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Bounds for the Convex Combination of Contra-harmonic and Harmonic Means by the Generalized Logarithmic Mean

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Abstract. We verify the optimal upper bound and optimal lower bound for the convex combination of contra-harmonic and harmonic means by the generalized logarithmic mean L_p when p is of the linear form $p = 2(1-c)\alpha + c$ and p is of the reciprocal of linear form $p = 1/[2(1-c)\alpha + c]$ respectively. We prove that

1) $L_{4\alpha-1} = \min_c \left\{ L_{2(1-c)\alpha+c} \mid L_{2(1-c)\alpha+c} > \alpha C + (1-\alpha)H \right\}$ for $\alpha \in (0, 1/2)$,
2) $L_{\frac{7}{13-12\alpha}} = \max_c \left\{ L_{\frac{1}{2(1-c)\alpha+c}} \,\middle|\, L_{\frac{1}{2(1-c)\alpha+c}} < \alpha C + (1-\alpha)H \right\}$ for $\alpha \in (1/2, 1)$

where $C(a,b)$ and $H(a,b)$ are contra-harmonic and harmonic means.

2020 Mathematics Subject Classifications: 26D15, 26D20, 26E20

Key Words and Phrases: Inequality, Generalized logarithmic mean, Weighted arithmetic mean, Harmonic mean, Contra-harmonic mean

1. Introduction

For any real number p, generalized logarithmic mean $L_p(a,b)$ of two positive numbers a and b is defined by

$$L_p(a,b) = \begin{cases} \left[\dfrac{a^{p+1} - b^{p+1}}{(p+1)(a-b)} \right]^{1/p}, & a \neq b,\ p \neq 0,\ p \neq -1; \\ \dfrac{1}{e} \left(\dfrac{b^b}{a^a} \right)^{1/(b-a)}, & a \neq b,\ p = 0; \\ \dfrac{b-a}{\log b - \log a}, & a \neq b,\ p = -1; \\ a, & a = b. \end{cases}$$

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Mean L_p is continuous and strictly increasing with respect to p. It has many applications in physics involving a heat conductor problem and a mean temperature between two points at different temperature [6, 10].

Many classical bivariate means are special cases of generalized logarithmic means such as

$$G(a,b) = L_{-2}(a,b), \quad L(a,b) = L_{-1}(a,b), \quad N(a,b) = L_{-1/2}(a,b),$$
$$I(a,b) = L_0(a,b), \quad \text{and} \quad A(a,b) = L_1(a,b),$$

where G, L, N, I, and A are geometric, logarithmic, square-root, identric and arithmetic means, respectively.

In view of generalized logarithmic means we have a well-known string of inequalities

$$\min\{a,b\} < H(a,b) < L_{-2}(a,b) < L_1(a,b) < S(a,b) < C(a,b) < \max\{a,b\}$$

for all distinct positive numbers a, b; here

$$H(a,b) = \frac{2ab}{a+b}, \quad S(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \quad C(a,b) = \frac{a^2+b^2}{a+b}$$

are harmonic, root-square, and contra-harmonic means, respectively.

Generalized logarithmic mean has been the subject of intensive research in particular those involving inequalities and monotonicity [1–4, 11–13, 18, 19]. Below we present some recent works concerning the optimal bound of certain means by generalized logarithmic means in one direction and weighted means by generalized logarithmic means in another.

For a problem of finding sharp double inequalities between generalized logarithmic means and other means, recently it was found possible for Neuman-Sándor mean M and Yang mean U which are defined by

$$M(a,b) = \begin{cases} \dfrac{a-b}{2\sinh^{-1}\left(\frac{a-b}{a+b}\right)}, & a \neq b; \\ a, & a = b, \end{cases} \qquad U(a,b) = \begin{cases} \dfrac{a-b}{\sqrt{2}\tan^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)}, & a \neq b; \\ a, & a = b. \end{cases}$$

In case of Neuman-Sándor mean, Li, Long and Chu [8] in 2012 found the best largest value $p = 1.8435\ldots$ and smallest value $q = 2$, where p is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$ such that the double inequalities

$$L_p(a,b) < M(a,b) < L_q(a,b)$$

hold for all distinct positive numbers a, b.

In case of Yang mean, Qian and Chu [14] in 2016 found the best possible parameters $p = 0.5451\ldots$ and $q = 2$, where p is the unique solution of the equation $(p+1)^{1/p} = \sqrt{2}\pi/2$ such that the double inequalities

$$L_p(a,b) < U(a,b) < L_q(a,b)$$

hold for all distinct positive numbers a, b.

For a problem of finding optimal bound of weight either geometric or arithmetic means by generalized logarithmic means, there are many recent works in this direction.

In case of weighted geometric mean, for $\alpha, \beta, \gamma \in (0,1)$ and $\alpha+\beta+\gamma=1$, Chu and Long [4] in 2010 found the optimal bound for $A^{\alpha}(a,b)G^{\beta}(a,b)H^{\gamma}(a,b)$. That is, they discovered that the largest value $p = 6\alpha+3\beta-5$ and the smallest value $q = -2/(2\alpha+\beta)$ are the optimal values such that the double inequalities

$$L_p(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{\gamma}(a,b) < L_q(a,b)$$

hold for all distinct positive numbers a, b.

In 2011, Qian and Long [16] presented the sharp upper and lower bound for the weighted geometric mean of geometric and harmonic means by generalized logarithmic means: for all positive numbers a and b

1) $L_{3\alpha-5}(a,b) = G^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{-2/\alpha}(a,b)$ for $\alpha = 2/3$,

2) $L_{3\alpha-5}(a,b) \geqslant G^{\alpha}(a,b)H^{1-\alpha}(a,b) \geqslant L_{-2/\alpha}(a,b)$ for $\alpha \in (0,2/3)$,

and $L_{3\alpha-5}(a,b) \leqslant G^{\alpha}(a,b)H^{1-\alpha}(a,b) \leqslant L_{-2/\alpha}$ for $\alpha \in (2/3,1)$, with equality if and only if $a = b$, and the parameters $3\alpha-5$ and $-2/\alpha$ cannot be improved in either case.

Chunrong and Siqi [5] established the optimal bounds for $G^{\alpha}(a,b)N^{1-\alpha}(a,b)$ in term of $L_p(a,b)$. They found that for any positive numbers a and b

1) $L_{-(1+3\alpha)/2}(a,b) = G^{\alpha}(a,b)N^{1-\alpha}(a,b) = L_{2/(\alpha-2)}(a,b)$ for $\alpha = 2/3$,

2) $L_{-(1+3\alpha)/2}(a,b) > G^{\alpha}(a,b)N^{1-\alpha}(a,b) > L_{2/(\alpha-2)}(a,b)$ for $\alpha \in (0,2/3)$,

and $L_{-(1+3\alpha)/2}(a,b) < G^{\alpha}(a,b)N^{1-\alpha}(a,b) < L_{2/(\alpha-2)}(a,b)$ for $\alpha \in (2/3,1)$, and the parameters $-(1+3\alpha)/2$ and $2/(\alpha-2)$ cannot be improved in either case.

In the case of weighted arithmetic mean, Long and Chu [9] in 2010 proposed the inequalities involving generalized logarithmic means and weighted arithmetic means of arithmetic and geometric means:

1) $L_{3\alpha-2}(a,b) = \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha = 1/2$,

2) $L_{3\alpha-2}(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in (0,1/2)$,

3) $L_{3\alpha-2}(a,b) > \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in (1/2,1)$.

Moreover, in each case, the bound $L_{3\alpha-2}(a,b)$ for the sum of $\alpha A(a,b)+(1-\alpha)G(a,b)$ is optimal.

The harmonic and contra-harmonic means have recently been used to investigate the optimal bounds for means inequalities as mentioned in the following.

In 2017, Qian, Zhang and Chu [17] discovered the greatest values α and λ, and the smallest values β and μ in [0,1/2] such that

$$H[\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a] < TQ(a,b) < H[\beta a+(1-\beta)b, \beta b+(1-\beta)a],$$

$$G[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a] < TQ(a,b) < H[\mu a+(1-\mu)b,\mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a\neq b$, where $TQ(a,b)=\frac{2}{\pi}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}d\theta$ is the Toader-Qi mean of a and b.

