EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 3, 2022, 924-937
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Łukasiewicz fuzzy BE-algebras and BE-filters

Young Bae Jun[1], Sun Shin Ahn[2,*]

[1] *Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*

[2] *Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. By applying the concept of Łukasiewicz fuzzy set to BE-algebras, the notions of Łukasiewicz fuzzy BE-algebra and Łukasiewicz fuzzy BE-filter are introduced, and their properties are investigated. Characterizations of Łukasiewicz fuzzy BE-algebra and Łukasiewicz fuzzy BE-filter are discussed, and the relationship between fuzzy BE-algebra (resp., fuzzy BE-filter) and Łukasiewicz fuzzy BE-algebra (resp., Łukasiewicz fuzzy BE-filter) is established. The conditions for the $\in$-set, q-set and O-set of Łukasiewicz fuzzy set to be BE-subalgebras are explored. Lukasiewicz fuzzy BE-filter is created by using BE-filter.

2020 Mathematics Subject Classifications: 03G25, 06F35, 08A72

Key Words and Phrases: Łukasiewicz fuzzy BE-algebra, Łukasiewicz fuzzy BE-filter, $\in$-set, q-set, O-set.

---

1. Introduction

BCK-algebra and BCI-algebra, introduced by Y. Imai, K. Iséki and S. Tanaka in 1966, are algebraic structures of universal algebra which describe fragments of propositional calculus related to implications known as BCK and BCI-logic. After that, various generalizations were attempted, and BCC-algebras, BCH-algebras, BH-algebras, d-algebras etc. appeared. In 2007, H. S. Kim and Y. H. Kim [3] introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra. They defined and studied the concept of a filter in BE-algebras. In [7] and [6], S. S. Ahn et al. and A. Rezaei et al. studied fuzzy BE-algebras. G. Dymek and A. Walendziak [1] developed the theory of fuzzy filters in BE-algebras. In the website https://plato.stanford.edu/entries/lukasiewicz/, we can see that Jan Łukasiewicz (1878–1956) was a Polish logician and philosopher who introduced mathematical logic into Poland, became the earliest founder of the Warsaw school of logic, and one of the principal architects and teachers of that school. His most famous achievement was to give the first rigorous formulation of many-valued logic. He introduced many improvements in propositional logic, and became the first historian of logic to treat the

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i3.4446

Email addresses: skywine@gmail.com (Y. B. Jun), sunshine@dongguk.edu (S. S. Ahn)

subject's history from the standpoint of modern formal logic. Łukasiewicz logic, which is the logic of the Łukasiewicz t-norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic. Using the idea of Łukasiewicz t-norm, Y. B. Jun [2] constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras.

In this paper, we apply the concept of Łukasiewicz fuzzy set to BE-algebras. We introduce the notion of Łukasiewicz fuzzy BE-algebra and Łukasiewicz fuzzy BE-filter, and investigate several properties. We discuss the characterization of Łukasiewicz fuzzy BE-algebra and Łukasiewicz fuzzy BE-filter. We conside the relationship between fuzzy BE-algebra (resp., fuzzy BE-filter) and Łukasiewicz fuzzy BE-algebra (resp., Łukasiewicz fuzzy BE-filter). We explore the conditions for the $\in$-set, q-set and O-set of Łukasiewicz fuzzy set to be BE-subalgebras. We use BE-filter to create Lukasiewicz fuzzy BE-filter.

2. Preliminary

A *BE-algebra* (see [3]) is defined to be a set X together with a binary operation "$*$" and a special element "1" satisfying the conditions:

(BE1) $(\forall a \in X)\ (a * a = 1)$,

(BE2) $(\forall a \in X)\ (a * 1 = 1)$,

(BE3) $(\forall a \in X)\ (1 * a = a)$,

(BE4) $(\forall a, b, c \in X)\ (a * (b * c) = b * (a * c))$.

The order relation "$\leq$" in a BE-algebra X is defined as follows:

$$(\forall a, b \in X)(a \leq b \ \Leftrightarrow\ a * b = 1). \tag{1}$$

Every BE-algebra X satisfies the following conditions (see [3]):

$$(\forall a, b \in X)\,(a * (b * a) = 1)\,. \tag{2}$$

$$(\forall a, b \in X)\,(a * ((a * b) * b) = 1)\,. \tag{3}$$

A subset A of a BE-algebra X is called

- a *BE-subalgebra* of X if it satisfies:

$$(\forall a, b \in A)(a * b \in A), \tag{4}$$

- a *BE-filter* of X (see [3]) if it satisfies:

$$1 \in A, \tag{5}$$

$$(\forall a, b \in X)(a * b \in A,\ a \in A \ \Rightarrow\ b \in A). \tag{6}$$

A fuzzy set ξ in a BE-algebra X is called

- a *fuzzy BE-algebra* of X (see [7]) if it satisfies:

$$(\forall a, b \in X)(\xi(a * b) \geq \min\{\xi(a), \xi(b)\}). \tag{7}$$

- a *fuzzy BE-filter* of X (see [1]) if it satisfies:

$$(\forall a \in X)(\xi(1) \geq \xi(a)), \tag{8}$$

$$(\forall a, b \in X)(\xi(b) \geq \min\{\xi(a * b), \xi(a)\}). \tag{9}$$

A fuzzy set ξ in a set X of the form

$$\xi(b) := \begin{cases} t \in (0,1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $[a/t]$.

For a fuzzy set ξ in a set X, we say that a fuzzy point $[a/t]$ is

(i) *contained* in ξ, denoted by $[a/t] \in \xi$, (see [5]) if $\xi(a) \geq t$.

(ii) *quasi-coincident* with ξ, denoted by $[a/t]\, q\, \xi$, (see [5]) if $\xi(a) + t > 1$.

If $[a/t]\, \alpha\, \xi$ is not established for $\alpha \in \{\in, q\}$, it is denoted by $[a/t]\, \overline{\alpha}\, \xi$.

