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On 2-Resolving Dominating Sets in the Join, Corona and Lexicographic Product of two Graphs

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Abstract. Let G be a connected graph. An ordered set of vertices $\{v_1, ..., v_l\}$ is a 2-resolving set for G if, for any distinct vertices $u, w \in V(G)$, the lists of distances $(d_G(u, v_1), ..., d_G(u, v_l))$ and $(d_G(w, v_1), ..., d_G(w, v_l))$ differ in at least 2 positions. A 2-resolving set $S \subseteq V(G)$ which is dominating is called a 2*-resolving dominating set* or simply $2R$-dominating set in G. The minimum cardinality of a 2-resolving dominating set in G, denoted by $\gamma_{2R}(G)$, is called the $2R$*-domination number* of G. Any $2R$-dominating set of cardinality $\gamma_{2R}(G)$ is then referred to as a γ_{2R}*-set* in G. This study deals with the concept of 2-resolving dominating set of a graph. It characterizes the 2-resolving dominating set in the join, corona and lexicographic product of two graphs and determine the bounds or exact values of the 2-resolving dominating number of these graphs.

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Key Words and Phrases: 2-resolving set, 2-resolving dominating set, $2R$-domination number, join, corona, lexicographic product of two graphs

1. Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater [7], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [3] where metric generators were called resolving sets.

Bailey and Yero in [6] demonstrated a construction of error-correcting codes from graphs by means of k-resolving sets, and present a decoding algorithm which makes use of covering designs.

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The distance between two vertices u and v of a graph is the length of a shortest path between u and v, and we denote this by $d_G(u,v)$. In recent years, much attention has been paid to the *metric dimension* of graphs: this is the smallest size of a subset of vertices (called a *resolving set*) with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex and is denoted by $\dim(G)$.

According to the paper of Saenpholphat et al. [8], for an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v in G, the k-vector (ordered k-tuple)

$$r(v/W) = (d_G(v, w_1), d_G(v, w_2), ..., d_G(v, w_k))$$

is referred to as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representation with respect to W. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r(v/W) : v \in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality is called a *minimum resolving set* or a *basis*, and the cardinality of a basis for G is the *dimension* $\dim(G)$ of G.

In the paper of Rara and Cabaro [4], an ordered set of vertices $W = \{w_1, ..., w_l\}$ is a 2-*resolving set* for G if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r(u/W)$ and $r(v/W)$ of u and v, respectively differ in at least 2 positions. Then W is said to be a 2-resolving set for G. If G has a 2-resolving set, the minimum cardinality $\dim_2(G)$ is called the 2-*metric dimension* of G. If $k = 2$ is the largest integer for which G has a 2-resolving set, then we say that G is a 2-*metric dimensional* graph.

In this paper, the concept of 2-resolving dominating set in the join, corona and lexicographic product of two graphs is discussed.

2. Preliminary Results

In this study, we consider finite, simple and connected undirected graphs. For basic graph-theoretic concepts, we refer readers to [5].

Theorem 1. [2] Let G and H be two nontrivial graphs such that G is connected. Then the following assertions hold for any $a, c \in V(G)$ and $b, d \in V(H)$ such that $a \neq c$.

(i) $N_{G[H]}(a, b) = (\{a\} \times N_H\{b\}) \cup \{N_G\{a\} \times V(H)\}$

(ii) $d_{G[H]}((a, b),(c, d)) = d_G(a, c)$

(iii) $d_{G[H]}(a, b),(a, d) = \min\{d_H(b, d), 2\}$.

Proposition 1. [1] Let G be a connected graph of order $n \geq 2$. Then $\dim_2(G) = 2$ if and only if $G \cong P_n$.

Proposition 2. $\dim_2(K_n) = n$ for $n \geq 2$.

Remark 1. For any connected graph G of order $n \geq 2$,

$$1 < \gamma_{2R}(G) \leq n.$$

Remark 2. For any connected graph G of order $n \geq 2$, $\dim_2(G) \leq \gamma_{2R}(G)$.

Remark 3. For $n \geq 2$, $\gamma_{2R}(K_n) = n$.

Theorem 2. Let G be a nontrivial connected graph of order $n \geq 2$. Then $\gamma_{2R}(G) = 2$ if and only if $G \cong P_n$, for $2 \leq n \leq 4$.