In 2018, Xu, Chu and Qian [20] found the optimal parameters $\alpha_i,\beta_i\in(0,1)$ $(i=1,2,3,4)$ to ensure that four double inequalities

$$\begin{aligned}
&C^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b) < R_{SA}(a,b) < C^{\beta_1}(a,b)A^{1-\beta_1}(a,b),\\
&C^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < R_{AS}(a,b) < C^{\beta_2}(a,b)A^{1-\beta_2}(a,b),\\
&\alpha_3\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]+(1-\alpha_3)C^{1/3}(a,b)A^{2/3}(a,b)\\
&\quad< R_{SA}(a,b) < \beta_3\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]+(1-\beta_3)C^{1/3}(a,b)A^{2/3}(a,b),\\
&\alpha_4\left[\frac{1}{6}C(a,b)+\frac{5}{6}A(a,b)\right]+(1-\alpha_4)C^{1/6}(a,b)A^{5/6}(a,b)\\
&\quad< R_{AS}(a,b) < \beta_4\left[\frac{1}{6}C(a,b)+\frac{5}{6}A(a,b)\right]+(1-\beta_4)C^{1/6}(a,b)A^{5/6}(a,b)
\end{aligned}$$

hold for all distinct positive numbers a,b and

$$\begin{aligned}
R_{SA}(a,b)&=\frac{1}{2}A(a,b)\left[\sqrt{1+u^2}+\frac{\sinh^{-1}(u)}{u}\right],\\
R_{AS}(a,b)&=\frac{1}{2}A(a,b)\left[1+\frac{(1+u^2)\tan^{-1}(u)}{u}\right],
\end{aligned}$$

where $a>b>0$ and $u=(a-b)/(a+b)$.

In 2019, Qian, He, Zhang and Chu [15] found the best values $\lambda_1=\lambda_1(\nu),\mu_1=\mu_1(\nu),\lambda_2=\lambda_2(\nu)$ and $\mu_2=\mu_2(\nu)$ on the interval $[1/2,1]$ such that the double inequalities

$$\begin{aligned}
W_{\lambda_1,\nu}(a,b) < R_{SA}(a,b) < W_{\mu_1,\nu}(a,b),\\
W_{\lambda_2,\nu}(a,b) < R_{AS}(a,b) < W_{\mu_2,\nu}(a,b)
\end{aligned}$$

hold for all distinct positive numbers a,b and $\nu\geq 1/2$ where

$$W_{\lambda,\nu}(a,b)=C^{\nu}[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a]A^{1-\nu}(a,b).$$

In 2022, Li, Miao and Guo [7] discovered the largest values α_i and the smallest values β_i $(i=1,2,3)$ such that the inequalities

$$\frac{\alpha_1}{C(a,b)}+\frac{1-\alpha_1}{A(a,b)} < \frac{1}{M(a,b)} < \frac{\beta_1}{C(a,b)}+\frac{1-\beta_1}{A(a,b)},$$

$$\frac{\alpha_2}{C^2(a,b)}+\frac{1-\alpha_2}{A^2(a,b)} < \frac{1}{M^2(a,b)} < \frac{\beta_2}{C^2(a,b)}+\frac{1-\beta_2}{A^2(a,b)},$$

and

$$\alpha_3 C^2(a,b) + (1-\alpha_3)A^2(a,b) < M^2(a,b) < \beta_3 C^2(a,b) + (1-\beta_3)A^2(a,b)$$

hold for all positive real numbers a and b with $a \neq b$.

The purpose of this paper is to present the inequalities with optimal upper bound and optimal lower bound of weighted arithmetic means of contra-harmonic and harmonic means by generalized logarithmic means L_p when p is of the linear form $p = 2(1-c)\alpha + c$ and p is of the reciprocal of linear form $p = 1/[2(1-c)\alpha + c]$ respectively and c is the value to be determined in both cases. Precisely, we prove that

1) $L_{4\alpha-1} = \min_c \left\{ L_{2(1-c)\alpha+c} \mid L_{2(1-c)\alpha+c} > \alpha C + (1-\alpha)H \right\}$ for $\alpha \in (0, 1/2)$,

2) $L_{\frac{7}{13-12\alpha}} = \max_c \left\{ L_{\frac{1}{2(1-c)\alpha+c}} \,\middle|\, L_{\frac{1}{2(1-c)\alpha+c}} < \alpha C + (1-\alpha)H \right\}$ for $\alpha \in (1/2, 1)$.

Details of the results are Theorem 1 and Theorem 2 in section 3. Some complicated computations are carried out using MatlabR2021a software computer system.

2. Preliminaries

In this section, we present four lemmas necessary in the proof of our main results in section 3. More specifically Lemma 1 is used in all Theorems whereas Lemma 2 to Lemma 4 are used only in Theorem 1.

Lemma 1. *If* $p \in \mathbb{R}$, $t > 1$ *and*

$$\begin{aligned} F(t) := &\frac{1}{p}\left[\ln\left(t^{p+1}-1\right) - \ln(p+1) - \ln(t-1)\right] \\ &- \ln\left[\alpha(t^2+1) + 2(1-\alpha)t\right] + \ln(t+1), \end{aligned} \tag{1}$$

then

$$F'(t) = \frac{G(t)}{p\left(t^{p+1}-1\right)\left(t^2-1\right)\left[\alpha(t^2+1)+2(1-\alpha)t\right]}, \tag{2}$$

where

$$\begin{aligned} G(t) = &(-3\alpha p + 2p - \alpha)\left(t^{p+3}-1\right) + (5\alpha p - 2p + \alpha - 2)\left(t^{p+2}-t\right) \\ &+ (-\alpha p + \alpha - 2)\left(t^{p+1}-t^2\right) - \alpha(p+1)\left(t^p - t^3\right). \end{aligned}$$

Furthermore, $G(1) = G'(1) = G''(1) = 0$.

Proof. Differentiating $F(t)$ yields (2) and by taking derivative of G, we obtain

$$\begin{aligned} G'(t) = &(-3\alpha p + 2p - \alpha)(p+3)t^{p+2} + (5\alpha p - 2p + \alpha - 2)\left[(p+2)t^{p+1} - 1\right] \\ &+ (-\alpha p + \alpha - 2)\left[(p+1)t^p - 2t\right] - \alpha(p+1)\left(pt^{p-1} - 3t^2\right), \\ G''(t) = &(-3\alpha p + 2p - \alpha)(p+3)(p+2)t^{p+1} + (5\alpha p - 2p + \alpha - 2)(p+2)(p+1)t^p \\ &+ (-\alpha p + \alpha - 2)\left[(p+1)pt^{p-1} - 2\right] - \alpha(p+1)\left[p(p-1)t^{p-2} - 6t\right], \end{aligned}$$

respectively. It follows immediately that $G(1) = G^{'}(1) = G^{''}(1) = 0$.

Lemma 2. *Let $\alpha \in (0, 1/8)$ and $t > 1$ and*

$$f(t) = \frac{1}{t}\exp\left[\frac{t^2-1}{\alpha(t^2+1)+2(1-\alpha)t}\right].$$

Function f is strictly increasing for t satisfying the inequality

$$\alpha^2t^2 - 2(\alpha^2-3\alpha+1)t + \alpha^2 < 0.$$

Proof. Differentiating $f(t)$ with respect to t yields

$$f'(t) = t^{-1}e^{\frac{t^2-1}{\alpha(t^2+1)+2(1-\alpha)t}}\left\{\frac{\big[\alpha(t^2+1)+2(1-\alpha)t\big](2t)-(t^2-1)\big[2\alpha t+2(1-\alpha)\big]}{\big[\alpha(t^2+1)+2(1-\alpha)t\big]^2}\right\}$$
$$- t^{-2}e^{\frac{t^2-1}{\alpha(t^2+1)+2(1-\alpha)t}}.$$

Simplifying $f'(t)$ and setting $f'(t) > 0$, we obtain

$$2[\alpha(t^2+1)+2(1-\alpha)t]t^2 - t(t^2-1)\big[2\alpha t+2(1-\alpha)\big] - \big[\alpha(t^2+1)+2(1-\alpha)t\big]^2 > 0$$

or

$$\alpha^2t^2 - (2\alpha^2-6\alpha+2)t + \alpha^2 < 0.$$

Lemma 3. *For $a, b > 0$ and $\alpha, \beta \in (0, 1)$ with $\alpha > \beta$, we have*

$$\alpha C(a,b) + (1-\alpha)H(a,b) > \beta C(a,b) + (1-\beta)H(a,b).$$

Proof. Because $C(a,b) > H(a,b)$ and $\alpha > \beta$, the result follows immediately from the inequality $(\alpha-\beta)C(a,b) > (\alpha-\beta)H(a,b)$.