Let ξ be a fuzzy set in a set X and let $\varepsilon \in (0,1)$. A function

$$Ł_\xi^\varepsilon : X \to [0,1],\ x \mapsto \max\{0, \xi(x) + \varepsilon - 1\}$$

is called the *Łukasiewicz fuzzy set* of ξ in X.

For the Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X and $t \in (0,1]$, consider the sets

$$(Ł_\xi^\varepsilon, t)_\in := \{x \in X \mid [x/t] \in Ł_\xi^\varepsilon\},$$

$$(Ł_\xi^\varepsilon, t)_q := \{x \in X \mid [x/t]\, q\, Ł_\xi^\varepsilon\},$$

which are called the $\in$-*set* and q-*set*, respectively, of $Ł_\xi^\varepsilon$ (with value t). Also, consider a set:

$$O(Ł_\xi^\varepsilon) := \{x \in X \mid Ł_\xi^\varepsilon(x) > 0\} \tag{10}$$

which is called an O-*set* of $Ł_\xi^\varepsilon$. It is observed that

$$O(Ł_\xi^\varepsilon) = \{x \in X \mid \xi(x) + \varepsilon - 1 > 0\}.$$

3. Łukasiewicz fuzzy BE-algebras

In what follows, let X and ξ be a BE-algebra and a fuzzy set in X respectively, and ε is an element of $(0,1)$ unless otherwise specified.

Definition 1. *The Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X is called a Łukasiewicz fuzzy BE-algebra of X if it satisfies:*

$$[x/t_a] \in Ł_\xi^\varepsilon,\ [y/t_b] \in Ł_\xi^\varepsilon\ \Rightarrow [(x * y)/\min\{t_a, t_b\}] \in Ł_\xi^\varepsilon \tag{11}$$

for all $x, y \in X$ and $t_a, t_b \in (0,1]$.

Example 1. *Consider a set $X = \{1, b_1, b_2, b_3, b_4, b_5\}$ with a binary operation "$*$" given in the table below.*

$*$	1	b_1	b_2	b_3	b_4	b_5
1	1	b_1	b_2	b_3	b_4	b_5
b_1	1	1	b_1	b_3	b_3	b_4
b_2	1	1	1	b_3	b_3	b_3
b_3	1	b_1	b_2	1	b_1	b_2
b_4	1	1	b_1	1	1	b_1
b_5	1	1	1	1	1	1

Then $(X, *, 1)$ is a BE-algebra (see [7]). Define a fuzzy set ξ in X as follows:

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.76 & \text{if } x \in \{1, b_1, b_2\}, \\ 0.52 & \text{otherwise.} \end{cases}$$

Given $\varepsilon := 0.67$, the Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X is given as follows:

$$Ł_\xi^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{1, b_1, b_2\}, \\ 0.19 & \text{otherwise.} \end{cases}$$

It is routine to verify that $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-algebra of X.

We provide a characterization of Łukasiewicz fuzzy BE-algebra.

Theorem 1. *Given the Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X, the following assertions are equivalent.*

(i) *$Ł_\xi^\varepsilon$ satisfies $Ł_\xi^\varepsilon(x * y) \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\}$ for all $x, y \in X$.*

(ii) *$Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-algebra of X.*

Proof. (i) $\Rightarrow$ (ii). Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in Ł_\xi^\varepsilon$ and $[y/t_b] \in Ł_\xi^\varepsilon$. Then $Ł_\xi^\varepsilon(x) \geq t_a$ and $Ł_\xi^\varepsilon(y) \geq t_b$, which implis that

$$Ł_\xi^\varepsilon(x * y) \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} \geq \min\{t_a, t_b\}.$$

Therefore $[(x * y)_{\min\{t_b, t_b\}}] \in Ł_\xi^\varepsilon$, and consequently $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-algebra of X.

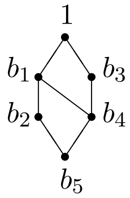
(ii) $\Rightarrow$ (i). Let $x, y \in X$. It is clear that $[x/Ł_\xi^\varepsilon(x)] \in Ł_\xi^\varepsilon$ and $[y/Ł_\xi^\varepsilon(y)] \in Ł_\xi^\varepsilon$. Hence $[(x * y)/\min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\}] \in Ł_\xi^\varepsilon$ by (11), that is, $Ł_\xi^\varepsilon(x * y) \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\}$.

Proposition 1. *If ξ is order preserving or order reversing in X, then its Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ is also order preserving or order reversing in X.*

Proof. Straightforward.

In Poposition 1, the converse may not be true as seen in the following example.

Example 2. *Consider the BE-algebra X given in Example 1. Its Hasse diagram is given as follows:*



(1) Let ξ be a fuzzy set in X defined as follows:

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.88 & \text{if } x = 1, \\ 0.78 & \text{if } x = b_1, \\ 0.63 & \text{if } x = b_2, \\ 0.48 & \text{if } x = b_3, \\ 0.55 & \text{if } x = b_4, \\ 0.47 & \text{if } x = b_5. \end{cases}$$

Given $\varepsilon := 0.43$, the Łukasiewicz fuzzy set L_ξ^ε of ξ in X is given as follows:

$$L_\xi^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.31 & \text{if } x = 1, \\ 0.21 & \text{if } x = b_1, \\ 0.06 & \text{if } x = b_2, \\ 0.00 & \text{if } x = b_3, \\ 0.00 & \text{if } x = b_4, \\ 0.00 & \text{if } x = b_5. \end{cases}$$

Then L_ξ^ε is order preversing in X, but ξ is not order preserving in X since $b_4 \leq b_3$ and $\xi(b_4) \geq \xi(b_3)$.

