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Annihilator Hyperideals in Strong Bounded Dual Distributive Meet-hyperlattice

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Abstract. In this paper, we study the properties of annihilator hyperideals in the class of strong bounded dual distributive meet-hyperlattice. We show that the set of all closed hyperideals forms a Boolean algebra. We introduce the concept of homomorphism, which preserves the annihilator hyperideal. Suitable conditions for preserving annihilator hyperideals are obtained. Representation and characterization theorems of annihilator hyperideals in sub-meet-hyperlattice and product meet-hyperlattice are proved.

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Key Words and Phrases: Annihilator,Boolean algebra, Homomorphism, Hyperideal, Hyperlattice

1. Introduction

The approach to the theory of hyberlattices was first made by M. Konstantinidou and J. Mittas in 1977, [12]. Modular distributive and complemented classes of hyperlattices were studied by M. Konstantinidou in [10] and [9]. Ideals of hyperlattices were introduced by Rahnamai-Barghi in [15], where he considered the prime ideal theorem for distributive hyperlattices. M. Amiri Bideshki and et al. defined the notions of hyperideals and hyperfilters in strong meet-hyperlattices in [11]. They also introduced the concept of annihilator hyperideals. Annihilators have been started in ring theory over many classes of rings, as examples refer to [19] and [8]. In that sense, the annihilator of a certain set A means the set of killer elements that make each element of A tends to zero by multiplication operation. The mention of annihilators in lattices was first introduced by M.Mandelker in 1970, [13]. He defined the relative annihilator as a generalization of relative pseudocomplementation. M.Mandelker introduced the relation between prime ideal conditions and annihilator conditions on distributive lattices. W. H. Cornish investigated the annihilator

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properties of distributive lattices in [18]. He defined the annihilator of a set A as a set of all elements whose elements tend to zero by the meet operation. The main result- that Cornish proved- is the normality of lattice equivalents to any two elements with zero meeting have a comaximal annihilator of their principal ideals. Moreover, the normality of the lattice equivalents to the annihilator of the principle ideal of the meeting of any two elements equals the joining of annihilators of their principle ideals. In [1], B. A. Davey and Nieminen studied the annihilators in the class of modular lattices. They proved that the weakly atomic modular lattice has necessary and sufficient conditions for its annihilators. As a generalization of lattices Halaš [6] studied annihilators in ordered sets. This was followed by a lot of studies and research on the concept of annihilator in many algebraic structures and classes, for instance: almost distributive lattices [3],0-almost distributive lattices [4], distributive dual weakly complemented lattice [17], BCK-algebras [7], standard QBCC algebras [14], C-algebra[16], and many other classes.

In this paper, the properties of annihilator hyperideals of strong bounded dual distributive meet-hyperlattice are investigated. Main terminologies and properties are recalled in Section 2. Important properties of annihilator hyperideals are proved. Moreover, the structure of the set of all closed hyperideals is investigated in Section 3. In Section 4, the conditions of homomorphism map to preserve Annihilator hyperideals are discussed. The proof of the preservation of annihilator hyperideals under the effects of these conditions is given. Finally, in section 5, we show that annihilator hyperideals are inherited for sub-meet-hyperlattice and product meet-hyperlattice.

2. Backgrounds

We recall here the basic terminologies and concepts of hyperlattices. The reader must be familiar with lattice theory. For more details about lattice theory, see [2] and [5].

Definition 1. *[11] Let L be a nonempty set, $\mathcal{P}^*(L)$ is the set of all nonempty subsets of L, $\bar{\wedge} : L \times L \to \mathcal{P}^*(L)$ is a hyperoperation and $\vee : L \to L$ is a binary operation. Then $\mathfrak{L} =< L; \bar{\wedge}, \vee >$ is called a meet-hyperlattice if:*

H1) $a \in a \,\bar{\wedge}\, a$, $a = a \vee a$;

H2) $a \,\bar{\wedge}\, b = b \,\bar{\wedge}\, a$, $a \vee b = b \vee a$;

H3) $a \,\bar{\wedge}\, (b \,\bar{\wedge}\, c) = (a \,\bar{\wedge}\, b) \bar{\wedge}\, c$, $a \vee (b \vee c) = (a \vee b) \vee c$;

H4) $a \in (a \,\bar{\wedge}\, (a \vee b)) \cap (a \vee (a \,\bar{\wedge}\, b))$, *for all* $a, b, c \in L$.

The meet-hyperlatticce $\mathfrak{L}$ is called strong, if it satisfies that:

$$\text{if } a \in a \,\bar{\wedge}\, b \quad \text{then} \quad a \vee b = b \,, \quad \text{for all } a, b \in \mathfrak{L}.$$

Consider an order relation $\leq$ on $\mathfrak{L}$ as: $a \leq b$ iff $a \vee b = b$, for all $a, b \in \mathfrak{L}$. Accordingly, meet-hyperlattice $\mathfrak{L}$ is bounded if there exist two elements $0, 1 \in \mathfrak{L}$ such that $0 \leq a \leq 1$, for all $a \in \mathfrak{L}$.

For subsets $A, B \subseteq L$:

$$A \bar{\curlywedge} B = \cup\{a \bar{\curlywedge} b : a \in A, b \in B\},$$
$$A \vee B = \{a \vee b : a \in A, b \in B\}.$$

Proposition 1. *[11] Let $\mathfrak{L}$ be a bounded strong meet-hyperlattice. Then the following conditions hold:*

(i) If $a, b \neq 1$ and $a \vee b = 1$, then $a, b \notin a \bar{\curlywedge} b$;

(ii) If $a \bar{\curlywedge} b = L$ or $a, b \in a \bar{\curlywedge} b$, then $a = b$;

(iii) For all $a \in \mathfrak{L}$: $a \in a \bar{\curlywedge} 1$ and $0 \in a \bar{\curlywedge} 0$.