Proof. Suppose that $\gamma_{2R}(G) = 2$. Let $S = \{x, y\}$ be a γ_{2R}-set in G. By Remark 2 and Proposition 1, $G \cong P_n$. Moreover, x and y are the end vertices in G. Since S is a dominating set in G, $2 \leq n \leq 4$.

The converse follows immediately from Proposition 1. □

Theorem 3. Let G be a connected graph of order $n \geq 2$. If $G \in \{K_n, F_3, P_2 + P_3, K_{m,n-m}\}$, where $m \geq 2$, then $\gamma_{2R}(G) = n$.

Example 1. The sets $S_1 = \{b, e, g\}$ and $S_2 = \{a, d, e, g\}$ in Figure 1 are 2-resolving dominating sets in G. Moreover, S_1 is a γ_{2R}-set in G. Thus, $\gamma_{2R}(G) = 3$.

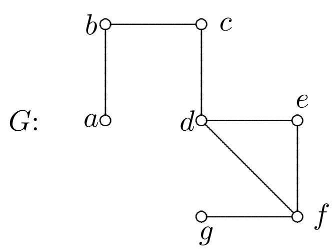

Figure 1: A graph G with $\gamma_{2R}(G) = 3$

Example 2. Consider the graph G in Figure 2. The ordered set of vertices $W = \{u_1, u_2, u_3\}$ is a 2-resolving set for the graph G since the representations $r_G(u_1/W_3) = (0, 1, 2)$, $r_G(u_2, W_3) = (1, 0, 1)$, $r_G(u_3/W_3) = (2, 1, 0)$, $r_G(u_4/W_3) = (2, 2, 1)$, $r_G(u_5/W_3) = (1, 2, 2)$ and $r_G(u_6/W_3) = (3, 3, 2)$ differ in at least 2 positions. But W is not a dominating set of G.

3. 2-Resolving Dominating Sets in the Join of Graphs

Definition 1. Let G be any nontrivial connected graph and $S \subseteq V(G)$. A set $S \subset V(G)$ is a 2-*locating set* of G if it satisfies the following conditions:

(i) $|(N_G(x) \triangle N_G(y)) \cap S| \geq 2$, for all $x, y \in V(G) \backslash S$ with $x \neq y$

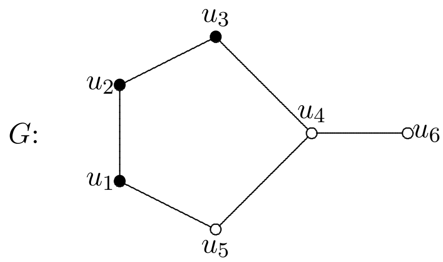

Figure 2: A graph G with $\dim_2(G) = 3$

(ii) $(N_G(v)\backslash N_G(w)) \cap S \neq \varnothing$ or $(N_G(w)\backslash N_G[v]\backslash S \neq \varnothing$, for all $v \in S$ and for all $w \in v(G)\backslash S$.

The 2-*locating number* of G, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of G. A 2-locating set of G of cardinality $ln_2(G)$ is referred to as an ln_2-set of G.

Definition 2. Let G be any nontrivial connected graph and $S \subseteq V(G)$. S is a $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *set* in G if S is 2-locating and $|N_G(y) \cap S| \leq |S| - 2$ ($|N_G(y) \cap S| \leq |S| - 1$, respectively), for all $y \in V(G)$. The $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *number* of G, denoted by $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively), is the smallest cardinality of a $(2,2)$-locating ($(2,1)$-locating, respectively) set in G. A $(2,2)$-locating ($(2,1)$-locating, respectively) set in G of cardinality $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively) is referred to as an $ln_{(2,2)}$-*set* ($ln_{(2,1)}$-*set*, respectively) in G.

Theorem 4. [4] Let G be a connected graph of order greater than 3 and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving set of $K_1 + G$ if and only if either $v \notin S$ and S is a $(2,2)$-locating set in G or $S = \{v\} \cup T$, where T is a $(2,1)$-locating set in G.

Theorem 5. [4] Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a 2-resolving set in $G+H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating set or S_G and S_H are $(2,1)$-locating sets.

