ik}) - \log\left((\theta+1) - (\theta+1+\frac{\theta^2 x_i}{\alpha})e^{-\frac{\theta x_i}{\alpha}}\right) + \log(\theta+1)\right)^2,$$

$k = 1, 2, 3, 4$. By solving

$$\frac{\partial}{\partial\alpha}E_k(\alpha,\theta) = 0$$
$$\frac{\partial}{\partial\theta}E_k(\alpha,\theta) = 0.$$

$k = 1, 2, 3, 4$.
We denoted by

$$A(x_i,\alpha,\theta) = x_i(\alpha+\theta^2 x_i)\log\left(-e^{-\frac{\theta x_i}{\alpha}}(\frac{\theta^2 x_i}{\alpha}+\theta+1)+\theta+1\right),$$
$$B_k(x_i,\alpha,\theta) = x_i(\alpha+\theta^2 x_i)\log(u_{ik}),$$
$$C(x_i,\alpha,\theta) = \alpha(\theta+1)(e^{\frac{\theta x_i}{\alpha}}-1) - \theta^2 x_i,$$
$$D_k(x_i,\alpha,\theta) = \frac{x_i[\log(u_{ik}) + \log(\theta+1) - \log(-e^{-\frac{\theta x_i}{\alpha}}(\frac{\theta^2 x_i}{\alpha}+\theta+1)+\theta+1)]}{(\theta+1)[\alpha(\theta+1)(e^{\frac{\theta x_i}{\alpha}}-1) - \theta^2 x_i},$$

where $\hat{\alpha}$ is the estimator of the parameter α with this method. Using some algebraic manipulations, $\hat{\theta}$ satisfies the following equation:

$$\hat{\theta} = \exp\left(\frac{\sum_{i=1}^{n} w_{ik}\frac{A(x_i,\alpha,\theta)}{C_k(x_i,\alpha,\theta)} - \sum_{i=1}^{n} w_{ik}\frac{B_k(x_i,\alpha,\theta)}{C(x_i,\alpha,\theta)}}{\sum_{i=1}^{n} w_{ik}\frac{x_i(\alpha+\theta^2 x_i)}{C(x_i,\alpha,\theta)}}\right) - 1,$$

and

$$\hat{\alpha} = \sum_{i=1}^{n}\frac{(\theta+1)\theta^2 x_i w_{ik} D_k(x_i,\alpha,\theta)}{w_{ik}D_k(x_i,\alpha,\theta)},$$

for $k = 1, 2, 3, 4$. These four WLS methods via u_{ik}, $k = 1, 2, 3, 4$ and weighting factors w_{ik}, will be referred to as Methods 7 through 10 respectively.

5.5. Genetic algorithm

In the genetic algorithm process is as follows:

Step 1. Determine the number of chromosomes, generation, and mutation rate and crossover rate value. The number of chromosomes is 2 (α and θ), the number of generations is 10000, and the mutation rate is 0.3.

Step 2. Generate chromosome-chromosome number of the population, and the initialization value of the genes chromosome-chromosome with a random value

Step 3. Process steps 4-7 until the number of generations is met

Step 4. Evaluation of fitness value of chromosomes by calculating objective function. The fitness values is defined by:

$$f_i = \frac{1}{i},$$

where i is current chromosomes.

Step 5. Chromosomes selection. We used the roulette wheel selection. The probability of choosing individual i is equal to

$$p_i = \frac{f_i}{\sum_{i=1}^{N} f_i}$$

where f_i is the fitness value of i and N is the size of the current generation.

Step 6. Crossover. A point on both parents' chromosomes is picked by chromosome 2.

Step 7. Mutation. The chromosome i for $i = 1, 2$ are mutated as follows: we random $u_i \in (0, 1)$, if $u_i <$ Mutation rate, then we mutated the chromosome i.

Step 8. Solution (Best Chromosomes)

This method will be referred to as Method 11.

6. A simulated Study

In this section. We use two data sets to find the good parameter estimation. The data are waiting time (in minutes) of 100 bank customers [16] as Table 1, and (in days) of 72 guinea pigs infected with virulent tubercle bacilli [14] as Table 2. The comparison of new Sushila distribution with Lindley and Sushila distribution is proposed in Table 5.