The meet-hyperlattice $\mathfrak{L}$ is distributive, if $a \vee (b \bar{\curlywedge} c) = (a \vee b) \bar{\curlywedge} (a \vee c)$, for all $a, b, c \in \mathfrak{L}$. Dually, $\mathfrak{L}$ is dual distributive if $a \bar{\curlywedge} (b \vee c) = (a \bar{\curlywedge} b) \vee (a \bar{\curlywedge} c)$.

Definition 2. *[11] Let I be a nonempty subset of a strong meet-hyperlattice $\mathfrak{L}$. I is called a hyperideal if:*

i) If $a, b \in I$, then $a \vee b \in I$;

ii) If $a \in I$ and $b \in \mathfrak{L}$, such that $b \leq a$, then $b \in I$.

The set of all hyperideals of strong bounded meet-hyperlattice $\mathfrak{L}$ is denoted by $I(\mathfrak{L})$. The meet operation on hyprideals is the usual sets intersection $\cap$ and the join operation is defined as $I \vee J = \{x \in \mathfrak{L} : x \leq a \vee b, a \in I, b \in J\}$. Then, we get the following theorem:

Theorem 1. *[11] The structure $(I(\mathfrak{L}); \cap, \vee, 0, L)$ forms a bounded distributive lattice.*

Definition 3. *[11] Let $\mathcal{L} = (L, \bar{\curlywedge}, \vee)$ be a strong bounded dual distributive meet-hyperlattice and $A \subseteq \mathcal{L}$. Then the hyperideal*

$$A^r = \{x \in L : 0 \in x \bar{\curlywedge} a, \text{ forall } a \in A\},$$

is called annihilator hyperideal or for brevity (annihilator).

When A is a singleton subset $\{a\}$ of $\mathcal{L}$, its annihilator is defined as:

$$a^r = \{x \in L : 0 \in a \bar{\curlywedge} x\}.$$

Proposition 2. *[11]*
Let $\mathcal{L} = (L, \bar{\curlywedge}, \vee)$ be a strong bounded dual distributive meet-hyperlattice. Then:

(i) $0^r = L$;

(ii) If $a, b \in L$ and $a \leq b$, then $b^r \subseteq a^r$;

(iii) $A^r = \cap\{a^r : \quad a \in A\}$;

(iv) $A^r \cap B^r = (A \cup B)^r$.

Notice that (ii) can be generalized to any subsets A and B of $\mathfrak{L}$ *as:*

$$\text{If} \;\; A, B \subseteq \mathcal{L} \;\; \text{and} \;\; A \subseteq B \;\; \text{then} \;\; B^r \subseteq A^r.$$

In all of the following, the meet-hyperlattice $\mathfrak{L} =< L; \bar{\wedge}, \vee, 0, 1 >$ is considered that strong bounded and dual distributive.

3. Annihilator Hyperideals

The core of this section is that we prove the main theory, which states that the set of all closed hyperideals forms a Boolean algebra.

Theorem 2. *Let* I, J *be subsets of meet-hyperlattice* $\mathfrak{L}$*. Then*

(i) $I \subseteq I^{rr}$;

(ii) $I^{rrr} = I^r$;

(iii) $I \cap J \subseteq (I^r \vee J^r)^r$;

(iv) $I^r \cap J^r = (I \vee J)^r$;

(v) $(I \cap J)^{rr} \subseteq I^{rr} \cap J^{rr}$;

(vi) $1^r \subseteq I^r$ *for all* $I \subseteq L$;

(vii) $I^r \cap I^{rr} = 1^r$;

(viii) *If* $I \subseteq J^r$ *then* $I \cap J^{rr} = 1^r$;

(ix) $1^{rr} = L$.

Proof.

(i) Since

$$\begin{aligned} I \cap I^{rr} &= I \cap \{x \in L : 0 \in x \bar{\wedge} i \;\text{ for all } i \in I^r\} \\ &= \{x \in I : 0 \in x \bar{\wedge} i \text{ for all } i \in I^r\} \\ &= I. \end{aligned}$$

Then, $I \subseteq I^{rr}$.

(ii) From (i) we have $I \subseteq I^{rr}$ and $I^r \subseteq I^{rrr}$. On the other side, we have $I^{rrr} \subseteq I^r$ from Proposition 2. Hence the equality is satisfied.

(iii) Let $x \in I \cap J$ and $y \in I^r \vee J^r$. Then $y \leq i \vee j$ for some $i \in I^r$ and $j \in J^r$ such that

$$\begin{aligned} x\bar{\wedge}y &\subseteq x\bar{\wedge}(i \vee j) \\ &= (x\bar{\wedge}i) \vee (x\bar{\wedge}j). \end{aligned}$$

Since $x \in I$, $x \in J$ then $0 \in (x\bar{\wedge}i)$ and $0 \in (x\bar{\wedge}j)$. Then $0 \in x\bar{\wedge}(i \vee j)$. From Proposition 2, we get $0 \in (x\bar{\wedge}y)$ .Therefore $x \in (I^r \vee J^r)^r$.