Theorem 6. Let G be a connected non-trivial graph and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving dominating set in $K_1 + G$ if and only if it is a 2-resolving set in $K_1 + G$.

Proof. Let $S \subseteq V(K_1 + G)$ be a 2-resolving dominating set in $K_1 + G$. Then, S is a 2-resolving set in $K_1 + G$ by the definition of 2-resolving dominating set.

Conversely, if S is a 2-resolving set in $K_1 + G$, then by Theorem 4, S is a 2-locating set. Hence, S is a dominating set in $K_1 + G$. Thus, S is a 2-resolving dominating set in $K_1 + G$. $\square$

Corollary 1. $\gamma_{2R}(K_1 + G) = dim_2(K_1 + G)$.

Theorem 7. Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a 2-resolving dominating set in $G+H$ if and only if it is a 2-resolving set in $G+H$.

Proof. Let $S \subseteq V(G+H)$ be a 2-resolving dominating set in $G+H$. Then, S is a 2-resolving set in $G+H$.

Conversely, if S is a 2-resolving set in $G+H$, then by Theorem 5, S is a 2-locating set. Hence, S is a dominating set in $G+H$. Thus, S is a 2-resolving dominating set in $G+H$. □

Corollary 2. Let G and H be connected nontrivial graphs. Then,

$$\gamma_{2R}(G+H) = \dim_2(G+H).$$

The set consisting of the shaded vertices in Figure 3 is a 2-resolving dominating set of the join $P_5 + P_6$.

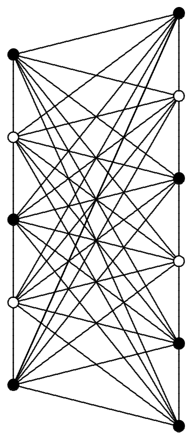

Figure 3: A graph $P_5 + P_6$ with $\gamma_{2R}(P_5 + P_6) = 7$

4. 2-Resolving Dominating Sets in the Corona of Graphs

Theorem 8. [4] Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a 2-resolving set of $G \circ H$ if and only if $S = A \cup B$, where $A \subseteq V(G)$ and

$$B = \bigcup \{S_v : S_v \text{ is a 2-resolving set of } H^v, \text{ for all } v \in V(G)\}.$$

Theorem 9. Let G and H be nontrivial connected graphs. Then $S \subseteq V(G \circ H)$ is a 2-resolving dominating set in $G \circ H$ if and only if $S = A \cup (\bigcup_{v \in V(G)} S_v)$, where $A \subseteq V(G)$, S_v is a 2-resolving set for each $v \in A$ and S_v is a 2-resolving dominating set for each $v \in V(G) \backslash A$.

Proof. Suppose S is a 2-resolving dominating set in $G \circ H$. Let $A = V(G) \cap S$ and $S_v = S \cap V(H^v)$ for all $v \in V(G)$. Then $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$. By Theorem 8, S_v is a 2-resolving set in H^v for each $v \in A$. If $v \in V(G) \backslash A$, then S_v is a 2-resolving dominating set in $G \circ H$.

Conversely, let $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ satisfying the given conditions. By Theorem 8, S is a 2-resolving set in $G \circ H$. Let $x \in V(G \circ H) \backslash S$ and let $v \in A$ such that $x \in V(v + H^v)$. Then $xv \in E(G \circ H)$. If $v \in V(G) \backslash A$, then there exists $y \in S_v$ such that $xy \in E(G \circ H)$. Therefore, S is a dominating set in $G \circ H$. Hence, S is a 2-resolving dominating set in $G \circ H$. □

Corollary 3. Let G and H be nontrivial connected graphs, where $|V(G)| = n$. Then $\gamma_{2R}(G \circ H) \leq \min \{n(1 + \dim_2(H)), n\gamma_{2R}(H)\}$.

The set consisting of the shaded vertices in Figure 4 is a 2-resolving dominating set of the corona $P_4 \circ C_5$.

