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Restrained 2-Resolving Sets in the Join, Corona and Lexicographic Product of two Graphs

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Abstract. Let G be a connected graph. An ordered set of vertices $\{v_1, ..., v_l\}$ is a 2-resolving set for G if, for any distinct vertices $u, w \in V(G)$, the lists of distances $(d_G(u, v_1), ..., d_G(u, v_l))$ and $(d_G(w, v_1), ..., d_G(w, v_l))$ differ in at least 2 positions. A set $S \subseteq V(G)$ is a *restrained* 2*-resolving set* in G if S is a 2-resolving set in G and $S = V(G)$ or $\langle V(G)\backslash S\rangle$ has no isolated vertex. The *restrained* 2*-resolving number* of G, denoted by $r\dim_2(G)$, is the smallest cardinality of a restrained 2-resolving set in G. A restrained 2-resolving set of cardinality $r\dim_2(G)$ is then referred to as an $rdim_2$*-set* in G. This study deals with the concept of restrained 2-resolving set of a graph. It characterizes the restrained 2-resolving set in the join, corona and lexicographic product of two graphs and determine the bounds or exact values of the 2-resolving dominating number of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: 2-resolving set, restrained 2-resolving set, restrained 2-resolving number, join, corona, lexicographic product of two graphs

1. Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater [8], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [4] where metric generators were called resolving sets.

Bailey and Yero in [1] demonstrated a construction of error-correcting codes from graphs by means of k-resolving sets, and present a decoding algorithm which makes use of

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covering designs. In [6], the explicit interpretation for F-index of different forms of corona products involving Zagreb indices, graph size and order are obtained.

The distance between two vertices u and v of a graph is the length of a shortest path between u and v, and we denote this by $d_G(u, v)$. In recent years, much attention has been paid to the *metric dimension* of graphs: this is the smallest size of a subset of vertices (called a *resolving set*) with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex and is denoted by $\dim(G)$.

According to the paper of Saenpholphat et al. [7], for an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v in G, the k-vector (ordered k-tuple)

$$r(v/W) = (d_G(v, w_1), d_G(v, w_2), ..., d_G(v, w_k))$$

is referred to as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representation with respect to W. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r(v/W) : v \in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality is called a *minimum resolving set* or a *basis*, and the cardinality of a basis for G is the *dimension* $\dim(G)$ of G.

In the paper of Rara and Cabaro [5], an ordered set of vertices $W = \{w_1, ..., w_l\}$ is a 2-*resolving set* for G if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r(u/W)$ and $r(v/W)$ of u and v, respectively differ in at least 2 positions. Then W is said to be a 2-resolving set for G. If G has a 2-resolving set, the minimum cardinality $\dim_2(G)$ is called the 2-*metric dimension* of G. If $k = 2$ is the largest integer for which G has a 2-resolving set, then we say that G is a 2-*metric dimensional* graph.

In this paper, the concept of restrained 2-resolving set in the join, corona and lexicographic product of two graphs is discussed.

2. Preliminary Results

In this study, we consider finite, simple and connected undirected graphs. For basic graph-theoretic concepts, we refer readers to [3].

Remark 1. Let G be a connected graph. Then every restrained 2-resolving set in G is 2-resolving. Hence, $\dim_2(G) \leq r\dim_2(G)$.

Proposition 1. [2] Let G be a connected graph of order $n \geq 2$. Then $\dim_2(G) = 2$ if and only if $G \cong P_n$.

Proposition 2. $\dim_2(K_n) = n$ for $n \geq 2$.

Proposition 3. Let G be any connected graph of order $n \geq 2$.

i. $r\dim_2(G) = 2$ if and only if $G \cong P_n$, $n \neq 3$.

ii. $r\dim_2(K_n) = n$.

Proof. i. Suppose $r\dim_2(G) = 2$. By Remark 1, $\dim_2(G) = 2$. Hence, by Proposition 1, $G = P_n$. Since $r\dim_2(P_3) = 3$, $G = P_n$ except $n = 3$.

Conversely, if $G = P_n = [v_1, v_2, ..., v_n]$, then $S = \{v_1, v_n\}$ is a restrained 2-resolving set of G. Hence, $r\dim_2(G) = 2$.

ii. By Proposition 2, $S = V(K_n)$ is the only 2-resolving set of K_n. Thus, $r\dim_2(K_n) = n$. □

3. Restrained 2-Resolving Sets in the Join of Graphs

Definition 1. Let G be any nontrivial connected graph and $S \subseteq V(G)$. A set $S \subset V(G)$ is a 2-*locating set* of G if it satisfies the following conditions:

(i) $|(N_G(x) \triangle N_G(y)) \cap S| \geq 2$, for all $x, y \in V(G) \backslash S$ with $x \neq y$

(ii) $(N_G(v) \backslash N_G(w)) \cap S \neq \varnothing$ or $(N_G(w) \backslash N_G[v] \backslash S \neq \varnothing$, for all $v \in S$ and for all $w \in v(G) \backslash S$.