(2) Let ζ be a fuzzy set in X defined as follows:

$$\zeta : X \to [0,1],\ x \mapsto \begin{cases} 0.34 & \text{if } x = 1, \\ 0.31 & \text{if } x = b_1, \\ 0.55 & \text{if } x = b_2, \\ 0.48 & \text{if } x = b_3, \\ 0.53 & \text{if } x = b_4, \\ 0.63 & \text{if } x = b_5. \end{cases}$$

Given $\delta := 0.62$, the Łukasiewicz fuzzy set L_ζ^δ of ζ in X is given as follows:

$$L_\zeta^\delta : X \to [0,1],\ x \mapsto \begin{cases} 0.00 & \text{if } x = 1, \\ 0.00 & \text{if } x = b_1, \\ 0.17 & \text{if } x = b_2, \\ 0.10 & \text{if } x = b_3, \\ 0.15 & \text{if } x = b_4, \\ 0.25 & \text{if } x = b_5. \end{cases}$$

Then L_ξ^ε is order reversing in X, but ζ is not order reversing in X since $b_1 \leq 1$ and $\zeta(b_1) \leq \zeta(1)$.

Lemma 1. *If* $Ł_\xi^\varepsilon$ *is a Łukasiewicz fuzzy BE-algebra of* X*, then* $Ł_\xi^\varepsilon(1) \geq Ł_\xi^\varepsilon(x)$ *for all* $x \in X$*.*

Proof. It can be induced by (BE1) and Theorem 1.

Proposition 2. *If a Łukasiewicz fuzzy BE-algebra* $Ł_\xi^\varepsilon$ *of* ξ *is order reversing in* X*, then it is constant.*

Proof. Let $Ł_\xi^\varepsilon$ be a Łukasiewicz fuzzy BE-algebra of X which is order reversing. Since $x \leq 1$ for all $x \in X$, we have $Ł_\xi^\varepsilon(x) \geq Ł_\xi^\varepsilon(1)$ for all $x \in X$. The combination of this and Lemma 1 induces $Ł_\xi^\varepsilon(x) = Ł_\xi^\varepsilon(1)$ for all $x \in X$. Hence $Ł_\xi^\varepsilon$ is a constant on X.

Theorem 2. *If* ξ *is a fuzzy BE-algebra of* X*, then its Łukasiewicz fuzzy set* $Ł_\xi^\varepsilon$ *is a Łukasiewicz fuzzy BE-algebra of* X*.*

Proof. Assume that ξ is a fuzzy BE-algebra of X. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in Ł_\xi^\varepsilon$ and $[y/t_b] \in Ł_\xi^\varepsilon$. Then $Ł_\xi^\varepsilon(x) \geq t_a$ and $Ł_\xi^\varepsilon(y) \geq t_b$, so

$$\begin{aligned} Ł_\xi^\varepsilon(x * y) &= \max\{0, \xi(x * y) + \varepsilon - 1\} \\ &\geq \max\{0, \min\{\xi(x), \xi(y)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{\xi(x) + \varepsilon - 1, \xi(y) + \varepsilon - 1\}\} \\ &= \min\{\max\{0, \xi(x) + \varepsilon - 1\}, \max\{0, \xi(y) + \varepsilon - 1\}\} \\ &= \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} \geq \min\{t_a, t_b\}. \end{aligned}$$

Hence $[(x * y)/\min\{t_a, t_b\}] \in Ł_\xi^\varepsilon$, and therefore $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-algebra of X.

The converse of Theorem 2 may not be true as shown in the following example.

Example 3. *Consider a set* $X = \{1, b_1, b_2, b_3, b_4\}$ *with a binary operation "*$*$*" given in the table below.*

$*$	1	b_1	b_2	b_3	b_4
1	1	b_1	b_2	b_3	b_4
b_1	1	1	b_2	b_3	b_4
b_2	1	b_1	1	b_3	b_3
b_3	1	1	b_2	1	b_2
b_4	1	1	1	1	1

Then $(X, *, 1)$ *is a BE-algebra (see [7]). Define a fuzzy set* ξ *in* X *as follows:*

$$\xi : X \to [0, 1],\ x \mapsto \begin{cases} 0.73 & \text{if } x = 1, \\ 0.42 & \text{if } x = b_1, \\ 0.59 & \text{if } x = b_2, \\ 0.46 & \text{if } x = b_3, \\ 0.68 & \text{if } x = b_4. \end{cases}$$

Given $\varepsilon := 0.41$, *the Łukasiewicz fuzzy set* $Ł_\xi^\varepsilon$ *of* ξ *in* X *is given as follows:*

$$Ł_\xi^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.14 & \text{if } x = 1, \\ 0.00 & \text{if } x = b_1, \\ 0.00 & \text{if } x = b_2, \\ 0.00 & \text{if } x = b_3, \\ 0.09 & \text{if } x = b_4. \end{cases}$$

It is routine to verify that $Ł_\xi^\varepsilon$ *is a Łukasiewicz fuzzy BE-algebra of* X. *But* ξ *is not a fuzzy BE-algebra of* X *because of* $\xi(b_2 * b_4) = \xi(b_3) = 0.46 \ngeq 0.59 = \min\{\xi(b_2), \xi(b_4)\}$.

Theorem 3. *Given a BE-subalgebra* F *of* X, *define a fuzzy set* ξ *in* X *as follows:*

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} t_0 & \text{if } x \in F, \\ t_1 & \text{if } x \notin F \end{cases} \tag{12}$$

where $t_0 > t_1$ *in* $[0,1]$. *Then the Łukasiewicz fuzzy set* $Ł_\xi^\varepsilon$ *of* ξ *is a Łukasiewicz fuzzy BE-algebra of* X.

Proof. It is easy to verify that the fuzzy set ξ given in (12) is a fuzzy BE-algebra of X. Hence the Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ is a Łukasiewicz fuzzy BE-algebra of X by Theorem 2.