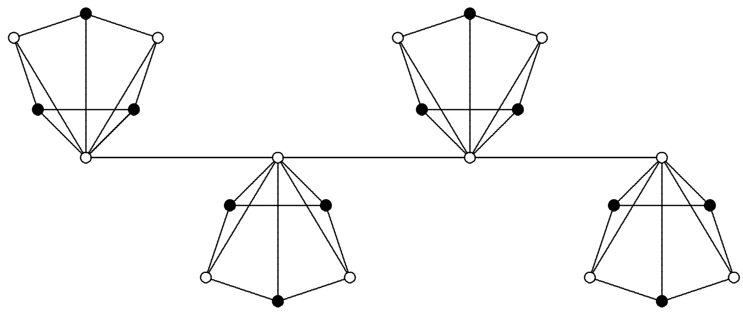

Figure 4: A graph $P_4 \circ C_5$ with $\gamma_{2R}(P_4 \circ C_5) = 12$

5. 2-Resolving Dominating Sets in the Lexicographic Product of Graphs

Definition 3. A vertex x is said to be 1-*equidistant* to y if $xy \in E(G)$ and $d_G(x, z) = d_G(y, z)$, for all $z \in V(G) \backslash \{x, y\}$ and it is 2-*equidistant* to y if $d_G(x, y) = 2$ and $d_G(x, z) = d_G(w, z)$, for all $z \in V(G) \backslash \{x, w\}$. A vertex is called a *free-vertex* in G if it is neither 1-equidistant nor 2-equidistant to any vertex. The set containing all 1-equidistant, 2-equidistant, and free-vertices in G are denoted by $EQ_1(G)$, $EQ_2(G)$ and $fr(G)$, respectively.

Theorem 10. Let G and H be non-trivial connected graphs. Then $W = \bigcup_{x \in S} \left[\{x\} \times T_x \right]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving set in $G[H]$ if and only if

(*i*) $S = V(G)$

(ii) T_x is a 2-locating set in H for every $x \in V(G)$;

(iii) T_x and T_y are $(2,1)$-locating sets or one of T_x and T_y is a $(2,2)$-locating set in H whenever $x, y \in EQ_1(G)$; and

(iv) T_x and T_y are (2-locating) dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in EQ_2(G)$.

Proof. Suppose $W = \bigcup_{x\in S} \left[\, \{x\} \times T_x \right]$ is a 2-resolving set in $G[H]$. Suppose there exists $x \in V(G)\backslash S$. Pick $a, b \in V(H)$, where $a \neq b$. Then $(x,a), (x,b) \notin W$ and $(x,a) \neq (x,b)$. Since $x \notin S$ and $d_{G[H]}((x,a),(y,p)) = d_{G[H]}((x,b),(y,p))$ for all $y \in V(G)\setminus\{x\}$ and for all $p \in V(H)$, $r_{G[H]}((x,a)/W) = r_{G[H]}((x,b)/W)$. This implies that W is not a 2-resolving set of $G[H]$, a contradiction to the assumption on W. Therefore, $S = V(G)$.

To prove (ii), let $x \in V(G)$ and $p, q \in V(H)$ where $p \neq q$. Then $(x,p) \neq (x,q)$. If $p, q \notin T_x$ or $[p \in T_x$ and $q \notin T_x]$, then $(x,p), (x,q) \notin W$ or $[(x,p) \in W$ and $(x,q) \notin W]$. Since W is a 2-resolving set in $G[H]$, $r_{G[H]}((x,p)/W)$ and $r_{G[H]}((x,q)/W)$ differ in at least 2 positions. Hence, by Theorem 1(iii) and Definition 1, T_x is a 2-locating set in H. Thus, (ii) follows.

To prove (iii), let x and y be adjacent vertices of G with $d_G(x,z) = d_G(y,z)$, for all $z \in V(G)\setminus\{x,y\}$. Let $a, b \in V(H)$, $a \neq b$. Since W is 2-resolving, $r_{G[H]}((x,a)/W)$ and $r_{G[H]}((y,b)/W)$ differ in at least 2 positions. By assumption, it is not possible that $N_H(a) \cap T_x = T_x$ and $N_H(b) \cap T_y = T_y$. If T_x or T_y is $(2,2)$-locating, then we are done. Otherwise, T_x and T_y are $(2,1)$-locating.

To prove (iv), let $x, y \in V(G)$ where $d_G(x,y) = 2$ and $d_G(x,z) = d_G(y,z)$, for all $z \in V(G)\setminus\{x,y\}$. Let $a, b \in V(H)$, $a \neq b$. Suppose one of T_x and T_y, say T_x is not a dominating set in H. Pick $a \in V(H)\backslash N_H[T_x]$ and let $b \in V(H)\backslash T_y$. Since $d_{G[H]}((x,a),(z,q)) = 2$, for all (z,q), it follows that $|N_H(b) \cap T_y| \geq 2$, i.e., T_y is a 2-dominating set.