(iv) Since $I, J \subseteq I \vee J$, then $I^r, J^r \supseteq (I \vee J)^r$. So, $(I \vee J)^r \subseteq I^r \cap J^r$. Conversely, let $y \in I^r \cap J^r$, thus $0 \in y\bar{\wedge}i$ for all $i \in I$ and $0 \in y\bar{\wedge}j$ for all $j \in I$ which indicates that

$$0 \in (y\bar{\wedge}i) \vee (y\bar{\wedge}j) = y\bar{\wedge}(i \vee j).$$

So for all $a \leq i \vee j$ meeting each side by y to get $y\bar{\wedge}a \subseteq y\bar{\wedge}(i \vee j)$. It implies that $0 \in y\bar{\wedge}a$ and then $y \in (I \vee J)^r$. Therefore $I^r \cap J^r \subseteq (I \vee J)^r$.

(v) Since

$$I \cap J \subseteq I \quad \text{and} \quad I \cap J \subseteq J$$

then

$$(I \cap J)^{rr} \subseteq I^{rr} \quad \text{and} \quad (I \cap J)^{rr} \subseteq J^{rr}.$$

Thus

$$(I \cap J)^{rr} \subseteq I^{rr} \cap J^{rr}.$$

(vi) Suppose $x \in 1^r$ and I is a nonempty subset of $\mathfrak{L}$, then $0 \in x\bar{\wedge}1$. From Proposition 2, we get $0 \in x\bar{\wedge}y$ for all $y \in I$. Therefore $x \in I^r$.

(vii) From (iv) in Proposition 2, we get $I^r \cap I^{rr} = (I \cup I^r)^r$. Since $a \leq 1$ for all $a \in I \cup I^r$, then $1^r \subseteq a^r$. It implies that $1^r \subseteq \cap\{a^r : a \in I \cup I^r\}$. Consequently, $1^r \subseteq (I \cup I^r)^r = I^r \cap I^{rr}$. The opposite direction is taken immediately from (vi).Therefore, $I^r \cap I^{rr} = 1^r$.

(viii) Let $I \subseteq J^r$. Intersect both sides by J^{rr}, we get $I \cap J^{rr} \subseteq J^r \cap J^{rr} = 1^r$. But, from (vi) we have $1^r \subseteq I \cap J^{rr}$.

(ix)

$$\begin{aligned} 1^{rr} &= \{x \in L : \ 0 \in x\bar{\wedge}a, \ \ a \in 1^r\} \\ &= \{x \in L : \ 0 \in x\bar{\wedge}a, \ \ 0 \in a\bar{\wedge}1\} \\ &= \{x \in L : \ 0 \in x\bar{\wedge}a, \ \ 0 \in a\bar{\wedge}y, \ \ \text{for all} \ \ y \in L\}. \end{aligned}$$

Definition 4. *A hyperideal I of meet-hyperlattice $\mathfrak{L}$ is called closed if $I = I^{rr}$.*

We denote the set of all closed hyperideals of $\wedge$- hyperlattice L by $H(\mathfrak{L})$.

Lemma 1. *Let I, J and K be closed hyperideals of meet-hyperlattice. Then:*

(i) $(I^r \vee J^r)^r = I \cap J$;

(ii) $(I \cap J)^{rr} = I^{rr} \cap J^{rr}$;

(iii) *If* $I \cap J^{rr} = 1^r$ *then* $I \subseteq J^r$;

(iv) $K \cap (I^r \cap J^r)^r \subseteq \big(I^r \cap (J \cap K)^r\big)^r$.

Proof.

(i) Since $I^r, J^r \subseteq I^r \vee J^r$ then, from Proposition 2, we have

$$(I^r \vee J^r)^r \subseteq I^{rr} = I,$$

and

$$(I^r \vee J^r)^r \subseteq J^{rr} = J,$$

Thus

$$(I^r \vee J^r)^r \subseteq I \cap J.$$

The equalty is obtained from (iii) in Theorem 2.

(ii) $I^{rr} \cap J^{rr} = I \cap J \subseteq (I \cap J)^{rr}$. On the other side, from (v) in Theorem 2. the equality is satisfied.

(iii) Let $I \cap J^{rr} = I \cap J = 1^r$. Then $I \subseteq J^r$ (from (vii) in Theorem 2).

(iv) It is clear that

$$K \cap I^r \cap (J \cap K)^r \subseteq I^r. \tag{1}$$

Consequently

$$\begin{aligned} J \cap K \cap \big(I^r \cap (J \cap K)^r\big) &= I^r \cap [(J \cap K) \cap (J \cap K)^r] \\ &= I^r \cap [(J^{rr} \cap K^{rr}) \cap (J \cap K)^r] \\ &= I^r \cap [(J \cap K)^{rr} \cap (J \cap K)^r], && \text{(From } ii)) \\ &= I^r \cap 1^r && \text{(From } vii) \text{ in Theorem 2.)} \\ &= 1^r, \end{aligned}$$

which implies that

$$K \cap I^r \cap (J \cap K)^r \subseteq J^r \tag{2}$$

Thus, from (1), (2) we get

$$K \cap I^r \cap (J \cap K)^r \subseteq I^r \cap J^r.$$

Hence

$$\big(K \cap I^r \cap (J \cap K)^r\big) \cap (I^r \cap J^r)^r = 1^r,$$

which equivalent,

$$I^r \cap (J \cap K)^r \cap \big(K \cap (I^r \cap J^r)^r\big) = 1^r.$$

As a result, we get

$$K \cap (I^r \cap J^r)^r \subseteq \big(I^r \cap (J \cap K)^r\big)^r.$$

For $I, J \in H(\mathfrak{L})$, we define two binary operations $I \wedge J = I \cap J$ and $I \veebar J = (I^r \cap J^r)^r$. We get

$$(I \cap J)^{rr} = I^{rr} \cap J^{rr} = I \cap J \in H(\mathfrak{L}),$$

and,

$$I \veebar J = (I^r \cap J^r)^r \in H(\mathfrak{L}).$$

Theorem 3. *Let* $\mathfrak{L} =< L; \bar{\wedge}, \vee >$ *be a meet-hyperlattice. Then* $< H(\mathfrak{L}); \cap, \veebar, ^r, 1^r, L >$ *forms a Boolean algebra.*