Lemma 4. *For $t > 1$, we have*

$$L_{-1/2}(1,t) > \frac{1}{4}C(1,t) + \frac{3}{4}H(1,t).$$

Proof. The proposed inequality is

$$\frac{1+2\sqrt{t}+t}{4} > \frac{1}{4}\left(\frac{t^2+1}{t+1}\right) + \frac{3}{4}\left(\frac{2t}{t+1}\right)$$

which is equivalent to $(\sqrt{t}-1)^2 > 0$.

3. Main Results

We first establish the optimal upper bound for the weighted arithmetic mean of contra-harmonic and harmonic means by generalized logarithmic means L_p where p has the linear form $p = 2(1-c)\alpha + c$ and $\alpha \in (0, 1/2)$. Precisely, we have

Theorem 1. *Let $a, b > 0$ with $a \neq b$. Then*

1) $L_{4\alpha-1}(a,b) = \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha = 1/2$*;*

2) $L_{4\alpha-1}(a,b) > \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha \in (0,1/2)$*, and the parameter* $4\alpha - 1$ *cannot be improved in the sense that*

$$L_{4\alpha-1} = \min_c \left\{ L_{2(1-c)\alpha+c} \mid L_{2(1-c)\alpha+c} > \alpha C + (1-\alpha)H \right\} \text{ for } \alpha \in (0,1/2)$$

i.e. $c = -1$*;*

3) $L_{4\alpha-1}(a,b) < \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha \in (1/2, 1)$.

Proof.

1) For $\alpha = 1/2$, on one hand we have

$$L_{4(\frac{1}{2})-1}(a,b) = L_1(a,b) = \frac{a+b}{2}.$$

On the other hand, we have

$$\frac{C(a,b) + H(a,b)}{2} = \frac{\frac{a^2+b^2}{a+b} + \frac{2ab}{a+b}}{2} = \frac{a+b}{2}.$$

2) Without loss of generality, we assume that $b > a > 0$ and set $t = b/a > 1$. The proposed inequality becomes

$$\left[\frac{t^{4\alpha} - 1}{4\alpha(t-1)}\right]^{1/(4\alpha-1)} > \alpha\left(\frac{t^2+1}{t+1}\right) + (1-\alpha)\left(\frac{2t}{t+1}\right) \quad \alpha \in (0, 1/2). \tag{3}$$

Inequality (3) is equivalent to $F(t) > 0$ in (1) with $p = 4\alpha - 1$. Using Lemma 1, we have a formula for $F'(t)$, $G'(t)$, $G''(t)$ where $G(t)$ is the numerator of $F'(t)$ appearing in (2). Taking derivative of $G''(t)$, we have

$$\begin{aligned}\frac{G'''(t)}{8\alpha t^{4\alpha-4}} &= (4\alpha+2)(4\alpha+1)(3\alpha-1)(1-2\alpha)t^3 - 2\alpha(4\alpha+1)(4\alpha-1)(3-5\alpha)t^2 \\ &+ 2(2\alpha^2 - \alpha + 1)(4\alpha-1)(1-2\alpha)t \\ &- \alpha(4\alpha-1)(1-2\alpha)(3-4\alpha) + 3\alpha t^{4-4\alpha}. \end{aligned} \tag{4}$$

We divide our proof into four cases: $\alpha \in [1/3, 1/2)$, $\alpha \in [1/8, 1/4)$, $\alpha \in [1/4, 1/3)$ and $\alpha \in (0, 1/8)$.

2.1) Case $\alpha \in [1/3, 1/2)$: Observe that the coefficients of t^3 and t in (4) are positive while that of t^2 and constant term are negative for $\alpha \in [1/3, 1/2)$. Since $t > 1$, we have $t^2 < t^{4-4\alpha} < t^3$ and consequently

$$\begin{aligned}\frac{G'''(t)}{8\alpha t^{4\alpha-4}} > &\Big[(4\alpha+2)(4\alpha+1)(3\alpha-1)(1-2\alpha) - 2\alpha(4\alpha+1)(4\alpha-1)(3-5\alpha) \\ &\quad + 3\alpha\Big] t^{4-4\alpha} \\ &+ \left[2(2\alpha^2-\alpha+1)(4\alpha-1)(1-2\alpha) - \alpha(4\alpha-1)(1-2\alpha)(3-4\alpha)\right] t \\ > &(1-2\alpha)(4\alpha-1)(-8\alpha^2+5\alpha+2)\left(t^{4-4\alpha}-t\right) > 0.\end{aligned}$$

Together with $G(1) = G'(1) = G''(1)$ from Lemma 1, we conclude that $F(t) > 0$ for all $\alpha \in [1/3, 1/2)$.

2.2) Case $\alpha \in [1/8, 1/4)$: Since monotonicity property of L_p implies that $L_{4\alpha-1}(1,t) \geq L_{-1/2}(1,t)$ for $\alpha \geq 1/8$, it is sufficient to prove instead the inequality $L_{-1/2}(1,t) > \alpha C(1,t) + (1-\alpha)H(1,t)$ or

$$(1-4\alpha)\sqrt{t}^2 + 4(1-2\alpha)\sqrt{t} + (1-4\alpha) > 0.$$

which is true for $t > 1$ and $\alpha \in [1/8, 1/4)$.

2.3) Case $\alpha \in [1/4, 1/3)$: Monotonicity of L_p and Lemma 3 imply that

$$\begin{aligned}&L_{4\alpha-1}(1,t) \geq L_0(1,t) > L_{-1/2}(1,t) \quad \text{for } \alpha \geq 1/4, \\ &L_0(1,t) = (1/e)t^{1+\frac{1}{t-1}} > t/e, \\ &(1/3)C(1,t) + (2/3)H(1,t) > \alpha C(1,t) + (1-\alpha)H(1,t) \quad \text{for } \alpha \in [1/4, 1/3).\end{aligned}$$

To prove this case, it is thus sufficient to show that there exist $t_1 < t_2 < t_3$ s.t.

(i) $L_{-1/2}(1,t) > (1/3)C(1,t) + (2/3)H(1,t), \quad 1 < t < t_2$;

(ii) $t/e > (1/3)C(1,t) + (2/3)H(1,t), \quad t_3 < t$;

(iii) $L_0(1,t) > (1/3)C(1,t) + (2/3)H(1,t), \quad t_1 < t \leq t_3$.

Since the inequality in (i) is just

$$\frac{1+2\sqrt{t}+t}{4} > \frac{t^2+4t+1}{3(t+1)} \quad \text{or} \quad \sqrt{t}^2 - 4\sqrt{t} + 1 < 0,$$

which is true for $1 < t < (2+\sqrt{3})^2$ and we choose $t_2 = (2+\sqrt{3})^2 \approx 13.92$ so that (i) is valid. Now, consider the inequality in (ii) or

$$\frac{t}{e} > \frac{t^2+4t+1}{3(t+1)} \quad \text{or} \quad (3-e)t^2 + (3-4e)t - e > 0,$$

which is true for

$$t > \frac{4e-3+\sqrt{12e^2-12e+9}}{6-2e} \approx 28.28.$$

Hence, we choose t_3 to be the right-side number so that (ii) is valid. Now, we consider $t \le t_3$. Since $t^{1/(t-1)}$ is a decreasing function for $t > 1$, we have

$$(1/e)t^{1+\frac{1}{t-1}} \ge t/c \quad \text{where} \quad c = e/{t_3}^{1/(t_3-1)}.$$

To find t_1 for (iii), it is enough to find it from

$$\frac{t}{c} > \frac{t^2+4t+1}{3(t+1)} \quad \text{or} \quad (3-c)t^2 + (3-4c)t - e > 0,$$

which is true for

$$t > \frac{4c-3+\sqrt{12c^2-12c+9}}{6-2c} \approx 11.47.$$

We therefore choose $t_1 = \frac{4c-3+\sqrt{12c^2-12c+9}}{6-2c}$ so that (iii) is valid. Now $t_1 < t_2 < t_3$ and satisfy (i), (ii) and (iii).