Proposition 3. *If* ξ *is a fuzzy BE-algebra of* X, *then its Łukasiewicz fuzzy set* $Ł_\xi^\varepsilon$ *satisfies:*

$$(\forall x, y \in X)\left(Ł_\xi^\varepsilon(y) = Ł_\xi^\varepsilon(1) \ \Leftrightarrow\ Ł_\xi^\varepsilon(x) \leq Ł_\xi^\varepsilon(x * y)\right). \tag{13}$$

Proof. If ξ is a fuzzy BE-algebra of X, then its Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-algebra of X (see Theoem 2). Assume that $Ł_\xi^\varepsilon(y) = Ł_\xi^\varepsilon(1)$ for all $y \in X$. Then

$$Ł_\xi^\varepsilon(x) = \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(1)\} = \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} \leq Ł_\xi^\varepsilon(x * y)$$

for all $x, y \in X$ by Theorem 1 and Lemma 1.

Conversely, suppose that $Ł_\xi^\varepsilon(x) \leq Ł_\xi^\varepsilon(x * y)$ for all $x, y \in X$. Then $Ł_\xi^\varepsilon(y) = Ł_\xi^\varepsilon(1 * y) \geq Ł_\xi^\varepsilon(1)$ by (BE3), and so $Ł_\xi^\varepsilon(y) = Ł_\xi^\varepsilon(1)$ for all $y \in X$.

Theorem 4. *If the Łukasiewicz fuzzy set* $Ł_\xi^\varepsilon$ *of* ξ *in* X *satisfies:*

$$[x/t_a] \in Ł_\xi^\varepsilon,\ [z/t_c] \in Ł_\xi^\varepsilon \ \Rightarrow\ [(x * y)/\min\{t_a, t_c\}] \in Ł_\xi^\varepsilon \tag{14}$$

for all $t_a, t_c \in (0,1]$ *and* $x, y, z \in X$ *with* $z \leq y$, *then* $Ł_\xi^\varepsilon$ *is a Łukasiewicz fuzzy BE-algebra of* X.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in Ł_\xi^\varepsilon$ and $[y/t_b] \in Ł_\xi^\varepsilon$. Since $y \leq y$ for all $y \in X$, it follows from (14) that $[(x * y)/\min\{t_a, t_b\}] \in Ł_\xi^\varepsilon$. Hence $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-algebra of X.

We consider the conditions for the $\in$-set and q-set of Łukasiewicz fuzzy set to be BE-subalgebras.

Theorem 5. *If $Ł_\xi^\varepsilon$ is the Lukasiewicz fuzzy set of ξ in X which satisfies:*

$$(\forall x, y \in X)\left(\min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} \leq \max\{Ł_\xi^\varepsilon(x*y), 0.5\}\right), \tag{15}$$

then the $\in$-set $(Ł_\xi^\varepsilon, t)_\in$ of $Ł_\xi^\varepsilon$ is a BE-subalgebra of X for the value $t \in (0.5, 1]$.

Proof. Assume that $Ł_\xi^\varepsilon$ satisfies the condition (15) and let $x, y \in X$ be such that $x, y \in (Ł_\xi^\varepsilon, t)_\in$ for $t \in (0.5, 1]$. Then $Ł_\xi^\varepsilon(x) \geq t$ and $Ł_\xi^\varepsilon(y) \geq t$, which imply from (15) that

$$\max\{Ł_\xi^\varepsilon(x*y), 0.5\} \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} \geq t > 0.5.$$

Hense $[(x*y)/t] \in Ł_\xi^\varepsilon$, i.e., $x*y \in (Ł_\xi^\varepsilon, t)_\in$, and therefore $(Ł_\xi^\varepsilon, t)_\in$ is a BE-subalgebra of X for $t \in (0.5, 1]$.

Theorem 6. *For the Lukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X, if its $\in$-set $(Ł_\xi^\varepsilon, t)_\in$ is a BE-subalgebra of X for the value $t \in (0.5, 1]$, then $Ł_\xi^\varepsilon$ satisfies the condition (15).*

Proof. Assume that $Ł_\xi^\varepsilon$ does not satisfy the condition (15). Then

$$\min\{Ł_\xi^\varepsilon(a), Ł_\xi^\varepsilon(b)\} > \max\{Ł_\xi^\varepsilon(a*b), 0.5\}$$

for some $a, b \in X$, and so $s \in (0.5, 1]$ and $[a/s]$, $[b/s] \in Ł_\xi^\varepsilon$, i.e., a, $b \in (Ł_\xi^\varepsilon, s)_\in$ where $s := \min\{Ł_\xi^\varepsilon(a), Ł_\xi^\varepsilon(b)\}$. Since $(Ł_\xi^\varepsilon, s)_\in$ is a BE-subalgebra of X by assumption, we have $a*b \in (Ł_\xi^\varepsilon, s)_\in$ and hence $[(a*b)//s] \in Ł_\xi^\varepsilon$, i.e.,

$$Ł_\xi^\varepsilon(a*b) \geq s = \min\{Ł_\xi^\varepsilon(a), Ł_\xi^\varepsilon(b)\}.$$

This is a contradiction. Hence $\min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} \leq \max\{Ł_\xi^\varepsilon(x*y), 0.5\}$ for all $x, y \in X$, that is, $Ł_\xi^\varepsilon$ satisfies the condition (15).

Theorem 7. *If the Lukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X is a Lukasiewicz fuzzy BE-algebra of X, then its q-set $(Ł_\xi^\varepsilon, t)_q$ is a BE-subalgebra of X for the value $t \in (0, 1]$.*

Proof. Let $t \in (0, 1]$ and $x, y \in (Ł_\xi^\varepsilon, t)_q$. Then $[x/t]\, q\, Ł_\xi^\varepsilon$ and $[y/t]\, q\, Ł_\xi^\varepsilon$, that is, $Ł_\xi^\varepsilon(x) + t > 1$ and $Ł_\xi^\varepsilon(y) + t > 1$. It follows from Theorem 1 that

$$Ł_\xi^\varepsilon(x*y) + t \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} + t = \min\{Ł_\xi^\varepsilon(x) + t, Ł_\xi^\varepsilon(y) + t\} > 1.$$

Thus $[(x*y)/t]\, q\, Ł_\xi^\varepsilon$, i.e., $x*y \in (Ł_\xi^\varepsilon, t)_q$. Hence $(Ł_\xi^\varepsilon, t)_q$ is a BE-subalgebra of X.