Proof. To demonstrate that $(H(L), \wedge, \veebar)$ forms a lattice, only associative and absorption identities are required, as idempotent and commutative identities are trivial.
Let $I, J, K \subseteq H(L)$. Then we have

$$\begin{aligned}(I \veebar J) \veebar K &= (I^r \cap J^r)^r \veebar K \\ &= \big((I^r \cap J^r)^{rr} \cap K^r\big)^r \\ &= \big((I^r \cap J^r) \cap K^r\big)^r \\ &= \big(I^r \cap (J^r \cap K^r)\big)^r \\ &= \big(I^r \cap (J^r \cap K^r)^{rr}\big)^r \\ &= I \veebar (J^r \cap K^r)^r \\ &= I \veebar (J \veebar K).\end{aligned}$$

It is easy to prove the second associative identity, $(I \cap J) \cap K = I \cap (J \cap K)$.
Now we are going to show the absorption identities. Since $I^r \subseteq (I \cap J)^r$, then

$$I \veebar (I \cap J) = \big(I^r \cap (I \cap J)^r\big)^r = I^{rr} = I.$$

Similarly, since $I^r \cap J^r \subseteq I^r$, then $I = I^{rr} \subseteq \big(I^r \cap J^r\big)^r$. Therefore

$$I \cap (I \veebar J) = I \cap (I^r \cap J^r)^r = I.$$

Notice that $I \subseteq 0^r = L$ and $1^r \subseteq I \in H(\mathfrak{L})$, consequently $(H(\mathfrak{L}), \wedge, \veebar)$ is a bounded lattice.

Clearly, I^r is the complement of I, because $I \cap I^r = 1^r$ and $I \veebar I^r = (I^r \cap I^{rr})^r = 1^{rr} = L$.

By using (iv) in Lemma 1, we get that:

$$K \cap (I \veebar J) \subseteq I \veebar (J \cap K) \tag{3}$$

Then the distributivity condition is proved by replacing K in (3) by $I \veebar K$ to get:

$$\begin{aligned}(I \veebar K) \cap (I \veebar J) &\subseteq I \veebar \big(J \cap (I \veebar K)\big)\\ &\subseteq I \veebar \big(I \veebar (K \cap J)\big) \text{ (by replacing } K \text{ in (3) by } J)\\ &= I \veebar (K \cap J).\end{aligned}$$

On the other hand, $J \cap K \subseteq (I \veebar J) \cap (I \veebar K)$ and $I \subseteq (I \veebar J) \cap (I \veebar K)$, which implies that

$$I \veebar (J \cap K) \subseteq (I \veebar J) \cap (I \veebar K). \tag{4}$$

Therefore $I \veebar (J \cap K) = (I \veebar J) \cap (I \veebar K)$ from (3) and (4).

Example 1. *Tables 1 and 2 represent the hyperoperation $\bar{\wedge}$ and operation $\vee$ of meet-hyberlattice $L = \{0, \alpha, \beta, \gamma, \delta, 1\}$. Figure 1 shows Boolean algebra $H(\mathfrak{L})$ of closed hyperideals of L.*

$\bar{\wedge}$	*0*	α	β	γ	δ	*1*
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
α	$\{0\}$	$\{0,\alpha\}$	$\{0\}$	$\{0,\alpha\}$	$\{0,\alpha\}$	$\{0,\alpha\}$
β	$\{0\}$	$\{0\}$	$\{\beta\}$	$\{\beta\}$	$\{0\}$	$\{\beta\}$
γ	$\{0\}$	$\{0,\alpha\}$	$\{\beta\}$	$\{\gamma\}$	$\{0,\alpha\}$	$\{\gamma\}$
δ	$\{0\}$	$\{0,\alpha\}$	$\{0\}$	$\{0,\alpha\}$	$\{\delta\}$	$\{\delta\}$
1	$\{0\}$	$\{0,\alpha\}$	$\{\beta\}$	$\{\gamma\}$	$\{\delta\}$	$\{1\}$

Table 1: Represents the hyperoperation $\bar{\wedge}$ of the meet-hyberlattice L

$\vee$	*0*	α	β	γ	δ	*1*
0	*0*	α	β	γ	δ	*1*
α	α	α	γ	γ	δ	*1*
β	β	γ	β	γ	*1*	*1*
γ	γ	γ	γ	γ	*1*	*1*
δ	δ	0	*1*	*1*	δ	*1*
1	*1*	*1*	*1*	*1*	*1*	*1*

Table 2: Represents operation $\vee$ of the meet-hyberlattice L

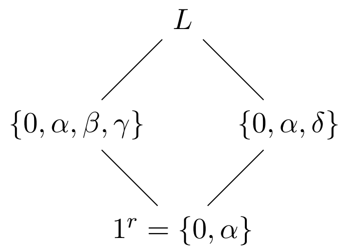

Figure1:Boolean Algebra $< H(\mathfrak{L}); \cap, \veebar, ^r, 1^r, L >$

4. Homomorphic Images of Annihilator Hyperideals

In this section, we define the homomorphism that is annihilator hyperideal preserving. Many properties related to the homomorphism of annihilator hyperideals are proven. Moreover, we show that homomorphic images and preimages of annihilator hyperideals are annihilator hyperideals.