2.4) Case $\alpha \in (0, 1/8)$: Monotonicity of L_p implies that $L_{4\alpha-1}(1,t) > L_{-1}(1,t)$ for $\alpha > 0$. Hence we will seek t which satisfies $L_{-1}(1,t) > \alpha C(1,t) + (1-\alpha)H(1,t)$ or

$$\frac{t-1}{\ln t} > \alpha\left(\frac{t^2+1}{t+1}\right) + (1-\alpha)\left(\frac{2t}{t+1}\right) \quad \text{or} \quad \frac{1}{t}\exp\left[\frac{t^2-1}{\alpha(t^2+1)+2(1-\alpha)t}\right] > 1.$$

Setting $f(t) := t^{-1}\exp\left[\frac{t^2-1}{\alpha(t^2+1)+2(1-\alpha)t}\right]$, we can see that $f(1) = 1$ and use Lemma 2 to conclude that $f'(t) > 0$ where $\alpha^2t^2 - 2(\alpha^2 - 3\alpha + 1)t + \alpha^2 < 0$. Function $f(t)$ is an increasing function when

$$t \in \Bigg(\frac{(\alpha^2 - 3\alpha + 1) - \sqrt{-6\alpha^3 + 11\alpha^2 - 6\alpha + 1}}{\alpha^2}, \frac{(\alpha^2 - 3\alpha + 1) + \sqrt{-6\alpha^3 + 11\alpha^2 - 6\alpha + 1}}{\alpha^2} \Bigg),$$

which implies that for $\alpha \in (0, 1/8)$,

$$f(t) > 1 \quad \text{when} \quad t \in \left(1, \frac{\alpha^2 - 3\alpha + 1}{\alpha^2}\right].$$

Hence

$$L_{4\alpha-1}(1,t) > \alpha C(1,t) + (1-\alpha)H(1,t) \quad \text{for} \quad 1 < t \leq (\alpha^2 - 3\alpha + 1)/\alpha^2. \tag{5}$$

Now, observe that the qualities

$$\left[\frac{t^{4\alpha} - 1}{4\alpha(t-1)}\right]^{\frac{1}{4\alpha-1}} > t\left(\frac{1}{4\alpha}\right)^{\frac{1}{4\alpha-1}} \quad \text{and} \quad \alpha t + (2-\alpha) > \frac{\alpha(1+t^2) + (1-\alpha)(2t)}{1+t}$$

hold for all $\alpha \in (0, 1/8)$. We will show that

$$t\left(\frac{1}{4\alpha}\right)^{\frac{1}{4\alpha-1}} > \alpha t + (2-\alpha) \quad \text{or} \quad t\left[\left(\frac{1}{4\alpha}\right)^{\frac{1}{4\alpha-1}} - \alpha\right] > 2 - \alpha$$

for all $t > (\alpha^2 - 3\alpha + 1)/\alpha^2$. It is sufficient to show that

$$\left(\frac{\alpha^2 - 3\alpha + 1}{\alpha^2}\right)\left[\left(\frac{1}{4\alpha}\right)^{\frac{1}{4\alpha-1}} - \alpha\right] > 2 - \alpha \quad \text{or} \quad \left(\frac{1}{4\alpha}\right)^{\frac{1}{4\alpha-1}} > \frac{\alpha - \alpha^2}{\alpha^2 - 3\alpha + 1}.$$

However, this is a direct consequence of a simple string of inequalities,

$$\left(\frac{1}{4\alpha}\right)^{\frac{1}{4\alpha-1}} > 2\alpha > \frac{\alpha - \alpha^2}{\alpha^2 - 3\alpha + 1} \quad \text{for all} \quad \alpha \in (0, 1/8).$$

Thus

$$L_{4\alpha-1}(1,t) > \alpha C(1,t) + (1-\alpha)H(1,t) \quad \text{for} \quad t > (\alpha^2 - 3\alpha + 1)/\alpha^2. \tag{6}$$

From inequalities (5) and (6), we conclude that for $\alpha \in (0, 1/8)$

$$L_{4\alpha-1}(1,t) > \alpha C(1,t) + (1-\alpha)H(1,t) \quad \text{for all} \quad t > 1. \tag{7}$$

Finally, we will prove that the parameter $4\alpha - 1$ cannot be improved in this case.

Suppose, to the contrary, that inequality (7) is true with parameter $2[1 - (-1-\epsilon)]\alpha + (-1-\epsilon)$ for a sufficiently small $\epsilon > 0$. That is

$$L_{2[1-(-1-\epsilon)]\alpha + (-1-\epsilon)}(1,t) - \big[\alpha C(1,t) + (1-\alpha)H(1,t)\big] > 0 \quad \text{for all} \quad t > 1.$$

Hence

$$\frac{L_{2[1-(-1-\epsilon)]\alpha+(-1-\epsilon)}(1,t)-\big[\alpha C(1,t)+(1-\alpha)H(1,t)\big]}{t}>0 \quad \text{for all} \quad t>1.$$

Taking limits to both sides of the above inequality leads to

$$\lim_{\alpha\to\left[\frac{\epsilon}{2(2+\epsilon)}\right]^+}\left(\lim_{t\to+\infty}\frac{L_{2[1-(-1-\epsilon)]\alpha+(-1-\epsilon)}(1,t)-\big[\alpha C(1,t)+(1-\alpha)H(1,t)\big]}{t}\right)$$
$$=-\frac{\epsilon}{2(2+\epsilon)}<0,$$

which is a contradiction.

3) As in 2), the proposed inequality becomes

$$\left[\frac{t^{4\alpha}-1}{4\alpha(t-1)}\right]^{1/(4\alpha-1)}<\alpha\left(\frac{t^2+1}{t+1}\right)+(1-\alpha)\left(\frac{2t}{t+1}\right) \quad \alpha\in(1/2,1). \tag{8}$$

Inequality (8) is equivalent to $F(t)<0$ and the term $G'''(t)$ is still that in (4). It is sufficient to show that $G'''(t)<0$ for all $\alpha\in(1/2,1)$. To that end, we divide our proof into three cases: $\alpha\in(1/2,3/5]$, $\alpha\in(3/5,3/4]$ and $\alpha\in(3/4,1)$.

(a) Case $\alpha\in(1/2,3/5]$: In (4), observe that the constant term is positive while the coefficients of t^3,t^2 and t are negative for $\alpha\in(1/2,3/5]$. Since $t>1$, it follows that $t^{4-4\alpha}<t^2<t^3$ and consequently

$$\begin{aligned}\frac{G'''(t)}{8\alpha t^{4\alpha-4}}&<\big[-(4\alpha+2)(4\alpha+1)(3\alpha-1)(2\alpha-1)\\&\qquad+2\alpha(5\alpha-3)(4\alpha+1)(4\alpha-1)+3\alpha\big]t^{4-4\alpha}\\&\quad+\big[-2(2\alpha^2-\alpha+1)(4\alpha-1)(2\alpha-1)\\&\qquad+\alpha(4\alpha-1)(2\alpha-1)(3-4\alpha)\big]\\&<(2\alpha-1)(4\alpha-1)\Big[-(-8\alpha^2+5\alpha+2)t^{4-4\alpha}-(8\alpha^2-5\alpha+2)\Big]\\&<(2\alpha-1)(4\alpha-1)\Big[-(-8\alpha^2+5\alpha+2)-(8\alpha^2-5\alpha+2)\Big]\\&=-4(2\alpha-1)(4\alpha-1)<0.\end{aligned}$$

(b) Case $\alpha\in(3/5,3/4]$: For such α, the coefficient of t^2 and the constant term in (4) are positive while the coefficients of t^3 and t are negative. Since $t>1$, we have $t^2<t^3$ and consequently

$$\begin{aligned}\frac{G'''(t)}{8\alpha t^{4\alpha-4}}&<\big[-(4\alpha+2)(4\alpha+1)(3\alpha-1)(2\alpha-1)\\&\qquad+2\alpha(5\alpha-3)(4\alpha+1)(4\alpha-1)\big]t^2\end{aligned}$$

$$
\begin{aligned}
&+ [-2(2\alpha^2-\alpha+1)(4\alpha-1)(2\alpha-1)+\alpha(4\alpha-1)(2\alpha-1)(3-4\alpha)] \\
&+ 3\alpha t^{4-4\alpha} \\
=& -2(1-\alpha)(4\alpha+1)(8\alpha^2-5\alpha+1)t^2 \\
&- (2\alpha-1)(4\alpha-1)(8\alpha^2-5\alpha+2)+3\alpha t^{4-4\alpha}.
\end{aligned}
$$

Now, consider the term on the right side of the above inequality. Since the coefficient $-2(1-\alpha)(4\alpha+1)(8\alpha^2-5\alpha+1)$ of t^2 is negative for $\alpha \in (3/5, 3/4]$ and $t^{4-4\alpha} < t^2$, we get

$$
\begin{aligned}
\frac{G'''(t)}{8\alpha t^{4\alpha-4}} &< \big[-2(1-\alpha)(4\alpha+1)(8\alpha^2-5\alpha+1)+3\alpha\big]t^{4-4\alpha} \\
&\quad - (2\alpha-1)(4\alpha-1)(8\alpha^2-5\alpha+2) \\
&= (2\alpha-1)(4\alpha-1)\big[(8\alpha^2-5\alpha-2)t^{4-4\alpha}-(8\alpha^2-5\alpha+2)\big] \\
&< (2\alpha-1)(4\alpha-1)\big[(8\alpha^2-5\alpha-2)-(8\alpha^2-5\alpha+2)\big] \\
&= -4(2\alpha-1)(4\alpha-1) < 0.
\end{aligned}
$$

