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A New Secant Type Method with Quadratic-Order Convergence

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Abstract. Using the development of the second approximation, a variation of the standard secant technique for nonlinear problems has been developed. The iterative formula is created using Taylor series expansion, which includes an estimate of the second derivative of $\Theta(\mu)$. It is demonstrated that the new approaches have quadratic convergence. In comparison with the Newton technique employing seven test functions in the practical application, it turns out that the performance of the method is efficient.

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1. Introduction

Consider the scalar function Θ, which is determined by a single independent variable, μ. Assume we wish to determine the value of μ where $\Theta(\mu)$ is the minimum value, as shown in [11].

Gradient-based minimization approaches look for spots that satisfy the optimality conditions to find a local minimum. Finding μ^* that satisfies the first-order optimality requirements, i.e.:

$$\Theta^{'}(\mu^*) = 0 \tag{1}$$

is equivalent to finding the roots of the function to be a minimum of its first derivative. Because of this, root finding methods are effective in function reduction and may be used to identify stationary locations. See [2][7] for further information.

In regard to different approximation approaches in terms of convergence, Newton's method is the most appealing, as illustrated in [10][1]. It works by locally approximating the objective function using a quadratic model that decides on it at a given place. The

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technique can be recurrent indefinitely until the approximation function is optimal. Newton's approach, on the other hand, is widely recognized for requiring assessment of the 1^{st} and 2^{nd} derivatives in each iteration, as shown by the iteration formula: See [14][13] for further information.

$$\mu_{i+1} = \mu_i - \frac{\Theta'(\mu_i)}{\Theta''(\mu_i)} \tag{2}$$

To avoid this, certain Newton-type methods are referred to as secant methods since they do not require the second derivative. Due to the time-consuming and difficulty involved in determining the second derivative of a function, the formula for iteration, often known as:

$$\mu_{i+1} = \mu_i - \frac{\Theta'(\mu_i)(\mu_i - \mu_{i-1})}{\Theta'(\mu_i) - \Theta'(\mu_{i-1})} \tag{3}$$

having been developed, more information may be found in [8]. The calculation of Θ and its derivatives consumes the majority of the computational time. Numerous studies have focused on nonlinear issues that do not require the second derivative (see [10][14][4] and [3], despite the fact that a lot of new modified techniques with a higher convergence order have been proposed (e.g. [5][6][15][9]).

The derivation of a secant approach based on the second-order Taylor expansion is presented in this study. In comparison to the Newton method, by employing seven test functions, the new method's performance is evaluated using the most common and extensively used criterion: the iterations' number.

2. A New Secant Type Method

An important category of nonlinear optimization problems is the second-order Taylor expansion. We have [12] for the second-order Taylor expansion around the point μ_i:

$$\Theta(\mu) \cong \Theta(\mu_i) + \Theta'(\mu_i)(\mu - \mu_i) + \frac{1}{2}\Theta''(c)(\mu - \mu_i)^2 \tag{4}$$

The minimal position at which the derivative of $\Theta(\mu)$ equals zero may be found by:

$$\Theta'(\mu) \cong \Theta'(\mu_i) + \Theta''(\mu_i)(\mu - \mu_i) = 0 \tag{5}$$

We got the following results from Eqs. (4) and (5):

$$\Theta''(\mu_i)(\mu - \mu_i)^2 \cong \Theta(\mu_i) - \Theta(\mu) - \frac{1}{2}\Theta'(\mu_i)(\mu - \mu_i) \tag{6}$$

This results in:

$$\Theta''(\mu_i) = \frac{\Theta(\mu_i) - \Theta(\mu) - \frac{1}{2}\Theta'(\mu_i)(\mu - \mu_i)}{(\mu - \mu_i)^2} \tag{7}$$

The preceding equation may be simplified by substituting μ_{i-1} for μ:

$$\Theta''(\mu_i) = \frac{\Theta(\mu_i) - \Theta(\mu_{i-1}) - \frac{1}{2}\Theta'(\mu_i)(\mu_{i-1} - \mu_i)}{(\mu_{i-1} - \mu_i)^2} \tag{8}$$

When we plug Eq. (8) into Eq. (2), we get:

$$\mu_{i+1} = \mu_i - \frac{\Theta'(\mu_i)(\mu_{i-1} - \mu_i)^2}{\Theta(\mu_i) - \Theta(\mu_{i-1}) - \frac{1}{2}\Theta'(\mu_i)(\mu_{i-1} - \mu_i)} \tag{9}$$

which is the new secant type method.