Definition 5. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two meet-hyperlattices. Then the map $\phi : L \to L'$ is called homomorphism if the following conditions hold:*

$$\phi(a \vee b) = \phi(a) \vee \phi(b),\ \phi(a \bar{\wedge} b) = \phi(a) \bar{\wedge} \phi(b).$$

Since for any $a \in L$: $\phi(a \vee 0) = \phi(a) \vee \phi(0) = \phi(a)$, then $\phi(0) = 0'$, where 0 and $0'$ are the zero elements of $\mathfrak{L}$ and $\mathfrak{L}'$ respectively. Obviously, if $a \leq b$ then $\phi(a) \leq \phi(b)$.

The kernal of the homomorphism ϕ is given by $ker(\phi) = \{a \in L : \phi(a) = 0'\}$. It is clear that $ker(\phi)$ is a hyperideal of $\mathfrak{L}'$. The set of images of ϕ is denoted by $Im(\phi)$. It forms sub-meet-hyperlattice of $\mathfrak{L}'$. If ϕ is one-to-one and onto, then $\mathfrak{L}$ and $\mathfrak{L}'$ are isomorphic and denoted by $\mathfrak{L} \cong \mathfrak{L}'$.

Example 2. *Tables 3 and 4 represent the hyperoperation $\bar{\wedge}$ and operation $\vee$ of meet-hyberlattice $L' = \{0', x, y, z, 1'\}$.*

$\bar{\wedge}$	$0'$	x	y	z	$1'$
$0'$	$\{0'\}$	$\{0'\}$	$\{0'\}$	$\{0'\}$	$\{0'\}$
x	$\{0'\}$	$\{x\}$	$\{0'\}$	$\{x\}$	$\{x\}$
y	$\{0'\}$	$\{0'\}$	$\{y\}$	$\{y\}$	$\{y\}$
z	$\{0'\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{z\}$
$1'$	$\{0'\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{1'\}$

Table 3: Represents the hyperoperation $\bar{\wedge}$ of meet-hyberlattice L'

$\vee$	$0'$	x	y	z	$1'$
$0'$	$0'$	x	y	z	$1'$
x	x	x	z	z	$1'$
y	y	z	y	z	$1'$
z	z	z	z	z	$1'$
$1'$	$1'$	$1'$	$1'$	$1'$	$1'$

Table 4: Represents the operation $\vee$ of the meet-hyberlattice L'

Consider the meet-hyperlattice L in Example 1 and the meet-hyperlattic L'. Define a homomorphism $f : L \to L'$ as:

$$f(0) = 0', \quad f(\beta) = x, \quad f(\alpha) = y, \quad f(\gamma) = z \quad \text{and} \quad f(\delta) = f(1) = 1'.$$

Proposition 3. *Let $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism between meet-hyperlattices. Then:*

(i) If ϕ is onto and I is a hyperideal of $\mathfrak{L}$, then $\phi(I)$ is a hyperideal of $\mathfrak{L}'$;

(ii) If J is a hyperideal of $\mathfrak{L}'$, then $\phi^{-1}(J)$ is a hyperideal of $\mathfrak{L}$ containing $ker(\phi)$;

(iii) If A is a nonempty subset of $\mathfrak{L}$, then $\phi(A^r) \subseteq (\phi(A))^r$.

Proof.

(i) Let $x, y \in \phi(I)$ then there exist $a, b \in I$ such that $x = \phi(a)$ and $y = \phi(b)$. Then $\phi(a \vee b) = \phi(a) \vee \phi(b) = x \vee y \subseteq \phi(I)$. Now, suppose $x, y \in \mathfrak{L}'$, $x \in \phi(I)$ and $y \leq x$. Hence $y = \phi(b) \in \phi(a)\bar{\wedge}\phi(b) = \phi(a\bar{\wedge}b)$, which indicates that $b \in a\bar{\wedge}b$ and $b \leq a$. Thus $b \in I$ and $y = \phi(b) \in \phi(I)$. Consequently $\phi(I)$ is a hyperideal.

(ii) Let $x, y \in J$. Then there exist $a, b \in \mathfrak{L}'$ such that $\phi^{-1}(x) = a$ and $\phi^{-1}(y) = b$. It implies $a \vee b = \phi^{-1}(x) \vee \phi^{-1}(y)$. By using the effect of ϕ on both sides we get $\phi(a \vee b) = \phi\big(\phi^{-1}(x) \vee \phi^{-1}(y)\big) = \phi\big(\phi^{-1}(x)\big) \vee \phi\big(\phi^{-1}(y)\big) = x \vee y$. Thus $\phi^{-1}\big(\phi(a \vee b)\big) = a \vee b = \phi^{-1}(x \vee y) \in \phi^{-1}(J)$. Let $x \in J$, $\phi^{-1}(x) = a$ and $y \in \mathfrak{L}$ such that $y \leq a = \phi^{-1}(x)$. Then $y \vee a = a$ which implies $\phi(y) \vee \phi(a) = \phi(y \vee a) = \phi(a)$. Thus $\phi(y) \leq \phi(a) = \phi\big(\phi^{-1}(x)\big) = x$. Therefore $\phi(y) \in J$ and $y \in \phi^{-1}(J)$. Clearly, $0' \in J$ which means $ker(\phi) = \phi^{-1}(0') \subseteq \phi^{-1}(J)$.