(c) Case $\alpha \in (3/4, 1)$: For α in this interval the coefficients of t^3 and t in (4) are negative while the coefficient of t^2 is positive. Because $t > 1$, we have $t^{4-4\alpha} < t^2 < t^3$. Thus

$$
\begin{aligned}
\frac{G'''(t)}{8\alpha t^{4\alpha-4}} &< \Big[-\alpha(4\alpha+2)(4\alpha+1)(3\alpha-1)(2\alpha-1) \\
&\quad + 2\alpha(5\alpha-3)(4\alpha+1)(4\alpha-1)\Big]t^2 \\
&\quad + \Big[-2(2\alpha^2-\alpha+1)(4\alpha-1)(2\alpha-1)+3\alpha\Big]t^{4-4\alpha} \\
&\quad - \alpha(4\alpha-1)(2\alpha-1)(4\alpha-3) \\
&= -2(1-\alpha)(4\alpha+1)(8\alpha^2-5\alpha+1)t^2 \\
&\quad + (-32\alpha^4+40\alpha^3-32\alpha^2+17\alpha-2)t^{4-4\alpha} - \alpha(4\alpha-1)(2\alpha-1)(4\alpha-3).
\end{aligned}
$$

Now consider the term on the right side of the last equality sign above. The coefficient $-2(1-\alpha)(4\alpha+1)(8\alpha^2-5\alpha+1)$ of t^2 is negative and $t^{4-4\alpha} < t^2$ as $\alpha \in (3/4, 1)$. Hence

$$
\begin{aligned}
\frac{G'''(t)}{8\alpha t^{4\alpha-4}} &< \big[-2(1-\alpha)(4\alpha+1)(8\alpha^2-5\alpha+1) \\
&\qquad + (-32\alpha^4+40\alpha^3-32\alpha^2+17\alpha-2)\big]t^{4-4\alpha} \\
&\quad - \alpha(4\alpha-1)(2\alpha-1)(4\alpha-3) \\
&= (2\alpha-1)(4\alpha-1)\big[-(-4\alpha^2+3\alpha+4)t^{4-4\alpha} \\
&\qquad - \alpha(4\alpha-1)(2\alpha-1)(4\alpha-3)\big] \\
&< (2\alpha-1)(4\alpha-1)\big[-(-4\alpha^2+3\alpha+4) \\
&\qquad - \alpha(4\alpha-1)(2\alpha-1)(4\alpha-3)\big] \\
&= -4(2\alpha-1)(4\alpha-1) < 0.
\end{aligned}
$$

The proof is complete.

The next theorem is concerned with determining the optimal lower bound for the weighted arithmetic mean of contra-harmonic and harmonic means by generalized logarithmic means L_p, where p is of the reciprocal of linear form $p = 1/[2(1-c)\alpha + c]$ where $\alpha \in (1/2, 1)$.

Theorem 2. *Let $a, b > 0$ with $a \neq b$. Then*

1) $L_{7/(13-12\alpha)}(a,b) = \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha = 1/2$;

2) $L_{7/(13-12\alpha)}(a,b) > \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha \in (0, 1/2)$;

3) $L_{7/(13-12\alpha)}(a,b) < \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha \in (1/2, 1)$, *and the parameter* $7/(13-12\alpha)$ *cannot be improved in the sense that*

$$L_{\frac{7}{13-12\alpha}} = \max_c \left\{ L_{\frac{1}{2(1-c)\alpha+c}} \,\middle|\, L_{\frac{1}{2(1-c)\alpha+c}} < \alpha C + (1-\alpha)H \right\} \quad \text{for } \alpha \in (1/2, 1)$$

i.e. $c = 13/7$.

Proof.

1) For $\alpha = 1/2$, we have

$$L_{7/[13-12(1/2)]}(a,b) = L_1(a,b) = \frac{a+b}{2} = \frac{C(a,b)+H(a,b)}{2}.$$

2) Treated as in Theorem 1.*2)*, the proposed inequality becomes

$$\left[\frac{t^{\frac{20-12\alpha}{13-12\alpha}} - 1}{\left(\frac{20-12\alpha}{13-12\alpha}\right)(t-1)}\right]^{(13-12\alpha)/7} > \alpha\left(\frac{t^2+1}{t+1}\right) + (1-\alpha)\left(\frac{2t}{t+1}\right) \quad \alpha \in (0, 1/2), \tag{9}$$

for $t = b/a > 1$. Inequality (9) is equivalent to $F(t) > 0$ in (1) with $p = 7/(13-12\alpha)$. Using Lemma 1, we have a formula for $F'(t)$, $G'(t)$, $G''(t)$ and by taking derivative of $G''(t)$, we obtain

$$G'''(t) = \frac{6At^{A-4}}{(13-12\alpha)^3} H(t), \tag{10}$$

where

$$A = \frac{20-12\alpha}{13-12\alpha}, \tag{11}$$

and

$$\begin{aligned} H(t) = {} & 2(1-2\alpha)(7-3\alpha)(23-18\alpha)(11-8\alpha)t^3 \\ & - 14(3\alpha^2 - 18\alpha + 10)(11-8\alpha)t^2 + 14(6\alpha^2 - 15\alpha + 13)(1-2\alpha)t \end{aligned}$$

$$- 7\alpha(1-2\alpha)(19-24\alpha) + \alpha(13-12\alpha)^3 t^{4-A}. \tag{12}$$

Note that the coefficients of t^3 and t^{4-A} in (12) are positive but that of t^2 is negative for $\alpha \in (0, 1/2)$. Since $t^{4-A} > t^2$ and $t^2 > 2t - 1$, we further have

$$\begin{aligned} H(t) &> 2(1-2\alpha)(7-3\alpha)(23-18\alpha)(11-8\alpha)t(2t-1) \\ &\quad + [\alpha(13-12\alpha)^3 - 14(3\alpha^2-18\alpha+10)(11-8\alpha)]t^2 \\ &\quad + 14(6\alpha^2-15\alpha+13)(1-2\alpha)t - 7\alpha(1-2\alpha)(19-24\alpha) \\ &= (1-2\alpha)\Big[(-864\alpha^3 + 6,072\alpha^2 - 10,723\alpha + 5,544)t^2 \\ &\qquad - 32(5-3\alpha)(9\alpha^2-29\alpha+21)t - 7\alpha(19-24\alpha)\Big]. \end{aligned}$$

Now consider the term on the right side of the above equality sign. The coefficient $(1 - 2\alpha)(-864\alpha^3 + 6,072\alpha^2 - 10,723\alpha + 5,544)$ of t^2 becomes positive for $\alpha \in (0, 1/2)$. Together with $t < t^2$, we finally have

$$\begin{aligned} H(t) &> (1-2\alpha)\Big[(-864\alpha^3 + 6,072\alpha^2 - 10,723\alpha + 5,544) \\ &\qquad - 32(5-3\alpha)(9\alpha^2-29\alpha+21) - 7\alpha(19-24\alpha)\Big]t \\ &= 168(1-\alpha)(13-12\alpha)(1-2\alpha)t > 0. \end{aligned}$$

Therefore $G'''(t) > 0$ for all $\alpha \in (0, 1/2)$.

3) Here the proposed inequality becomes

$$\left[\frac{t^{\frac{20-12\alpha}{13-12\alpha}} - 1}{\left(\frac{20-12\alpha}{13-12\alpha}\right)(t-1)}\right]^{(13-12\alpha)/7} < \alpha\left(\frac{t^2+1}{t+1}\right) + (1-\alpha)\left(\frac{2t}{t+1}\right) \quad \alpha \in (1/2, 1), \tag{13}$$

for $t = b/a > 1$. Inequality (13) is equivalent to $F(t) < 0$ in (1). Terms $G'''(t)$, A and $H(t)$ are still of the forms (10), (11) and (12), respectively. It is then sufficient to show that $H(t) < 0$ for all $\alpha \in (1/2, 1)$. To that end, we divide our proof into two cases $\alpha \in (8/9, 1)$ and $\alpha \in (1/2, 8/9]$.