The 2-*locating number* of G, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of G. A 2-locating set of G of cardinality $ln_2(G)$ is referred to as an ln_2-set of G.

Definition 2. Let G be any nontrivial connected graph and $S \subseteq V(G)$. S is a $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *set* in G if S is 2-locating and $|N_G(y) \cap S| \leq |S| - 2$ ($|N_G(y) \cap S| \leq |S| - 1$, respectively), for all $y \in V(G)$. The $(2,2)$-*locating* ($(2,1)$-*locating*, respectively) *number* of G, denoted by $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively), is the smallest cardinality of a $(2,2)$-locating ($(2,1)$-locating, respectively) set in G. A $(2,2)$-locating ($(2,1)$-locating, respectively) set in G of cardinality $ln_{(2,2)}(G)$ ($ln_{(2,1)}(G)$, respectively) is referred to as an $ln_{(2,2)}$-*set* ($ln_{(2,1)}$-*set*, respectively) in G.

Theorem 1. Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a 2-resolving set in $G + H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating set or S_G and S_H are $(2,1)$-locating sets.

Theorem 2. Let G and H be nontrivial connected graphs. A set $S \subseteq V(G + H)$ is a restrained 2-resolving set in $G + H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H, respectively where S_G or S_H is a $(2,2)$-locating set or S_G and S_H are $(2,1)$-locating sets and one of the following holds:

(i) $S_G = V(G)$ and S_H is a restrained 2-locating set in H;

(ii) $S_H = V(H)$ and S_G is a restrained 2-locating set in G;

(iii) $S_G \neq V(G)$ and $S_H \neq V(H)$.

Proof. Let $S \subseteq V(G+H)$ be a restrained 2-resolving set in $G+H$. Then by Theorem 1, $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating set or S_G and S_H are $(2,1)$-locating sets. To show that (i), (ii), and (iii) hold we consider the following cases:

Case 1. $S_G = V(G)$.

Suppose that $S_H \neq V(H)$. Since $\langle V(G+H)\backslash S\rangle = \langle V(H)\backslash S_H\rangle$ and $\langle V(G+H)\backslash S\rangle$ has no isolated vertex, it follows that $\langle V(H)\backslash S_H\rangle$ has no isolated vertex. Thus, S_H is a restrained 2-locating set in H. Hence, (i) holds.

Case 2. Suppose that $S_G \neq V(G)$.

If $S_H \neq V(H)$, then (iii) holds. Suppose that $S_H = V(H)$. Then $\langle V(G+H)\backslash S\rangle = \langle V(G)\backslash S_G\rangle$ has no isolated vertex. Hence, S_G is a restrained 2-locating set in G. Thus, (ii) holds.

For the converse, suppose that S_G and S_H are 2-locating sets in G and H, respectively, where S_G or S_H is a $(2,2)$-locating set or S_G and S_H are $(2,1)$-locating sets. Then by Theorem 1, S is a 2-resolving set in $G+H$. Suppose that $S_G = V(G)$. If $S_H = V(H)$, then $S = V(G+H)$ is a restrained 2-resolving set in $G+H$. If $S_H \neq V(H)$, then by (i) $\langle V(H)\backslash S_H\rangle$ has no isolated vertex. Since $V(G+H)\backslash S = V(H)\backslash S_H$, S is a restrained 2-resolving set in $G+H$. Similarly, if (ii) holds, then S is a restrained 2-resolving set in $G+H$. Finally, suppose that $S_G \neq V(G)$ and $S_H \neq V(H)$. Then clearly, S is a restrained 2-resolving set in $G+H$. □

Corollary 1. Let G and H be connected non-trivial graphs of order m and n, respectively. Then

$$r\dim_2(G+H) = \begin{cases} m+n, \text{ if } rln_2(G) = m \text{ and } rln_2(H) = n \\ \min\{rln_{(2,2)}(G) + rln_2(H), rln_2(G) + rln_{(2,2)}(H), \\ rln_{(2,1)}(G) + rln_{(2,1)}(H)\}, \text{ otherwise} \end{cases}$$

The set consisting of the shaded vertices in Figure 1 is a restrained 2-resolving set of the join $P_5 + P_6$.

Theorem 3. Let G be a connected non-trivial graph and let $K_1 = \{v\}$. Then $S \subseteq V(K_1+G)$ is a restrained 2-resolving set of K_1+G if and only if either $v \notin S$ and S is a $(2,2)$-locating set in G with $V(G) \neq S$ or $S = \{v\} \cup T$, where T is a restrained $(2,1)$-locating set in G.