(iii) Let $x \in A^r$ and $\phi(x) = b$. Then $0' = \phi(0) \in \phi(x \wedge a) = \phi(x)\bar{\wedge}\phi(a)$ for all $a \in A$. It means that $0' \in \phi(x)\bar{\wedge}\phi(a)$ for all $\phi(a) \in \phi(A)$ i.e., $\phi(x) \in \big(\phi(a)\big)^r$. Therefore $\phi(A^r) \subseteq \big(\phi(A)\big)^r$.

Definition 6. *Let $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism. Then ϕ is called annihilator hyperideal preserving if for any subset A of $\mathfrak{L}$: $\phi(A^r) = [\phi(A)]^r$.*

The homomrphism f in Example 2 is not an annihilator hyperideal preserving.

Example 3. *Tables 5 and 6 represent the hyperoperation $\bar{\wedge}$ and operation $\vee$ of meet-hyberlattice $L'' = \{0, \alpha, \beta, \gamma, \delta, 1\}$.*

$\bar{\wedge}$	0	α	β	γ	δ	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
α	$\{0\}$	$\{\alpha\}$	$\{0\}$	$\{\alpha\}$	$\{\alpha\}$	$\{\alpha\}$
β	$\{0\}$	$\{0\}$	$\{\beta\}$	$\{\beta\}$	$\{0\}$	$\{\beta\}$
γ	$\{0\}$	$\{\alpha\}$	$\{\beta\}$	$\{\gamma\}$	$\{\alpha\}$	$\{\gamma\}$
δ	$\{0\}$	$\{\alpha\}$	$\{0\}$	$\{\alpha\}$	$\{\delta\}$	$\{\delta\}$
1	$\{0\}$	$\{\alpha\}$	$\{\beta\}$	$\{\gamma\}$	$\{\delta\}$	$\{1\}$

Table 5: Represents the hyperoperation $\bar{\wedge}$ of the meet-hyberlattice L''

$\vee$	0	α	β	γ	δ	1
0	0	α	β	γ	δ	1
α	α	α	γ	γ	δ	1
β	β	γ	β	γ	1	1
γ	γ	γ	γ	γ	1	1
δ	δ	0	1	1	δ	1
1	1	1	1	1	1	1

Table 6: Represents operation $\vee$ of the meet-hyberlattice L''

Define a homomorphism $f : L'' \to L'$ as:

$$g(0) = 0', \quad g(\beta) = g(\delta) = y, \quad g(\alpha) = x, \quad g(\gamma) = z \quad \text{and} \quad g(1) = 1'.$$

Where L' is a meet-hyperlattice in Example 2. f is an annihilator hyperideal preserving.

Theorem 4. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two meet-hyperlattice, $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism and $ker(\phi) = \{0\}$. Then:*

(i) If ϕ is onto then:

(a) ϕ is annihilator hyperideal preserving;

(b) For any nonempty subsets A and B of $\mathfrak{L}$

$$A^r = B^r \quad \text{if and only if} \quad [\phi(A)]^r = [\phi(B)]^r.$$

(ii) ϕ^{-1} is annihilator hyperideal preserving.

Proof.

(i) (a) For a nonempty subset A of $\mathfrak{L}$, we have $\phi(A^r) \subseteq [\phi(A)]^r$, from Proposition 3. So we just need to prove that $[\phi(A)]^r \subseteq \phi(A^r)$. To do that, let $x \in [\phi(A)]^r \subseteq \mathfrak{L}'$ then there is $a \in L$ such that $\phi(a) = x$, but $0' \in x\bar{\wedge}\phi(b)$ for all $b \in A$. Then $0 \in a\bar{\wedge}b$ for all $b \in A$. Therefore $a \in A^r$ and then $x = \phi(a) \in \phi(A^r)$.

(b) Suppose that A and B are nonempty subsets of $\mathfrak{L}$ such that $A^r = B^r$. Then by using $a)$ we get $[\phi(A)]^r = \phi(A^r) = \phi(B^r) = [(\phi(B)]^r$. Conversely, Let $[\phi(A)]^r = [\phi(B)]^r$ and $x \in A^r$. Then $0 \in a\bar{\wedge}x$ for all $a \in A$. So $0' = \phi(0) \in \phi(a\bar{\wedge}x) = \phi(a)\bar{\wedge}\phi(x)$. It means $\phi(x) \in [\phi(A)]^r = [\phi(B)]^r$. Therefore $0' = \phi(0) \in \phi(x)\bar{\wedge}\phi(b)$ for all $b \in B$, which implies that $0 \in x\bar{\wedge}b$ for all $b \in B$. Accordingly $x \in B^r$ and hence $A^r \subseteq B^r$. Similarly we can prove that $B^r \subseteq A^r$.