Corollary 1. *If ξ is a fuzzy BE-algebra of X, then the q-set $(Ł_\xi^\varepsilon, t)_q$ of $Ł_\xi^\varepsilon$ is a BE-subalgebra of X for the value $t \in (0, 1]$.*

Theorem 8. *For the Lukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X, if the q-set $(Ł_\xi^\varepsilon, t)_q$ is a BE-subalgebra of X, then $Ł_\xi^\varepsilon$ satisfies:*

$$x \in (Ł_\xi^\varepsilon, t_a)_q,\ y \in (Ł_\xi^\varepsilon, t_b)_q \ \Rightarrow\ x*y \in (Ł_\xi^\varepsilon, \max\{t_a, t_b\})_\in \tag{16}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 0.5]$.

Proof. Assume that the q-set $(Ł_\xi^\varepsilon, t)_q$ is a BE-subalgebra of X. Let $x, y \in X$ and $t_a, t_b \in (0, 0.5]$ be such that $x \in (Ł_\xi^\varepsilon, t_a)_q$ and $y \in (Ł_\xi^\varepsilon, t_b)_q$. Then $x, y \in (Ł_\xi^\varepsilon, \max\{t_a, t_b\})_q$, and hence $x * y \in (Ł_\xi^\varepsilon, \max\{t_a, t_b\})_q$ by hypothesis. It follows that

$$Ł_\xi^\varepsilon(x * y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\}$$

since $\max\{t_a, t_b\} \leq 0.5$. Therefore $[(x*y)/\max\{t_a, t_b\}] \in Ł_\xi^\varepsilon$, that is, $x*y \in (Ł_\xi^\varepsilon, \max\{t_a, t_b\})_\in$.

Theorem 9. *If the Lukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ is a Łukasiewicz fuzzy BE-algebra of X, then its O-set $O(Ł_\xi^\varepsilon)$ is a BE-subalgebra of X.*

Proof. Assume that $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-algebra of X and let $x, y \in O(Ł_\xi^\varepsilon)$. Then $\xi(x) + \varepsilon - 1 > 0$ and $\xi(y) + \varepsilon - 1 > 0$ which implies that

$$Ł_\xi^\varepsilon(x * y) \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} = \min\{\xi(x) + \varepsilon - 1, \xi(y) + \varepsilon - 1\} > 0$$

by Theoem 1. Hence $x * y \in O(Ł_\xi^\varepsilon)$, and therefore $O(Ł_\xi^\varepsilon)$ is a BE-subalgebra of X.

Corollary 2. *If ξ is a fuzzy BE-algebra of X, then the O-set $O(Ł_\xi^\varepsilon)$ of $Ł_\xi^\varepsilon$ is a BE-subalgebra of X.*

Theorem 10. *If the Lukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X satisfies:*

$$x \in (Ł_\xi^\varepsilon, t_a)_\in,\ y \in (Ł_\xi^\varepsilon, t_b)_\in \ \Rightarrow\ x * y \in (Ł_\xi^\varepsilon, \max\{t_a, t_b\})_q \tag{17}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then its O-set $O(Ł_\xi^\varepsilon)$ is a BE-subalgebra of X.

Proof. Assume that $Ł_\xi^\varepsilon$ satisfies the condition (17) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$. Let $x, y \in O(Ł_\xi^\varepsilon)$. Then $\xi(x) + \varepsilon - 1 > 0$ and $\xi(y) + \varepsilon - 1 > 0$. Since $x \in (Ł_\xi^\varepsilon, Ł_\xi^\varepsilon(x))_\in$ and $y \in (Ł_\xi^\varepsilon, Ł_\xi^\varepsilon(y))_\in$, it follows from (17) that

$$x * y \in (Ł_\xi^\varepsilon, \max\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\})_q. \tag{18}$$

If $x * y \notin O(Ł_\xi^\varepsilon)$, then $Ł_\xi^\varepsilon(x * y) = 0$ and so

$$\begin{aligned}
&Ł_\xi^\varepsilon(x * y) + \max\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} = \max\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\} \\
&= \max\{\max\{0, \xi(x) + \varepsilon - 1\}, \max\{0, \xi(y) + \varepsilon - 1\}\} \\
&= \max\{\xi(x) + \varepsilon - 1, \xi(y) + \varepsilon - 1\} \\
&= \max\{\xi(x), \xi(y)\} + \varepsilon - 1 \\
&\leq 1 + \varepsilon - 1 = \varepsilon \leq 1,
\end{aligned}$$

that is, $[(x * y)/\max\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\}]\, \overline{q}\, Ł_\xi^\varepsilon$ which shows that (18) is not valid. This is a contradiction, and thus $x * y \in O(Ł_\xi^\varepsilon)$. Hence $O(Ł_\xi^\varepsilon)$ is a BE-subalgebra of X.

Theorem 11. *If the Łukasiewicz fuzzy set* $Ł_\xi^\varepsilon$ *of* ξ *in* X *satisfies the condition* (16) *for all* $x, y \in X$ *and* $t_a, t_b \in (0,1]$, *then its O-set* $O(Ł_\xi^\varepsilon)$ *is a BE-subalgebra of* X.