3.1) Case $\alpha \in (8/9, 1)$: For such α, we have $4 - A \in (-4, 0)$ and

$$\begin{aligned} t^{4-A} &< 1 < 1 + (4-A)(t-1) + \frac{(4-A)(3-A)}{2}(t-1)^2 \\ &= \frac{(432\alpha^2 - 726\alpha + 304)t^2 + (-432\alpha^2 + 600\alpha - 192)t + (144\alpha^2 - 186\alpha + 57)}{(13-12\alpha)^2}. \end{aligned}$$

Hence

$$\begin{aligned} H(t) < 2(2\alpha-1)\Big[&-(7-3\alpha)(23-18\alpha)(11-8\alpha)t^3 \\ &+2(-648\alpha^3+1,509\alpha^2-1,191\alpha+385)t^2 \\ &-(-1,296\alpha^3+2,598\alpha^2-1,353\alpha+91)t \\ &-2\alpha(24\alpha-19)(9\alpha-8)\Big]. \end{aligned}$$

Now consider the term on the right side of the above inequality sign. As the coefficient of t^3 is negative for $\alpha \in (1/2, 1)$ and $t^3 > t(2t-1)$, we have

$$\begin{aligned} H(t) &< 2(2\alpha-1)\Big[-(7-3\alpha)(23-18\alpha)(11-8\alpha)t(2t-1) \\ &\qquad +2(-648\alpha^3+1,509\alpha^2-1,191\alpha+385)t^2 \\ &\qquad -(-1,296\alpha^3+2,598\alpha^2-1,353\alpha+91)t \\ &\qquad -2\alpha(24\alpha-19)(9\alpha-8)\Big] \\ &= 2(2\alpha-1)\Big[(-432\alpha^3-1,290\alpha^2+4,484\alpha-2,772)t^2 \\ &\qquad +(864\alpha^3-444\alpha^2-2,080\alpha+1,680)t \\ &\qquad -2\alpha(24\alpha-19)(9\alpha-8)\Big]. \end{aligned}$$

Examine the term on the right side of the equality sign above. The coefficient $-432\alpha^3 - 1,290\alpha^2 + 4,484\alpha - 2,772$ is an increasing function of $\alpha \in (1/2, 1)$ with negative value (-10) at $\alpha = 1$. Since the coefficient of t^2 is negative for $\alpha \in (8/9, 1)$ and $t^2 > 2t - 1$, we obtain

$$\begin{aligned} H(t) &< 2(2\alpha-1)\Big[(-432\alpha^3-1,290\alpha^2+4,484\alpha-2,772)(2t-1) \\ &\qquad +(864\alpha^3-444\alpha^2-2,080\alpha+1,680)t \\ &\qquad -2\alpha(24\alpha-19)(9\alpha-8)\Big] \\ &= 2(2\alpha-1)\Big[-168(1-\alpha)(23-18\alpha)t+252(1-\alpha)(11-8\alpha)\Big]. \end{aligned}$$

Look at the term on the right side of the above inequality sign. Since the coefficient $-168(1-\alpha)(23-18\alpha)$ of t is negative and $t > 1$, we finally have

$$\begin{aligned} H(t) &< 2(2\alpha-1)\Big[-168(1-\alpha)(23-18\alpha)t+252(1-\alpha)(11-8\alpha)\Big] \\ &< -168(2\alpha-1)(1-\alpha)(13-12\alpha) < 0. \end{aligned}$$

3.2) Case $\alpha \in (1/2, 8/9]$: For α in this interval, the coefficient of t^3 in (12) is negative and $4 - A \in [0, 2)$. Hence $t^{4-A} < t^2$. These consequences and $t^3 > t(2t-1)$ imply that

$$\begin{aligned} H(t) < & -2(2\alpha-1)(7-3\alpha)(23-18\alpha)(11-8\alpha)t(2t-1) \\ & + \left[\alpha(13-12\alpha)^3 - 14(3\alpha^2-18\alpha+10)(11-8\alpha)\right]t^2 \\ & - 14(6\alpha^2-15\alpha+13)(2\alpha-1)t - 7\alpha(2\alpha-1)(24\alpha-19) \\ = & \ (864\alpha^3 - 6,072\alpha^2 + 10,723\alpha - 5,544)t^2 \\ & + 32(5-3\alpha)(9\alpha^2-29\alpha+21)t - 7\alpha(24\alpha-19). \end{aligned}$$

Examine the term on the right side of the inequality sign shown above. Notice that the coefficient $864\alpha^3 - 6,072\alpha^2 + 10,723\alpha - 5,544$ is an increasing function for $\alpha \in (1/2, 1)$ with negative value (-29) at $\alpha = 1$. The coefficient of t^2 is negative for $\alpha \in (1/2, 8/9]$. We have $t^2 > t$ and then

$$\begin{aligned} H(t) < & \left[(864\alpha^3 - 6,072\alpha^2 + 10,723\alpha - 5,544) \right. \\ & \left. + 32(5-3\alpha)(9\alpha^2-29\alpha+21)\right]t - 7\alpha(24\alpha-19) \\ = & \ (-1,848\alpha^2 + 4,067\alpha - 2,184)t - 7\alpha(24\alpha-19). \end{aligned}$$

Consider the term on the right side of the equality sign above. Coefficient $-1,848\alpha^2 + 4,067\alpha - 2,184$ is increasing for $\alpha \in (1/2, 8/9]$ with negative value (-29.03) at $\alpha = 8/9$, it is therefore negative on the whole interval $(1/2, 8/9]$. Since $t > 1$, we get

$$\begin{aligned} H(t) < & \ (-1,848\alpha^2 + 4,067\alpha - 2,184) - 7\alpha(24\alpha-19) \\ < & -168(1-\alpha)(13-12\alpha) < 0. \end{aligned}$$

Finally, we will prove that the parameter $7/(13-12\alpha)$ cannot be improved in this case. Suppose, to the contrary, that inequality (13) is true for the parameter

$$\frac{1}{2[1-(1+k+\epsilon)]\alpha + (1+k+\epsilon)}$$

for a sufficiently small $\epsilon > 0$. That is

$$L_{\frac{1}{2[1-(1+k+\epsilon)]\alpha+(1+k+\epsilon)}}(1,t) < \alpha C(1,t) + (1-\alpha)H(1,t) \quad \text{for all} \quad t > 1.$$

Taking logarithm of both sides of the above inequality, we get

$$\ln\left[L_{\frac{1}{2[1-(13/7+\epsilon)]\alpha+(13/7+\epsilon)}}(1,t)\right] - \ln\left[\alpha C(1,t) + (1-\alpha)H(1,t)\right] < 0.$$

With the notation in Lemma 1, this is just

$$F(t) < 0 \quad \text{for all } t > 1 \quad \text{where} \quad p = \frac{1}{2\left[1-(13/7+\epsilon)\right]\alpha + (13/7+\epsilon)}. \tag{14}$$

From Lemma 1, $G(1) = G'(1) = G''(1) = 0$. Taking derivative of $G''(t)$, we have

$$\begin{aligned} G'''(t) &= (-3\alpha p + 2p - \alpha)(p+3)(p+2)(p+1)t^p \\ &\quad + (5\alpha p - 2p + \alpha - 2)(p+2)(p+1)pt^{p-1} \\ &\quad + (-\alpha p + \alpha - 2)(p+1)p(p-1)t^{p-2} \\ &\quad - \alpha(p+1)\left[p(p-1)(p-2)t^{p-3} - 6\right]. \end{aligned}$$

Hence

$$G'''(1) = 2(p+1)p(p - 12\alpha + 5) \quad \text{for } \alpha \in (1/2, 1).$$

However,

$$\lim_{\alpha \to 1^-} 2(p+1)p(p-12\alpha+5) = \frac{686(7\epsilon - 8)\epsilon}{(7\epsilon - 1)^3} > 0.$$

Therefore $G(t) < 0$ or $F(t) < 0$ in a small neighborhood of 1 if $\epsilon < 1/7$. This contradict to statement (14).

The proof is complete.

Remark 1. *In statement* 3) *of Theorem 1,* $L_{4\alpha-1}$ *is not the optimal lower bound of the considered weighted arithmetic mean for* $\alpha \in (1/2, 1)$*. Neither is* $L_{7/(13-12\alpha)}$ *the optimal upper bound of the one for* $\alpha \in (0, 1/2)$ *in statement* 2) *of Theorem 2.*

Due to monotonicity property of generalized logarithmic means, we expect a sharper result. Partial results are shown in the following theorem.

Theorem 3. *Let* $a, b > 0$ *with* $a \neq b$ *and* $k = 2/(2\ln 2 - 1)$*. Then*

1) $L_{1/[-2k\alpha+(k+1)]}(a,b) = \alpha C(a,b) + (1-\alpha)H(a,b) = L_{2k\alpha+(1-k)}(a,b)$ *for* $\alpha = 1/2$*;*
2) $L_{1/[-2k\alpha+(k+1)]}(a,b) > \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha \in (0, 2/k) \approx (0, 0.38)$*;*
3) $L_{2k\alpha+(1-k)}(a,b) < \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha \in ((k+2)/2k, 1) \approx (0.7, 1)$.