Corollary 2. Let G be a connected nontrivial graph of order m. Then

$$r\dim_2(K_1+G) = \begin{cases} 1+m, \text{ if } ln_{(2,2)}(G) = m \text{ and } rln_{(2,1)}(G) = m \\ \min\left\{ln_{(2,2)}(G), rln_{(2,1)}(G)+1\right\}, \text{ otherwise.} \end{cases}$$

$P_5 + P_6$:

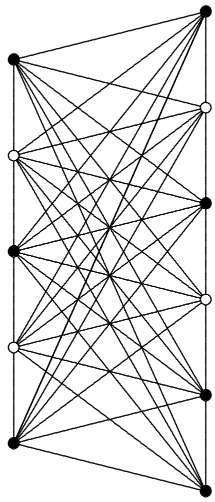

Figure 1: A graph $P_5 + P_6$ with $r\dim_2(P_5 + P_6) = 7$

4. Restrained 2-Resolving Sets in the Corona of Graphs

Theorem 4. Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a 2-resolving set of $G \circ H$ if and only if $S = A \cup B$, where $A \subseteq V(G)$ and

$$B = \bigcup \{S_v : S_v \text{ is a 2-resolving set of } H^v, \text{ for all } v \in V(G)\}.$$

Theorem 5. Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a restrained 2-resolving set in $G \circ H$ if and only if $S = A \cup (\bigcup_{v \in V(G)} S_v)$ satisfying the following conditions.

(i) $A \subseteq V(G)$

(ii) S_v is a 2-resolving set for each $v \in V(G) \backslash A$

(iii) S_v is a restrained 2-resolving set for each $v \in A$

(iv) $w \in N_G(V(G) \backslash A)$ for each $w \in V(G) \backslash A$ with $S_w = V(H^w)$.

Proof. Suppose S is a restrained 2-resolving set in $G \circ H$. Let $A = V(G) \cap S$ and $S_v = S \cap V(H^v)$ for all $v \in V(G)$. Then $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for each $v \in V(G)$. By Theorem 4, S_v is a 2-resolving set in H^v for every $v \in V(G)$. Since S is a restrained 2-resolving set in $G \circ H$, $S = V(G \circ H)$ or $\langle V(G \circ H) \backslash S \rangle$ has no isolated vertex. Let $v \in A$. If $S_v = V(H^v)$, then S_v is a restrained 2-resolving set of H^v. Suppose $S_v \neq V(H^v)$. Since $v \in A$, $\langle V(H^v) \backslash S_v \rangle$ must have no isolated vertex. Hence, S_v is a restrained 2-resolving set in H^v. Next, let $w \in V(G) \backslash A$ with $S_w = V(H^w)$. Since $V(G \circ H) \backslash S$ has no isolated vertex, $w \in N_G(V(G) \backslash A)$. Hence, (i)-(iv) hold.

Conversely, let $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$, where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for each $v \in V(G)$ satisfying (i)-(iv). By Theorem 4, S is a 2-resolving set in $G \circ H$. Moreover, because of (i)-(iv), S is a restrained 2-resolving set in $G \circ H$. $\square$

Corollary 3. Let G and H be nontrivial connected graphs, where $|V(G)| = n$. Then $r\dim_2(G \circ H) = n \cdot \dim_2(H)$.

The set consisting of the shaded vertices in Figure 2 is a restrained 2-resolving set of the corona $P_4 \circ C_5$.

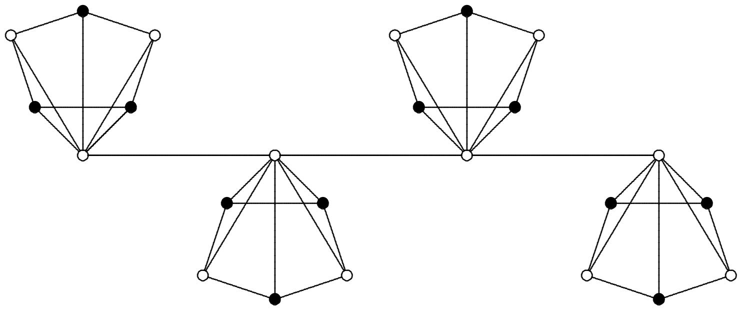

Figure 2: A graph $P_4 \circ C_5$ with $r\text{dim}_2(P_4 \circ C_5) = 12$

5. Restrained 2-Resolving Sets in the Lexicographic Product of Graphs

Definition 3. A vertex x is said to be 1-*equidistant* to y if $xy \in E(G)$ and $d_G(x,z) = d_G(y,z)$, for all $z \in V(G) \setminus \{x,y\}$ and it is 2-*equidistant* to y if $d_G(x,y) = 2$ and $d_G(x,z) = d_G(w,z)$, for all $z \in V(G) \setminus \{x,w\}$. A vertex is called a *free-vertex* in G if it is neither 1-equidistant nor 2-equidistant to any vertex. The set containing all 1-equidistant, 2-equidistant, and free-vertices in G are denoted by $EQ_1(G)$, $EQ_2(G)$ and $fr(G)$, respectively.