Proof. Let $x, y \in O(Ł_\xi^\varepsilon)$. Then $\xi(x) + \varepsilon - 1 > 0$ and $\xi(y) + \varepsilon - 1 > 0$. Hence

$$Ł_\xi^\varepsilon(x) + 1 = \max\{0, \xi(x) + \varepsilon - 1\} + 1 = \xi(x) + \varepsilon - 1 + 1 = \xi(x) + \varepsilon > 1$$

and

$$Ł_\xi^\varepsilon(y) + 1 = \max\{0, \xi(y) + \varepsilon - 1\} + 1 = \xi(y) + \varepsilon - 1 + 1 = \xi(y) + \varepsilon > 1,$$

that is, $x \in (Ł_\xi^\varepsilon, 1)_q$ and $y \in (Ł_\xi^\varepsilon, 1)_q$. It follows from (16) that

$$x * y \in (Ł_\xi^\varepsilon, \max\{1, 1\})_\in = (Ł_\xi^\varepsilon, 1)_\in.$$

Thus $Ł_\xi^\varepsilon(x * y) + 1 > 1$, and so $Ł_\xi^\varepsilon(x * y) > 0$, i.e., $x * y \in O(Ł_\xi^\varepsilon)$. Therefore $O(Ł_\xi^\varepsilon)$ is a BE-subalgebra of X.

4. Łukasiewicz fuzzy BE-filters

Definition 2. *The Łukasiewicz fuzzy set* $Ł_\xi^\varepsilon$ *of* ξ *in* X *is called a Łukasiewicz fuzzy BE-filter of* X *if it satisfies:*

$$x \in (Ł_\xi^\varepsilon, t_a)_\in \;\Rightarrow\; 1 \in (Ł_\xi^\varepsilon, t_a)_\in, \tag{19}$$

$$x * y \in (Ł_\xi^\varepsilon, t_a)_\in,\; x \in (Ł_\xi^\varepsilon, t_b)_\in \;\Rightarrow\; y \in (Ł_\xi^\varepsilon, \min\{t_a, t_b\})_\in \tag{20}$$

for all $x, y \in X$ *and* $t_a, t_b \in (0,1]$.

Example 4. *Consider a set* $X = \{1, b_1, b_2, b_3\}$ *with a binary operation "$*$" given in the table below.*

$*$	1	b_1	b_2	b_3
1	1	b_1	b_2	b_3
b_1	1	1	b_2	b_2
b_2	1	b_1	1	b_1
b_3	1	1	1	1

Then $(X, *, 1)$ *is a BE-algebra (see [4]). Define a fuzzy set* ξ *in* X *as follows:*

$$\xi : X \to [0,1],\; x \mapsto \begin{cases} 0.73 & \text{if } x = 1, \\ 0.62 & \text{if } x = b_1, \\ 0.48 & \text{if } x \in \{b_2, b_3\}. \end{cases}$$

Given $\varepsilon := 0.62$, *the Łukasiewicz fuzzy set* $Ł_\xi^\varepsilon$ *of* ξ *in* X *is given as follows:*

$$Ł_\xi^\varepsilon : X \to [0,1],\; x \mapsto \begin{cases} 0.35 & \text{if } x = 1, \\ 0.24 & \text{if } x = b_1, \\ 0.10 & \text{if } x \in \{b_2, b_3\}. \end{cases}$$

It is routine to verify that $Ł_\xi^\varepsilon$ *is a Łukasiewicz fuzzy BE-filter of* X.

We discuss relationship between fuzzy BE-filter and Łukasiewicz fuzzy BE-filter.

Theorem 12. *If ξ is a fuzzy BE-filter of X, then its Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X.*

Proof. Assume that ξ is a fuzzy BE-filter of X and let $Ł_\xi^\varepsilon$ be its Łukasiewicz fuzzy set in X. Let $x \in X$ and $t_a \in (0, 1]$ be such that $x \in (Ł_\xi^\varepsilon, t_a)_\in$. Then

$$Ł_\xi^\varepsilon(1) = \max\{0, \xi(1) + \varepsilon - 1\} \geq \max\{0, \xi(x) + \varepsilon - 1\} = Ł_\xi^\varepsilon(x) \geq t_a,$$

and so $1 \in (Ł_\xi^\varepsilon, t_a)_\in$. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $x * y \in (Ł_\xi^\varepsilon, t_a)_\in$ and $x \in (Ł_\xi^\varepsilon, t_b)_\in$. Then $Ł_\xi^\varepsilon(x * y) \geq t_a$ and $Ł_\xi^\varepsilon(x) \geq t_b$, which imply that

$$\begin{aligned} Ł_\xi^\varepsilon(y) &= \max\{0, \xi(y) + \varepsilon - 1\} \geq \max\{0, \min\{\xi(x * y), \xi(x)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{\xi(x * y) + \varepsilon - 1, \xi(x) + \varepsilon - 1\}\} \\ &= \min\{\max\{0, \xi(x * y) + \varepsilon - 1\}, \max\{0, \xi(x) + \varepsilon - 1\}\} \\ &= \min\{Ł_\xi^\varepsilon(x * y), Ł_\xi^\varepsilon(x)\} \geq \min\{t_a, t_b\}. \end{aligned}$$

Hence $[y/\min\{t_a, t_b\}] \in Ł_\xi^\varepsilon$, that is, $y \in (Ł_\xi^\varepsilon, \min\{t_a, t_b\})_\in$. Therefore $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X.

In Theorem 12, the converse may not be true as shown in the following example.

Example 5. *Consider the BE-algebra $(X, *, 1)$ in Example 4 and let ξ be a fuzzy set in X defined by*

$$\xi : X \to [0, 1],\ x \mapsto \begin{cases} 0.73 & \text{if } x = 1, \\ 0.51 & \text{if } x = b_1, \\ 0.62 & \text{if } x = b_2, \\ 0.47 & \text{if } x = b_3. \end{cases}$$

Then ξ is not a fuzzy BE-filter of X since

$$\xi(b_3) = 0.47 \ngeq 0.51 = \min\{\xi(b_1 * b_3), \xi(b_1)\}.$$

Given $\varepsilon := 0.49$, the Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X is calculated as follows:

$$Ł_\xi^\varepsilon : X \to [0, 1],\ x \mapsto \begin{cases} 0.22 & \text{if } x = 1, \\ 0.00 & \text{if } x = b_1, \\ 0.11 & \text{if } x = b_2, \\ 0.00 & \text{if } x = b_3, \end{cases}$$

and it is a Łukasiewicz fuzzy BE-filter of X.