(ii) Let $x \in [\phi^{-1}(B)]^r$ which means $0 \in x\bar{\wedge}b$ for all $b \in \phi^{-1}(B)$. Then $0 \in x\bar{\wedge}b$ for all $\phi(b) \in B$. It implies $\phi(0) = 0' \in \phi(x\bar{\wedge}b) = \phi(x)\bar{\wedge}\phi(b)$ for all $\phi(b) \in B$. Hence $\phi(x) \in B^r$, i.e., $x \in \phi^{-1}(B^r)$. As a result, $[\phi^{-1}(B)]^r \subseteq \phi^{-1}(B^r)$. Conversely, let $x \in \phi^{-1}(B^r)$ and $b \in \phi^{-1}(B)$. Then $\phi(x) \in B^r$ and $\phi(b) \in B$. Thus $\phi(0) = 0' \in \phi(x)\bar{\wedge}\phi(b) = \phi(x\bar{\wedge}b)$. As a result, $0 \in x \wedge b$ for all $b \in \phi^{-1}(B)$. Hence, $x \in [\phi^{-1}(B)]^r$ and then $\phi^{-1}(B^r) \subseteq [\phi^{-1}(B)]^r$.

Theorem 5. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two meet-hyperlattice, $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism and $ker(\phi) = \{0\}$. Then:*

(i) If ϕ is annihilator hyperideal preserving and onto then the homomorphic image $\phi(I)$ of annihilator hyperideal I of $\mathfrak{L}$ is an annihilator hyperideal of $\mathfrak{L}'$;

(ii) If ϕ^{-1} is annihilator hyperideal of $\mathfrak{L}'$ preserving then the preimage $\phi^{-1}(J)$ of annihilator hyperideal J is an annihilator hyperideal of $\mathfrak{L}$;

(iii) If ϕ is annihilator hyperideal preserving and onto then $ker(\phi)$ is an annihilator hyperideal of $\mathfrak{L}$.

Proof.

(i) Let I be an annihilator hyperideal of $\mathfrak{L}$. Then by using $i)$ in Proposition 3, $\phi(I)$ is a hyperideal. From the assumption of ϕ is annihilator hyperideal preserving we get

$$[\phi(I)]^{rr} = \phi(I^{rr}) = \phi(I).$$

Consequently, $\phi(I)$ is an annihilator ideal of $\mathfrak{L}'$.

(ii) Let J be an annihilator hyperideal of $\mathfrak{L}'$. Then by using $ii)$ in Proposition 3, $\phi^{-1}(I)$ is a hyperideal. From the assumption of ϕ^{-1} is annihilator hyperideal preserving we get

$$[\phi^{-1}(I)]^{rr} = \phi^{-1}(I^{rr}) = \phi^{-1}(I).$$

Therefore, $\phi^{-1}(I)$ is an annihilator ideal of $\mathfrak{L}$.

(iii) We know that $ker(\phi) = \phi^{-1}(\{0'\})$ and $\{0'\} = 1^r$ is annihilator hyperideal. Then from $ii)$, $ker(\phi)$ is annihilator hyperideal.

Corollary 1. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two meet-hyperlattice, $\phi : \mathfrak{L} \to \mathfrak{L}'$ be an annihilator hyperideal preserving. Then $H(\mathfrak{L})$ and $H(\mathfrak{L}')$ are isomorphic. Symbolically writes $H(\mathfrak{L}) \cong H(\mathfrak{L}')$.*

Example 4. *The meet-hyperlattice L' in Example 3 has the Boolean algebra of its closed hyperideals which is given by Figure 2. Clearly, its isomorphic to the Boolean algebra in Figure 3 which represents the closed hyperideals of the meet-hyperlattice in Example 2.*

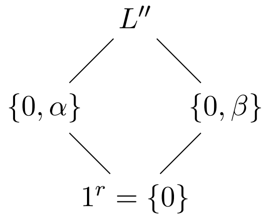

Figure2: Boolean Algebra $< H(\mathfrak{L}''); \cap, \underline{\vee},^r, 1^r, L'' >$

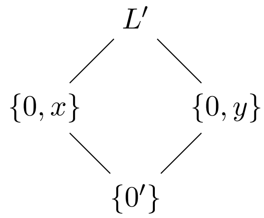

Figure3: Boolean Algebra $< H(\mathfrak{L}'); \cap, \underline{\vee},^r, 1^r, L' >$

5. Sub-meet-hyperlattice and Product

Representations of annihilator hyperideals in sub-meet-hyperlattice and product meet-hyperlattice, are investigated in the following.

Definition 7. *A subset $S \subseteq \mathfrak{L}$, of meet-hyperlattice $\mathfrak{L}$, is a sub-meet-hyperlattice iff it is close under the same operations of $\mathfrak{L}$.*

Clearly, sub-meet-hyperlattice of strong bounded dual distributive meet-hyperlattice is also too.

Theorem 6. *If I is an annihilator hyperideal of L and S is a sub-meet hyperlattice of L. Then $I \cap S$ is an annihilator hyperideal of S.*

Proof. Let I be an annihilator hyperideal of L and S be a sub-meet hyperlattice of L. Then there exist a subset K of L such that $I = K^r = \{x \in L : 0 \in x\bar{\wedge}a \text{ for all } a \in K\}$. Thus $I \cap S = \{x \in L : 0 \in x\bar{\wedge}a \text{ for all } a \in K \cap S\} = (K \cap S)^r$.