Proof.

1) This is obvious after inserting $\alpha = 1/2$ into the left and right sides of the statement.

2) Treated as in Theorem 1(2), the proposed inequality becomes

$$\left[\frac{t^{1+\left\{1/[-2k\alpha+(k+1)]\right\}} - 1}{\Big(1 + \left\{1/\left[-2k\alpha + (k+1)\right]\right\}\Big)(t-1)}\right]^{-2k\alpha+(k+1)} > (\alpha)\frac{t^2+1}{t+1} + (1-\alpha)\frac{2t}{t+1} \tag{15}$$

for all $\alpha \in (0, 2/k)$ and $t = b/a > 1$. Inequality (15) is equivalent to $F(t) > 0$ in (1) with $p = 1/\left[-2k\alpha + (k+1)\right]$. Using Lemma 1, we have formulae for $F'(t)$, $G'(t)$, $G''(t)$. Taking derivative of $G''(t)$, we have

$$G'''(t) = \frac{2 + k - 2k\alpha}{(2k\alpha - k - 1)^4} t^{\frac{-2+(6\alpha-3)k}{-1+(2\alpha-1)k}} J(t),$$

where

$$\begin{aligned}J(t) &= (1-2\alpha)(2-k\alpha)(6k\alpha-3k-4)(4k\alpha-2k-3)t^3\\&\quad-(2\alpha^2k-5k\alpha-6\alpha+2k+4)(-4k\alpha+2k+3)t^2\\&\quad+(1-2\alpha)k(2\alpha^2k-5k\alpha+2k+2)t-\alpha(1-2\alpha)k(-4k\alpha+2k+1)\\&\quad+6\alpha(1+k-2k\alpha)^3t^{\frac{-2+(6\alpha-3)k}{-1+(2\alpha-1)k}}.\end{aligned}$$

For $\alpha \in (0, 2/k)$, exponent $[-1+(4\alpha-2)k]/[-1+(2\alpha-1)k] > 1$ and hence

$$\begin{aligned}t^{\frac{-2+(6\alpha-3)k}{-1+(2\alpha-1)k}} &= t\left[t^{\frac{-1+(4\alpha-2)k}{-1+(2\alpha-1)k}}\right]\\&> t\left\{1+\left[\frac{-1+(4\alpha-2)k}{-1+(2\alpha-1)k}\right](t-1)\right\}\\&= \left[\frac{-1+(4\alpha-2)k}{-1+(2\alpha-1)k}\right]t^2+\left[\frac{(1-2\alpha)k}{-1+(2\alpha-1)k}\right]t.\end{aligned}$$

Furthermore coefficient $6\alpha(1+k-2k\alpha)^3 > 0$ for such α. Therefore

$$\begin{aligned}J(t) > (1-2\alpha)\Big[&(2-k\alpha)(6k\alpha-3k-4)(4k\alpha-2k-3)t^3\\&+(48\alpha^3k^3-48\alpha^2k^3-64\alpha^2k^2+12\alpha k^3+40\alpha k^2+39k\alpha-4k^2-14k-12)t^2\\&-k(24\alpha^3k^2-24\alpha^2k^2-26\alpha^2k+6\alpha k^2+17k\alpha+6\alpha-2k-2)t\\&-\alpha k(-4k\alpha+2k+1)\Big].\end{aligned}$$

Consider the term on the right side of the inequality sign above. Since $t^2 > 2t-1$ and coefficient $(2-k\alpha)(6k\alpha-3k-4)(4k\alpha-2k-3)$ of t^3 is positive, we have

$$\begin{aligned}J(t) > (1-2\alpha)\Big[&(100\alpha^2k^2-90\alpha k^2-121k\alpha+20k^2+54k+36)t^2\\&+(-56\alpha^2k^2+48\alpha k^2+74k\alpha-10k^2-32k-24)t-\alpha k(-4k\alpha+2k+1)\Big].\end{aligned}$$

Because $t^2 > t$ and coefficient $100\alpha^2k^2-90\alpha k^2-121k\alpha+20k^2+54k+36$ of t^2 on the right side of the above inequality is positive for $\alpha \in (0, 2/k)$, we obtain

$$\begin{aligned}J(t) > (1-2\alpha)\Big[&(44\alpha^2k^2-42\alpha k^2-47k\alpha+10k^2+22k+12)t\\&-\alpha k(-4k\alpha+2k+1)\Big].\end{aligned}$$

Observe that coefficient $44\alpha^2k^2-42\alpha k^2-47k\alpha+10k^2+22k+12$ of t is positive for such α. Since $t > 1$, we finally have

$$J(t) > 2(1-2\alpha)(12k\alpha-5k-6)(2k\alpha-k-1) > 0.$$

Therefore $G'''(t) > 0$ for all $\alpha \in (0, 2/k)$.

3) Treated as in Theorem 1(2), the proposed inequality becomes

$$\left\{\frac{t^{2k\alpha+(2-k)}-1}{\big[2k\alpha+(2-k)\big](t-1)}\right\}^{1/[2k\alpha+(1-k)]} < \alpha\left(\frac{t^2+1}{t+1}\right)+(1-\alpha)\left(\frac{2t}{t+1}\right) \tag{16}$$

for all $\alpha \in ((k+2)/2k, 1)$ and $t = b/a > 1$. Inequality (16) is equivalent to $F(t) > 0$ in (1) with $p = 2k\alpha + (1-k)$. Using Lemma 1, we have formulae for $F'(t)$, $G'(t)$, $G''(t)$. Taking derivative of $G''(t)$, we have

$$G'''(t) = (2k\alpha - k + 2)t^{2k\alpha-k-2}K(t),$$

where

$$\begin{aligned}K(t) = &-(2\alpha-1)(3k\alpha-2k+2)(2k\alpha-k+4)(2k\alpha-k+3)t^3\\ &+(10\alpha^2k-9k\alpha+6\alpha+2k-4)(2k\alpha-k+3)(2k\alpha-k+1)t^2\\ &-(2\alpha^2k-k\alpha+2)(2k\alpha-k+1)(2\alpha-1)kt\\ &-\alpha(2k\alpha-k+1)(2k\alpha-k-1)(2\alpha-1)k+6\alpha t^{k-2k\alpha+2}.\end{aligned} \tag{17}$$

Since $t^3 > t(2t-1)$ and the coefficient of t^3 in (17) is negative for $\alpha \in ((k+2)/2k, 1)$, we have

$$\begin{aligned}K(t) < &-(2k\alpha-k+3)\\ &(4\alpha^3k^2-12\alpha^2k^2+42\alpha^2k+9\alpha k^2-49k\alpha-2k^2+26\alpha+14k-2)t^2\\ &+2(2\alpha-1)(2k\alpha-k+2)(2\alpha^2k^2-3\alpha k^2+10k\alpha+k^2-7k+6)t\\ &-\alpha(2k\alpha-k+1)(2k\alpha-k-1)(2\alpha-1)k+6\alpha t^{k-2k\alpha+2}.\end{aligned} \tag{18}$$

Since $t^2 > 2t-1$ and the coefficient of t^2 in (18) is negative for $\alpha \in ((k+2)/2k, 1)$, it follows that

$$\begin{aligned}K(t) < &(16\alpha^3k^3-96\alpha^3k^2-24\alpha^2k^3+184\alpha^2k^2+12\alpha k^3-228\alpha^2k\\ &-112\alpha k^2-2k^3+250k\alpha+22k^2-108\alpha-68k+48)t\\ &+(-16\alpha^3k^3+96\alpha^3k^2+24\alpha^2k^3-176\alpha^2k^2-12\alpha k^3+180\alpha^2k\\ &+104\alpha k^2+2k^3-198k\alpha-20k^2+78\alpha+54k-36)+6\alpha t^{k-2k\alpha+2}.\end{aligned} \tag{19}$$

The coefficient of t in (19) is negative and $t^{k-2k\alpha+2} < 1$ for $\alpha \in ((k+2)/2k, 1)$. Therefore,

$$\begin{aligned}K(t) &< (8\alpha^2k^2-48\alpha^2k-8\alpha k^2+52k\alpha+2k^2-30\alpha-14k+12)+6\alpha t^{k-2k\alpha+2}\\ &< (8\alpha^2k^2-48\alpha^2k-8\alpha k^2+52k\alpha+2k^2-30\alpha-14k+12)+6\alpha\\ &= 2(k-6)(2\alpha-1)(2k\alpha-k+1) < 0.\end{aligned}$$

As a result, $G'''(t) < 0$ for all $\alpha \in ((k+2)/2k, 1)$.