Theorem 13. *The Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ is a Łukasiewicz fuzzy BE-filter of X if and only if it satisfies:*

$$Ł_\xi^\varepsilon(1) \text{ is an upper bound of } \{Ł_\xi^\varepsilon(x) \mid x \in X\}, \tag{21}$$

$$(\forall x, y \in X)(Ł_\xi^\varepsilon(y) \geq \min\{Ł_\xi^\varepsilon(x * y), Ł_\xi^\varepsilon(x)\}). \tag{22}$$

Proof. Assume that $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X. Since $x \in (Ł_\xi^\varepsilon, Ł_\xi^\varepsilon(x))_\in$ for all $x \in X$, it follows from (19) that $1 \in (Ł_\xi^\varepsilon, Ł_\xi^\varepsilon(x))_\in$. Hence $Ł_\xi^\varepsilon(1) \geq Ł_\xi^\varepsilon(x)$ for all $x \in X$, and thus (21) is valid. Since $x * y \in (Ł_\xi^\varepsilon, Ł_\xi^\varepsilon(x*y))_\in$ and $x \in (Ł_\xi^\varepsilon, Ł_\xi^\varepsilon(x))_\in$ for all $x, y \in X$, we have $y \in (Ł_\xi^\varepsilon, \min\{Ł_\xi^\varepsilon(x*y), Ł_\xi^\varepsilon(x)\})_\in$ by (20). Hence $Ł_\xi^\varepsilon(y) \geq \min\{Ł_\xi^\varepsilon(x*y), Ł_\xi^\varepsilon(x)\}$ for all $x, y \in X$.

Conversely, suppose that $Ł_\xi^\varepsilon$ satisfies (21) and (22). Let $x, y \in X$ and $t_a, t_b \in (0, 1]$. If $x \in (Ł_\xi^\varepsilon, t_a)_\in$, then $Ł_\xi^\varepsilon(1) \geq Ł_\xi^\varepsilon(x) \geq t_a$ and so $1 \in (Ł_\xi^\varepsilon, t_a)_\in$. Assume that $x * y \in (Ł_\xi^\varepsilon, t_a)_\in$ and $x \in (Ł_\xi^\varepsilon, t_b)_\in$. Then $Ł_\xi^\varepsilon(x*y) \geq t_a$ and $Ł_\xi^\varepsilon(x) \geq t_b$ It follows from (22) that $Ł_\xi^\varepsilon(y) \geq \min\{Ł_\xi^\varepsilon(x*y), Ł_\xi^\varepsilon(x)\} \geq \min\{t_a, t_b\}$, i.e., $[y/\min\{t_a, t_b\}] \in Ł_\xi^\varepsilon$. Hence $y \in (Ł_\xi^\varepsilon, \min\{t_a, t_b\})_\in$. Therefore $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X.

Corollary 3. *If ξ is a fuzzy BE-filter of X, then its Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ satisfies* (21) *and* (22).

Theorem 14. *The Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ is a Łukasiewicz fuzzy BE-filter of X if and only if it satisfies the condition* (21) *and*

$$(\forall x, y, z \in X)(Ł_\xi^\varepsilon(x*z) \geq \min\{Ł_\xi^\varepsilon(x*(y*z)), Ł_\xi^\varepsilon(y)\}). \tag{23}$$

Proof. Assume that $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X. The condition (21) was verified by the proof of Theorem 13. Using (BE4) and (22), we get

$$Ł_\xi^\varepsilon(x*z) \geq \min\{Ł_\xi^\varepsilon(y*(x*z)), Ł_\xi^\varepsilon(y)\} = \min\{Ł_\xi^\varepsilon(x*(y*z)), Ł_\xi^\varepsilon(y)\}.$$

Conversely, suppose that $Ł_\xi^\varepsilon$ satisfies the conditions (21) and (23). If we take $x := 1$ in (23) and use (BE3), then

$$Ł_\xi^\varepsilon(z) = Ł_\xi^\varepsilon(1*z) \geq \min\{Ł_\xi^\varepsilon(1*(y*z)), Ł_\xi^\varepsilon(y)\} = \min\{Ł_\xi^\varepsilon(y*z), Ł_\xi^\varepsilon(y)\}$$

for all $y, z \in X$. Therefore $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X by Theorem 13.

Corollary 4. *If ξ is a fuzzy BE-filter of X, then its Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ satisfies* (23).

Theorem 15. *The Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ is a Łukasiewicz fuzzy BE-filter of X if and only if it satisfies:*

$$(\forall x, y \in X)\left(Ł_\xi^\varepsilon(x*y) \geq Ł_\xi^\varepsilon(y)\right), \tag{24}$$

$$(\forall x, y, z \in X)\left(Ł_\xi^\varepsilon((x*(y*z))*z) \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\}\right). \tag{25}$$

Proof. Assume that $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X and let $x, y, z \in X$. Then

$$\begin{aligned} Ł_\xi^\varepsilon(x*y) &\geq \min\{Ł_\xi^\varepsilon(y*(x*y)), Ł_\xi^\varepsilon(y)\} = \min\{Ł_\xi^\varepsilon(x*(y*y)), Ł_\xi^\varepsilon(y)\} \\ &= \min\{Ł_\xi^\varepsilon(x*1), Ł_\xi^\varepsilon(y)\} = \min\{Ł_\xi^\varepsilon(1), Ł_\xi^\varepsilon(y)\} = Ł_\xi^\varepsilon(y) \end{aligned}$$

by (BE1), (BE2), (BE4) and Theorem 13. Also, we have

$$\begin{aligned}Ł_\xi^\varepsilon((x*(y*z))*z) &\geq \min\{Ł_\xi^\varepsilon((x*(y*z))*(y*z)), Ł_\xi^\varepsilon(y)\}\\ &\geq \min\{\min\{Ł_\xi^\varepsilon(x*((x*(y*z))*(y*z)), Ł_\xi^\varepsilon(x))\}, Ł_\xi^\varepsilon(y)\}\\ &= \min\{\min\{Ł_\xi^\varepsilon(1), Ł_\xi^\varepsilon(x)\}, Ł_\xi^\varepsilon(y)\}\\ &= \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\}\end{aligned}$$

by (3), Theorem 13 and Theorem 14.