It can be observed from Table 3 that the estimates obtained via UWLS (Method 3-6) are very same, also the estimates obtained via WLS (Method 7-10), although the estimates obtained via ME (Method 1) is not the best estimates, this method is good estimate, directly calculate. Moreover, it is not an iterative method like the others. The best estimate can be obtained using GA (Method 11).

Table 1: Waiting time (in minutes) of 100 bank customers.

Waiting time (minutes)	Observed frequency
0 - 4.9	30
5 - 9.9	32
10 - 14.9	19
15 - 19.9	10
20 - 24.9	5
25 - 29.9	1
30 - 34.9	2
35 - 39.9	1
Total	100

Table 2: Data of survival time (in days) of 72 guinea pigs infected with virulent tubercle bacilli.

Survival time (days)	Observed frequency
0 - 79	8
80 - 159	30
160 - 239	18
240 - 319	8
320 - 399	4
400 - 479	3
480 - 559	1
Total	72

It can be observed from Table 4 that the estimates obtained via UWLS (Method 3-6) and WLS (Method 7-10) are very same, although the estimates obtained via ME (Method 1) have the largest χ^2, this estimates is directly calculated and it is not an iterative method like the others. MLE can not calculate the parameters. The best performance can be obtained using GA (Method 11).

Table 5 shows comparison parameter estimation for the waiting time data set. The estimator of the Lindley distribution has been obtained by the moment estimation (ME). The estimators of the Sushila distribution have been obtained by the moment estimation (ME) and the maximum likelihood estimates (MLE). The estimator of new Sushila distribution has been obtained by the genetic algorithm (GA). The results show that new Sushila distribution gives better performance than the Lindley and Sushila distributions for waiting time data set.

The survival time data set in Table 2, the estimator of the Sushila distribution is obtained by the moment estimation (ME). We found that the parameter $\hat{\theta} < 0$, the $\hat{\theta}$ does not satisfy the definition of Sushila distribution. Therefore we can not find the estimators for Sushila distribution. However, the new Sushila distibution can find the estimators for this data as the following Table 4.

Table 3: ME, MLE, UWLS, WLS, and GA estimators of α and θ for waiting time

Method	α	θ	χ^2
1	28.099565	5.366360	2.317336
2	0.000398	0.000032	10.140794
3	2.977747	0.408350	7.017687
4	5.230973	0.772538	4.996613
5	3.334206	0.437684	6.528164
6	5.470875	0.796321	4.878338
7	4.055956	0.644292	7.033345
8	10.583034	1.606832	3.627089
9	2.576107	0.303251	7.314763
10	1.670607	0.187810	7.806489
11	22	4	2.083757

Table 4: ME, MLE, UWLS, WLS, and GA estimators of α and θ for survival time

Method	α	θ	χ^2
1	389.388353	0.267269	27.842259
2	-	-	-
3	319.472049	3.114969	14.267841
4	322.349246	3.125277	14.271553
5	321.338306	3.121736	14.269368
6	318.985074	3.113425	14.267277
7	318.245084	3.110803	14.267468
8	316.905403	3.106044	14.268873
9	315.503637	3.101055	14.271810
10	318.441567	3.111500	14.267377
11	9438.901726	103	7.936444

7. Conclusion

A two-parameter continuous distribution, called new Sushila distribution has been presented. Some properties of the distribution such as CDF, expected value, the r^{th} moment, and estimation of parameters by nonlinear least squares methods, the maximum likelihood estimation (MLE), the moment estimation (ME), and genetic algorithm (GA) have been discussed. We found that the distribution contains the Sushila distribution as a particular case $p = \frac{1}{2}$ $(\theta = 1)$.

Finally, the two popular methods to estimate the parameters of a probability distribution are maximum likelihood (MLE) and the method of moments (ME), and the other method of least squares (UWLS and WLS) with various u_{ik} and genetic algorithm (GA). We compare the 11 methods to estimate the parameters of the new Sushila distribution. In experiment, we observed that the best parameter estimation is obtained via GA. Besides, if we want a fast method to estimate the parameters, we introduce ME for estimating parameters; moreover, we present an application of new Sushila distribution for fitting the