Theorem 6. Let G and H be non-trivial connected graphs. Then $W = \bigcup_{x \in S} \left[\{x\} \times T_x \right]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a 2-resolving set in $G[H]$ if and only if

(i) $S = V(G)$

(ii) T_x is a 2-locating set in H for every $x \in V(G)$;

(iii) T_x and T_y are $(2,1)$-locating sets or one of T_x and T_y is a $(2,2)$-locating set in H whenever $x, y \in EQ_1(G)$; and

(iv) T_x and T_y are (2-locating) dominating sets in H or one of T_x and T_y is a 2-dominating set whenever $x, y \in EQ_2(G)$.

Theorem 7. Let G and H be non-trivial connected graphs. Then $W = \bigcup_{x\in S}\big[\{x\} \times T_x\big]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a restrained 2-resolving set in $G[H]$ if and only if

(i) $S = V(G)$

(ii) T_x is a 2-locating set in H for all $x \in V(G)$;

(iii) T_x is a restrained 2-locating set for each x with $T_y = V(H)$, for all $y \in N_G(x)$;

(iv) T_x and T_y are $(2,1)$-locating sets or one of T_x and T_y is a $(2,2)$-locating set in H whenever $x, y \in EQ_1(G)$; and

(v) T_x and T_y are (2-locating) dominating sets in H or if one of T_x and T_y, say T_x is not dominating, then T_y is 2-dominating whenever $x, y \in EQ_2(G)$.

Proof. Suppose $W = \bigcup_{x\in S}\big[\{x\}\times T_x\big]$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ is a restrained 2-resolving set in $G[H]$. Then by Theorem 6, (i), (ii), (iv) and (v) hold. Now, let $x \in V(G)$ with $T_y = V(H)$, for all $y \in N_G(x)$. Suppose that T_x is not restrained 2-locating set. Then $\langle V(H)\backslash T_x\rangle$ has an isolated vertex, say u. Thus, (x, u) is an isolated vertex in $\langle V(G[H])\backslash W\rangle$, contrary to the assumption that W is a restrained 2-resolving set in $G[H]$. Hence, T_x is a restrained 2-locating set.

For the converse, suppose that (i), (ii), (iii), and (iv) hold. Then by Theorem 6, $W = \bigcup_{x\in S}\big[\{x\}\times T_x\big]$ is a 2-resolving set in $G[H]$. If $W = V(G[H])$, then W is a restrained 2-resolving set in $G[H]$. Suppose that $W \neq V(G[H])$. Let $(x, a) \in V(G[H])\backslash W$. If $T_y \neq V(H)$, for all $y \in N_G(x)$, then $\langle V(G[H])\backslash W\rangle$ has no isolated vertex. If $T_y = V(H)$, for some $y \in N_G(x)$, then by (iii), T_x is a restrained 2-locating set. Thus, $V(H)\backslash T_x$ has no isolated vertex. Hence, $\langle V(G[H])\backslash W\rangle$ has no isolated vertex. Therefore, W is a restrained 2-resolving set in $G[H]$. □

The following corollaries are the direct consequences of Theorem 7.

Corollary 4. Let G and H be nontrivial connected graphs such that G is not free-equidistant. Then,

$$r\dim_2(G[H]) \leq n \cdot ln_{(2,1)}(H) + m \cdot \gamma_{2L}(H) + p \cdot rln_2(H),$$

where $n + m + p = |V(G)|$ with $|EQ_1(G)| = n$, $|EQ_2(G)| = m$ and $|fr(G)| = p$.

The following result follows from Theorem 7.

Corollary 5. Let G and H be non-trivial connected graphs such that G is free-equidistant. Then

$$r\dim_2(G[H]) = \begin{cases} |V(G)| \cdot ln_2(H), & \text{if } ln_2(H) \neq |V(H)| \\ |V(G)| \cdot rln_2(H), & \text{otherwise.} \end{cases}$$

The set consisting of the shaded vertices in Figure 3 is a restrained 2-resolving set of the lexicographic product $P_4[P_3]$.

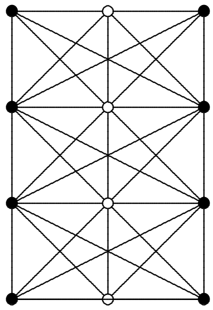

Figure 3: A graph $P_4[P_3]$ with $r\text{dim}_2\ P_4[P_3] = 8$

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