Conversely, suppose that $Ł_\xi^\varepsilon$ satisfies (24) and (25). If we take $y = x$ in (24) and use (BE1), then $Ł_\xi^\varepsilon(1) = Ł_\xi^\varepsilon(x*x) \geq Ł_\xi^\varepsilon(x)$ for all $x \in X$, that is, $Ł_\xi^\varepsilon(1)$ is an upper bound of $\{Ł_\xi^\varepsilon(x) \mid x \in X\}$. The combination of (BE1), (BE3) and (25) induces

$$Ł_\xi^\varepsilon(y) = Ł_\xi^\varepsilon(1*y) = Ł_\xi^\varepsilon(((x*y)*(x*y))*y) \geq \min\{Ł_\xi^\varepsilon(x*y), Ł_\xi^\varepsilon(x)\}$$

for all $x, y \in X$. It follows from Theorem 13 that $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X.

Corollary 5. *If ξ is a fuzzy BE-filter of X, then its Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ satisfies* (24) *and* (25).

Theorem 16. *The Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ is a Łukasiewicz fuzzy BE-filter of X if and only if it satisfies the condition* (21) *and*

$$(\forall x, y, z \in X)\left(x*(y*z) = 1 \;\Rightarrow\; Ł_\xi^\varepsilon(z) \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\}\right). \tag{26}$$

Proof. Assume that $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X. The condition (21) was verified by the proof of Theorem 13. Let $x, y, z \in X$ be such that $x*(y*z) = 1$. Using Theorem 13, we have

$$Ł_\xi^\varepsilon(y*z) \geq \min\{Ł_\xi^\varepsilon(x*(y*z)), Ł_\xi^\varepsilon(x)\} = \min\{Ł_\xi^\varepsilon(1), Ł_\xi^\varepsilon(x)\} = Ł_\xi^\varepsilon(x)$$

and so $Ł_\xi^\varepsilon(z) \geq \min\{Ł_\xi^\varepsilon(y*z), Ł_\xi^\varepsilon(y)\} \geq \min\{Ł_\xi^\varepsilon(x), Ł_\xi^\varepsilon(y)\}$.

Conversely, suppose that $Ł_\xi^\varepsilon$ satisfies the condition (21) and (26). Since $(x*y)*(x*y) = 1$ for all $x, y \in X$, we have $Ł_\xi^\varepsilon(y) \geq \min\{Ł_\xi^\varepsilon(x*y), Ł_\xi^\varepsilon(x)\}$ for all $x, y \in X$. It follows from Theorem 13 that $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X.

Corollary 6. *If ξ is a fuzzy BE-filter of X, then its Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ satisfies* (26).

We use BE-filter to create a Łukasiewicz fuzzy BE-filter.

Theorem 17. *Let F be a BE-filter of X and let $\alpha, \beta \in (0, 1]$ with $\alpha \geq \beta$. For every ε, define the Łukasiewicz fuzzy set $Ł_\xi^\varepsilon$ of ξ in X as follows:*

$$Ł_\xi^\varepsilon : X \to [0,1], x \mapsto \begin{cases} \alpha & \text{if } x \in F, \\ \beta & \text{otherwise.} \end{cases}$$

Then $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X.

Proof. Since $1 \in F$, we have $Ł_\xi^\varepsilon(1) = \alpha \geq Ł_\xi^\varepsilon(x)$ for all $x \in X$. Hence $Ł_\xi^\varepsilon(1)$ is an upper bound of $\{Ł_\xi^\varepsilon(x) \mid x \in X\}$. Let $x, y \in X$. If $y \in F$, then $Ł_\xi^\varepsilon(y) = \alpha \geq \min\{Ł_\xi^\varepsilon(x*y), Ł_\xi^\varepsilon(x)\}$. If $y \notin F$, then $x * y \notin F$ or $x \notin F$. Hence

$$\min\{Ł_\xi^\varepsilon(x * y), Ł_\xi^\varepsilon(x)\} = \beta = Ł_\xi^\varepsilon(y).$$

Therefore $Ł_\xi^\varepsilon$ is a Łukasiewicz fuzzy BE-filter of X by Theorem 13.

Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable suggestions.

References

[1] G.Dymek and A. Walendiziak. Fuzzy filters of be-algebras. *Math. Slovaca*, 63:935–946, 2013.

[2] Y. B. Jun. łukasiewicz fuzzy subalgebrs in bck-algebras and bci-algebras. *Ann. Fuzzy Math. Inform.*, 23(2):213–223, 2022.

[3] H. S. Kim and Y. H. Kim. On be-algebras. *Sci. Math. Jpn.*, 66:113–116, 2007.

[4] H. S. Kim and K. J. Lee. Extended upper sets in be-algebras. *Bull. Malays. Math. Sci. Soc.*, 34:511–520, 2011.

[5] P. M. Pu and Y. M. Liu. Fuzzy topology i, neighborhood structure of a fuzzy point and moore-smith convergence. *J. Math. Anal. Appl.*, 76:571–599, 1980.

[6] A. Rezaei and A. Borumand Saeid. On fuzzy subalgebras of be-algebras. *Afr. Mat.*, 22:115–127, 2011.

[7] Y. H. Kim S. S. Ahn and K. S. So. Fuzzy be-algebras., 2011.