Corollary 2. *If I is a closed hyperideal of L and S is a sub-meet hyperlattice of L. Then $I \cap S$ is a closed hyperideal of S.*

Example 5. *Tables 7 and 8 represent a sub-meet-hyperlattice S of meet-hyperlattice L in Example 1. Figure 4 shows Boolean algebra $H(\mathfrak{S})$ of closed hyperideals of S.*

$\bar{\wedge}$	0	α	β	γ
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
α	$\{0\}$	$\{0,\alpha\}$	$\{0\}$	$\{0,\alpha\}$
β	$\{0\}$	$\{0\}$	$\{\beta\}$	$\{\beta\}$
γ	$\{0\}$	$\{0,\alpha\}$	$\{\beta\}$	$\{\gamma\}$

Table 7: Represents the hyperoperation $\bar{\wedge}$ of the sub-meet-hyberlattice S

$\vee$	0	α	β	γ
0	0	α	β	γ
α	α	α	γ	γ
β	β	γ	β	γ
γ	γ	γ	γ	γ

Table 8: Represents operation $\vee$ of the sub-meet-hyberlattice S

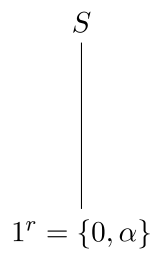

Figure4: Boolean Algebra $< H(\mathfrak{S}); \cap, \veebar,^{r}, 1^{r}, L >$

Definition 8. *Let $\mathfrak{L}_1 =< L_1; \bar{\wedge}_1, \vee_1, 0_1, 1_1 >$ and $\mathfrak{L}_2 =< L_2; \bar{\wedge}_2, \vee_2, 0_2, 1_2 >$ be two meet-hyperlattices. Then the product $\mathfrak{L}_1 \times \mathfrak{L}_2$ with respect to the pair-wise operations such that for any $(a,b), (a',b') \in \mathfrak{L}_1 \times \mathfrak{L}_2$:*

$$(a,b)\bar{\wedge}(a',b') = (a\bar{\wedge}_1 a', b\bar{\wedge}_2 b'), \quad \textit{and} \quad (a,b) \vee (a',b') = (a \vee_1 a', b \vee_2 b'),$$

forms meet-hyperlattice called the prouduct meet-hyperlattice of $\mathfrak{L}_1$ and $\mathfrak{L}_2$ with zero element $0 = (0_1, 0_2)$, and one element$1 = (1_1, 1_2)$.

Notice that, if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are strong bounded dual distributive meet-hyperlattices, then $\mathfrak{L}_1 \times \mathfrak{L}_2$ is also too.

Theorem 7. *For any two hyperideals I_1 and I_2 of two meet-hyperlattices $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively. I_1 and I_2 are annihilator hyperideals iff $I_1 \times I_2$ is an annihilator hyperideal of the product meet-hyperlattice $\mathfrak{L}_1 \times \mathfrak{L}_2$.*

Proof. Assume I_1 and I_2 be annihilator hyperideals of $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. It implies that $I_1 = {K_1}^r$ and $I_2 = {K_2}^r$ for some two sets $K_1 \subseteq \mathfrak{L}_1$ and $K_2 \subseteq \mathfrak{L}_2$. I.e., $I_i = \{x_i \in \mathfrak{L}_{\mathfrak{i}} : 0_i \in x_i \bar{\wedge} a_i \ \text{ for all } \ a_i \in K_i\}$, for $i = 1, 2$. Accordingly $I_1 \times I_2 = \{(x_1, x_2) \in L_1 \times L_2 : (0_1, 0_2) \in (x_1, x_2)\bar{\wedge}(a_1, a_2) \ \text{ for all } \ (a_1, a_2) \in K_1 \times K_2\}$. Therefore, $I_1 \times I_2$ is an annihilator hyperideal of $\mathfrak{L}_1 \times \mathfrak{L}_2$. Conversely, let I be an annihilator hyperideal of $\mathfrak{L}_1 \times \mathfrak{L}_2$. It means there exist subset $K \subseteq \mathfrak{L}_1 \times \mathfrak{L}_2$ such that $I = K^r$. We define the projections $\pi_i : \mathfrak{L}_1 \times \mathfrak{L}_2 \to \mathfrak{L}_i$ for $i = 1, 2$. Let I_1 and I_2 be the projections of I on $\mathfrak{L}_1$ and $\mathfrak{L}_1$ respectively. In addition to K_1 and K_2 are the projections of K on $\mathfrak{L}_1$ and $\mathfrak{L}_2$. So $\pi_i(I) = I_i$ and $\pi_i(K) = K_i$ for $i = 1, 2$. We get $I = \{(x_1, x_2) \in L_1 \times L_2 : (0_1, 0_2) \in (x_1, x_2)\bar{\wedge}(a_1, a_2) \text{ for all } (a_1, a_2) \in K_1 \times K_2\}$. Thus $\pi_i(I) = \{x_i \in L_1 : 0_i \in x_i\bar{\wedge}_i a_i \text{ for all } a_i \in K_i\}$. Therefore $I \cong \pi_1(I) \times \pi_2(I)$.

Corollary 3. *For any two hyperideals I_1 and I_2 of two meet-hyperlattices $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively. I_1 and I_2 are closed hyperideals iff $I_1 \times I_2$ is a closed hyperideal of the product meet-hyperlattice $\mathfrak{L}_1 \times \mathfrak{L}_2$.*

6. Conclusion

In our work, the properties of annihilator hyperideals in the class of strong bounded dual distributive meet-hyperlattice were studied and proved. The connection between closed hyperideals and annihilators was discovered. In general, the set of closed hyperideals is a subset of the set of all annihilators. It also forms a Boolean algebra. We introduce the concept of homomorphisms, which preserve the annihilator hyperideal. Suitable conditions for preserving annihilator hyperideals are obtained. Under these conditions, homomorphic images and preimages of annihilators are also annihilators. Accordingly, if there exists an annihilator hyperideal preserving map between two meet-hyperlattices, then there is a one-to-one correspondence between their Boolean algebras of closed hyperideals. Representation and characterization of annihilator hyperideals under product and sub-structure of meet-hyperlattice are shown.

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