Based on a new secant type technique, the following algorithm is presented:

1. Put ϵ, μ_0 (initial point) and the function $\Theta(\mu_0)$, set $i = 0$.
2. Let $i = i + 1$.
3. Set $\mu_i = \mu_{i-1} - \frac{\Theta'(\mu_{i-1})}{\Theta''(\mu_{i-1})}$
4. Compute $\mu_{i+1} = \mu_i - \frac{\Theta'(\mu_i)(\mu_{i-1} - \mu_i)^2}{\Theta(\mu_i) - \Theta(\mu_{i-1}) - \frac{1}{2}\Theta'(\mu_i)(\mu_{i-1} - \mu_i)}$.
5. Stop, if the absolute value of $|\mu_{i+1} - \mu_i| \leq \epsilon$.

Convergence rate refers to how quickly an iterative process approaches the numerical solution.

3. Analysis of Convergence

A more explicit investigation of the convergence of the new secant iteration has been done since it is critical for the development of optimization algorithms.

Theorem 1. *Let a sufficiently differentiable function $\Theta : I \to R$, where I is an open interval and let $\mu^* \in I$ be a zero of Θ. If μ_0 converges to μ^*, then the new secant type approach has a quadratic convergence rate.*

Proof. The new secant's type method is as:

$$\mu_{i+1} = \mu_i - \frac{\Theta'(\mu_i)(\mu_{i-1} - \mu_i)^2}{\Theta(\mu_i) - \Theta(\mu_{i-1}) - \frac{1}{2}\Theta'(\mu_i)(\mu_{i-1} - \mu_i)} \tag{10}$$

We may get the following result by removing μ^* from both sides of Eq. (10):

$$e_{i+1} = e_i - \frac{\Theta'(\mu_i)(e_{i-1} - e_i)^2}{\Theta(\mu_i) - \Theta(\mu_{i-1}) - \frac{1}{2}\Theta'(\mu_i)(e_{i-1} - e_i)} \tag{11}$$

Where $e_i = \mu_{i+1} - \mu^*$, as the error after i iterations in the approximate minimum.

Now, using Taylor's theorem in its mean value form:

$$\Theta(\mu^*) = \Theta(\mu_i) + \Theta'(\mu_i)(\mu^* - \mu_i) + \frac{1}{2!}\Theta''(\mu_i)(\mu^* - \mu_i)^2 + \frac{1}{3!}\Theta'''(c)(\mu^* - \mu_i)^3 \tag{12}$$

For some c between μ^* and μ_i, and because the derivative of the given equation is zero, we get:

$$-\Theta^{'}(\mu_i) = -e_i\Theta^{''}(\mu_i) + \frac{e_i^2}{2}\Theta^{'''}(c) \tag{13}$$

By putting Eq. (13) into Eq. (11), we can get the following conclusion:

$$e_{i+1} = e_i^2 \frac{1}{2}\frac{\Theta^{'''}(c)}{\Theta^{''}(\mu_i)} \tag{14}$$

This indicates that the convergence order is quadratic.

4. Application Examples

In this study, the offered methodologies will be utilized to solve seven test functions. The functions are listed in the tables, along with the minimum points (μ^*), number of iterations (NI), and Execution time (T).

A comparison with the Newton Procedure (N- Procedure) is shown to assess the new procedure performance.

By using the Matlab program, the calculations have been performed with double precision. If $|\mu_{i+1} - \mu_i| \leq \epsilon$ where $\epsilon = 10^{-10}$ achieves in a numerical test, the search is stopped. Lastly, it is evident that the proposed approach is superior to the N-Procedure for all functions where fewer iterations are required to get the desired outcomes.

EXAMPLE 1. Function $\Theta(\mu) = cos(\mu) + (\mu - 2)^2$, **with** $\mu_0 = 2$.

Procedure	(NI)	μ_i	(T)
N- Procedure	4.0	2.3542	0.5790
New Procedure	4.0	2.4333	0.0780

EXAMPLE 2. Function $\Theta(\mu) = e^{\mu} - 3\mu^2$, **with** $\mu_0 = 0.25$.