Table 5: Comparing the estimation for waiting time

Waiting time (minutes)	Observed frequency	Lindley (ME)	Sushila (ME)	Sushila (MLE)	New Sushila (GA)
0 -4.9	30	29.8841	29.8295	32.3716	29.7298
5-9.9	32	30.2563	30.3055	21.7393	29.2572
10-14.9	19	18.8852	18.9084	14.5916	18.7491
15-19.9	10	10.0637	10.0653	9.7891	10.3586
20-24.9	5	4.9523	4.9468	6.5642	5.3034
25-29.9	1	2.3226	2.3168	4.3997	2.5920
30-34.9	2	1.0547	1.0506	2.9477	1.2277
35-39.9	1	0.4680	0.4655	1.9740	0.5685
Total	100	97.8870	97.8884	94.3772	97.7864
Paramters		$\hat{\theta} = 0.1907$	$\hat{\theta} = 17.9493$	$\hat{\theta} = 0.1863$	$\hat{\theta} = 4$
			$\hat{\alpha} = 213.1197$	$\hat{\alpha} = 0.9758$	$\hat{\alpha} = 22$
	χ^2	2.3074	2.3173	10.1382	2.0813

waiting time data and survival time. Furthermore, We compare new Sushila distribution with Lindley and Sushila distributions for the waiting time data. The results show that new Sushila gives better fit than both the Lindley and Suhila distribution. Besides, the Sushila distribution cannot find the estimator for the survival time data.

In future work, we will adjust the genetic algorithm for estimating parameter. On the other hand as there are other well established meta-heuristics, such as Bee and Ant Colony. We can use them for a better estimation.

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Appendix

The structure of genetic algorithm is presented, where the notation is used:
Nc is the number of generation,
Np is the number of population,
Mr is the mutation rate,
Cr is crossover rate,
Gen is a number of the population,
$[a_k^{(i)}, b_k^{(i)}]$ is the i current chromosomes and k is the current generation,
$f(X)$ is the objective function,
S^* is the best chromosomes,
$P_n(m,:)$ is the m^{th} row of matrix P_n

Initial the parameter (Nc, Np, Mr, Cr, Gen) and Set $k = 1$;
While $k \leq Gen$ do:
  Generate a random number of chromosomes $X_k^{(i)} = \begin{bmatrix} a_k^{(i)} & b_k^{(i)} & f(a_k^{(i)}, b_k^{(i)}) \end{bmatrix}$ and calculate $f(X_k^{(i)})$ for each $i = 1, 2, \ldots, Np$ and defined P_k at k iteration as follows: $P_k = \begin{bmatrix} X_k^{(1)} & X_k^{(2)} & \ldots & X_k^{(i)} & X_k^{(Np)} \end{bmatrix}'_{Np \times 3}$
  Calculate $i^* = \arg_i \min\{f(a_k^{(i)}, b_k^{(i)})\}$.
  Set $g^* = f(a_k^{(i^*)}, b_k^{(i^*)})$ and $P^* = \begin{bmatrix} a_k^{(i^*)} & b_k^{(i^*)} & f(a_k^{(i^*)}, b_k^{(i^*)}) \end{bmatrix}$.
  Calculate the fitness value $f^{(i)} = \frac{1}{i}$ for each population $i = 1, 2, \ldots, Np$.
  Calculate the roulette wheel selection $p_i = \frac{f^{(i)}}{\sum_{j=1}^{Np} f^{(j)}}$ for $i = 1, 2, \ldots, Np$.
    if $(f(a_k^{(i^*)}, b_k^{(i^*)}) < g^*)$
      $g^* = f(a_n^{(k)}, b_n^{(k)})$; $k = k + 1$;
    end
  while $j \leq Np/2$
    % **Selection**
    for $l = 1 : 2$
      generate a random number, $u_l \in (0, 1)$.
      if $(\sum_{i=1}^{m} p_i < u_l < \sum_{i=1}^{m+1} p_i)$ for $m = 1, 2, \ldots, Np$
        $Parent(l,:) = P_k(m,:)$
      end
    end
    %**Crossover**
    $offspring(1,:) = [Parent(1,1) \quad Parent(2,2)]$
    $offspring(2,:) = [Parent(2,1) \quad Parent(1,2)]$
  end
  $P_{k+1}(2j-1,:) = [offspring(1,:) \quad f(offspring(1,:))]$
  $P_{k+1}(2j,:) = [offspring(2,:) \quad f(offspring(2:))]$
  %**Mutation**

while $i \leq Np$ do:
  Perform mutation operation on P_{k+1} with Mr.
  Replaced the bad population with the best population $X_k^{(i^*)}$ in P_{k+1}.
end
end