Conversely, suppose (i),(ii), (iii) and (iv) hold. Let $(x,a), (y,b) \in V(G[H])$, $(x,a) \neq (y,b)$. Consider the following cases.

Case 1. $x = y$

Supppose $(x,a), (y,b) \notin W$. Then $a \neq b$ and $a, b \notin T_x = T_y$. By (ii), T_x is a 2-locating set. Hence, by Theore 1(iii) and by Definition 1, $r_{G[H]}((x,a)/W)$ and $r_{G[H]}((y,b)/W)$ differ in at least two positions. On the other hand, if $(x,a) \in W$, $(y,b) \notin W$, then $a \in T_x$, $b \notin T_y$. Using similar argument as in above, $r_{G[H]}((x,a)/W)$ and $r_{G[H]}((y,b)/W)$ differ in at least 2 positions.

Case 2. $x \neq y$.

Subcase 2.1 $xy \in E(G)$.

If $d_G(x,z) \neq d_G(y,z)$ for some $z \in V(G)\setminus\{x,y\}$, then $r_{G[H]}((x,a)/W)$ and $r_{G[H]}((y,b)/W)$ differ in at least 2 positions since H is nontrivial. Suppose $d_G(x,z) = d_G(y,z)$, for all $z \in V(G)\setminus\{x,y\}$. Then by (iii), T_x and T_y are $(2,1)$-locating sets in H or one of T_x and T_y is a $(2,2)$-locating set in H. Hence, by Definition 1, $r_{G[H]}((x,a)/W)$ and $r_{G[H]}((y,b)/W)$ differ in at least 2 positions.

Subcase 2.2 $xy \notin E(G)$

If $d_G(x, y) > 2$, then it follows that $r_{G[H]}((x, a)/W)$ and $r_{G[H]}((y, b)/W)$ differ in at least 2 positions. If $d_G(x, y) = 2$ and $d_G(x, z) \neq d_G(y, z)$ for some $z \in V(G) \setminus \{x, y\}$, then it follows that $r_{G[H]}((x, a)/W)$ and $r_{G[H]}((y, b)/W)$ differ in at least 2 positions. Suppose $d_G(x, y) = 2$ and $d_G(x, z) = d_G(y, z)$, for all $z \in V(G) \setminus \{x, y\}$. Suppose $(x, a), (y, b) \notin W$. Then $a \notin T_x$ and $y \notin T_y$. If T_x and T_y are both dominating, then $r_{G[H]}((x, a)/W)$ and $r_{G[H]}((y, b)/W)$ differ in at least 2 positions. If one, say T_y, is a 2-dominating set, then $r_{G[H]}((x, a)/W)$ and $r_{G[H]}((y, b)/W)$ differ in at least 2 positions. Similarly, if $(x, a) \in W$, $(y, b) \notin W$, then $r_{G[H]}((x, a)/W)$ and $r_{G[H]}((y, b)/W)$ differ in at least 2 positions.

Accordingly, W is a 2-resolving set of $G[H]$. □

Theorem 11. Let G and H be non-trivial connected graphs. Then $W = \bigcup_{x \in S} \big[\{x\} \times T_x\big]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving dominating set in $G[H]$ if and only if it is a 2-resolving set in $G[H]$.

Proof. The proof is similar to that of Theorem 10. □

Corollary 4. Let G and H be nontrivial connected graphs such that G is not free-equidistant. Then,

$$\gamma_{2R}(G[H]) = \dim_2(G[H]).$$

The set consisting of the shaded vertices in Figure 5 is a 2-resolving dominating set of the lexicographic product $P_4[P_3]$.

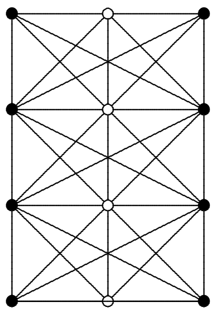

Figure 5: A graph $P_4[P_3]$ with $\gamma_{2R} P_4[P_3] = 8$

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