Conjecture 1. *Let $a, b > 0$ with $a \neq b$ and $k = 2/(2\ln 2 - 1)$.*

1) $L_{1/[-2k\alpha+(k+1)]}(a,b) > \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha \in (0, 1/2)$*;*

2) $L_{2k\alpha+(1-k)}(a,b) < \alpha C(a,b) + (1-\alpha)H(a,b)$ *for* $\alpha \in (1/2, 1)$*.*

If conjecture 1 is correct, $L_{1/[-2k\alpha+(k+1)]}$ will be the optimal upper bound for the considered weighted arithmetic mean for$\alpha \in (0, 1/2)$ and $L_{2k\alpha+(1-k)}$ will be the optimal lower bound for the considered weighted arithmetic mean for $\alpha \in (1/2, 1)$, as shown in the following theorem.

Theorem 4. *Let $a, b > 0$ with $a \neq b$ and $k = 2/(2\ln 2 - 1)$.*

1) *If $L_{1/[-2k\alpha+(k+1)]}(a,b) > \alpha C(a,b) + (1-\alpha)H(a,b)$ for $\alpha \in (0, 1/2)$, then the parameter $1/[-2k\alpha + (k+1)]$ cannot be improved in the sense that*

$$L_{\frac{1}{-2k\alpha+(k+1)}} = \min_c \left\{ L_{\frac{1}{2(1-c)\alpha+c}} \,\middle|\, L_{\frac{1}{2(1-c)\alpha+c}} > \alpha C + (1-\alpha)H \right\} \quad \textit{for } \alpha \in (0, 1/2)$$

i.e. $c = 1 + k$*;*

2) *If $L_{2k\alpha+(1-k)}(a,b) < \alpha C(a,b) + (1-\alpha)H(a,b)$ for $\alpha \in (1/2, 1)$, then the parameter $2k\alpha + (1-k)$ cannot be improved in the sense that*

$$L_{2k\alpha+(1-k)} = \max_c \left\{ L_{2(1-c)\alpha+c} \mid L_{2(1-c)\alpha+c} < \alpha C + (1-\alpha)H \right\} \quad \textit{for } \alpha \in (1/2, 1)$$

i.e. $c = 1 - k$*.*

Proof.

1) Suppose, to the contrary, that inequality (15) is true for the parameter

$$\frac{1}{2[1-(1+k+\epsilon)]\alpha + (1+k+\epsilon)}$$

for a sufficiently small $\epsilon > 0$. That is

$$L_{\frac{1}{2[1-(1+k+\epsilon)]\alpha+(1+k+\epsilon)}}(1,t) > \alpha C(1,t) + (1-\alpha)H(1,t) \quad \text{for all} \quad t > 1.$$

Hence

$$\frac{1}{t}\left\{L_{\frac{1}{2[1-(1+k+\epsilon)]\alpha+(1+k+\epsilon)}}(1,t)\right\} \geq \frac{1}{t}\left\{\alpha C(1,t) + (1-\alpha)H(1,t)\right\} \quad \text{for all} \quad t > 1.$$

Taking limits on both sides of the above inequality lead to

$$\lim_{t\to+\infty} \frac{1}{t}\left\{L_{\frac{1}{2[1-(1+k+\epsilon)]\alpha+(1+k+\epsilon)}}(1,t)\right\} \geq \lim_{t\to+\infty} \frac{1}{t}\Big[\alpha C(1,t) + (1-\alpha)H(1,t)\Big],$$

which is equivalent to

$$\left[\frac{(1-2\alpha)(k+\epsilon)+1}{(1-2\alpha)(k+\epsilon)+2}\right]^{(1-2\alpha)(k+\epsilon)+1} \geq \alpha.$$

Taking logarithm of both sides of the above inequality, we get

$$Q(\alpha) := [(1-2\alpha)(k+\epsilon)+1]\ln\left[\frac{(1-2\alpha)(k+\epsilon)+1}{(1-2\alpha)(k+\epsilon)+2}\right] - \ln\alpha \geq 0.$$

Sine $Q(\alpha) \geq 0$ for all $\alpha \in (0, 1/2)$ and $Q((1/2)^-) = 0$, it immediately follows that $Q'((1/2)^-) \leq 0$. However,

$$Q'(\alpha) = \frac{-2(k+\epsilon)}{(1-2\alpha)(k+\epsilon)+2} - 2(k+\epsilon)\ln\left[\frac{(1-2\alpha)(k+\epsilon)+1}{(1-2\alpha)(k+\epsilon)+2}\right] - \frac{1}{\alpha}$$

leading to

$$\lim_{\alpha\to(1/2)^-} Q'(\alpha) = \frac{2\epsilon}{k} > 0,$$

which is a contradiction.

2) Suppose, to the contrary, that inequality (16) is true for the parameter $2[1-(1-k-\epsilon)]\alpha + (1-k-\epsilon)$ for a sufficiently small $\epsilon > 0$. That is

$$L_{2[1-(1-k-\epsilon)]\alpha+(1-k-\epsilon)}(1,t) < \alpha C(1,t) + (1-\alpha)H(1,t) \quad \text{for all} \quad t > 1.$$

Hence

$$\frac{1}{t}\Big\{L_{2[1-(1-k-\epsilon)]\alpha+(1-k-\epsilon)}(1,t)\Big\} < \frac{1}{t}\big[\alpha C(1,t) + (1-\alpha)H(1,t)\big] \quad \text{for all} \quad t > 1.$$

Taking limits on both sides of the above inequality lead to

$$\lim_{t\to+\infty}\frac{1}{t}\left\{L_{2[1-(-1-\epsilon)]\alpha+(-1-\epsilon)}(1,t)\right\} \leq \lim_{t\to+\infty}\frac{1}{t}\Big[\alpha C(1,t) + (1-\alpha)H(1,t)\Big],$$

which is equivalent to

$$\left[\frac{1}{(2\alpha-1)(k+\epsilon)\alpha+2}\right]^{1/[(2\alpha-1)(k+\epsilon)\alpha+1]} \leq \alpha.$$

Taking logarithm of both sides of the above inequality, we get

$$R(\alpha) := -\ln[(2\alpha-1)(k+\epsilon)+2] - [(2\alpha-1)(k+\epsilon)+1]\ln(\alpha) \leq 0.$$

Sine $R(\alpha) \leq 0$ for all $\alpha \in (1/2, 1)$ and $R((1/2)^+) = 0$, it immediately follows that $R'((1/2)^+) \leq 0$. However,

$$R'(\alpha) = -\frac{(k+\epsilon)\Big\{4\alpha\big[1+(\alpha-1/2)(k+\epsilon)\big]\ln(\alpha)+4(\alpha-1/2)^2(k+\epsilon)+(8\alpha-3)\Big\}+2}{\alpha\left[(2\alpha-1)(k+\epsilon)+2\right]}$$

leading to

$$\lim_{\alpha\to(1/2)^+} R'(\alpha) = \frac{2\epsilon}{k} > 0,$$

which is a contradiction.

4. Conclusions

In this paper, we seek the optimal upper and lower bounds of weighted arithmetic means of contra-harmonic and harmonic means by generalized logarithmic means L_p when p is of the linear form $p = 2(1-c)\alpha + c$ and p is of the reciprocal of linear form $p = 1/[2(1-c)\alpha + c]$ respectively. When p has a linear form, we found that

$$L_{4\alpha-1} = \min_c \left\{L_{2(1-c)\alpha+c} \mid L_{2(1-c)\alpha+c} > \alpha C + (1-\alpha)H\right\} \text{ for } \alpha \in (0, 1/2).$$

When p has a reciprocal of linear form, we found that

$$L_{\frac{7}{13-12\alpha}} = \max_c \left\{ L_{\frac{1}{2(1-c)\alpha+c}} \,\middle|\, L_{\frac{1}{2(1-c)\alpha+c}} < \alpha C + (1-\alpha)H\right\} \text{ for } \alpha \in (1/2, 1).$$

We also show that, if conjecture 1 is correct, then $L_{1/[-2k\alpha+(k+1)]}$ will be the optimal upper bound for the considered weighted arithmetic mean for $\alpha \in (0, 1/2)$ and $L_{2k\alpha+(1-k)}$ will be the optimal lower bound for the considered weighted arithmetic mean for $\alpha \in (1/2, 1)$.

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