Procedure	(NI)	μ_i	(T)
N- Procedure	4.0	0.2045	0.2660
New Procedure	3.0	0.2036	0.0160

EXAMPLE 3. Function $\Theta(\mu) = e^{-\mu} + \mu^2$, **with** $\mu_0 = 1$.

Procedure	(NI)	μ_i	(T)
N- Procedure	5.0	0.3517	0.3130
New Procedure	4.0	0.2025	0.0780

EXAMPLE 4. Function $\Theta(\mu) = -\mu e^{-\mu}$, **with** $\mu_0 = 0$.

Procedure	(NI)	μ_i	(T)
N- Procedure	7.0	1	0.3900
New Procedure	5.0	0.2535	0.0790

EXAMPLE 5. Function $\Theta(\mu) = 0.65 - 0.75/(1 + \mu^2) - 0.65\mu\ tan^{-1}(1/\mu)$, **with** $\mu_0 = 0.1$.

Procedure	(NI)	μ_i	(T)
N- Procedure	6.0	0.4809	0.7970
New Procedure	5.0	0.2901	0.0620

EXAMPLE 6. Function $\Theta(\mu) = 0.5\mu^2 - sin(\mu)$, **with** $\mu_0 = 2$.

Procedure	(NI)	μ_i	(T)
N- Procedure	5.0	0.7391	0.4210
New Procedure	3.0	0.7262	0.0630

EXAMPLE 7. Function $\Theta(\mu) = \mu^4 + 2\mu^2 - \mu - 3$, **with** $\mu_0 = 1$.

Procedure	(NI)	μ_i	(T)
N- Procedure	7.0	0.2367	0.6870
New Procedure	5.0	0.7624	0.0780

5. Conclusion

This research aims to develop a new determined second derivative using a second-order Taylor expansion and a new secant equation. Therefore, it is acceptable to assume that the new secant type technique is more efficient than the Newton method. Some strategies may not always converge to the global minimum, which happens when they fail to converge to the minimum in some circumstances.

References

[1] Auwal Bala Abubakar, Maulana Malik, Poom Kumam, Hassan Mohammad, Min Sun, Abdulkarim Hassan Ibrahim, and Aliyu Ibrahim Kiri. A liu-storey-type conjugate gradient method for unconstrained minimization problem with application in motion control. *Journal of King Saud University-Science*, 34(4):101923, 2022.

[2] C Edwin and Z Stanislaw. *An introduction to optimization.* John Wiley & Sons, 2004.

[3] Marco Frontini and E Sormani. Modified newton's method with third-order convergence and multiple roots. *Journal of Computational and Applied Mathematics*, 156(2):345–354, 2003.

[4] Marco Frontini and E Sormani. Some variant of newton's method with third-order convergence. *Applied Mathematics and Computation*, 140(2-3):419–426, 2003.

[5] Herbert HH Homeier. A modified newton method with cubic convergence: the multivariate case. *Journal of Computational and Applied Mathematics*, 169(1):161–169, 2004.

[6] Herbert HH Homeier. On newton-type methods with cubic convergence. *Journal of Computational and Applied Mathematics*, 176(2):425–432, 2005.

[7] Emin Kahya. Modified secant-type methods for unconstrained optimization. *Applied Mathematics and Computation*, 181(2):1349–1356, 2006.

[8] Emin Kahya and Jinhai Chen. A modified secant method for unconstrained optimization. *Applied Mathematics and Computation*, 186(2):1000–1004, 2007.

[9] Issam AR Moghrabi. A non-secant quasi-newton method for unconstrained nonlinear optimization. *Cogent Engineering*, 9(1):2018929, 2022.

[10] Ahmet Yasar Özban. Some new variants of newton's method. *Applied Mathematics Letters*, 17(6):677–682, 2004.

[11] Singiresu S Rao. *Engineering optimization: theory and practice.* John Wiley & Sons, 2019.

[12] Ronald L Rardin. *Optimization in operations research*, volume 166. Prentice Hall Upper Saddle River, NJ, 1998.

[13] Ekhlass S Al-Rawi. A numerical method for accelerating the convergence of the power method. *Journal of Education and Science*, 22(2):132–146, 2009.

[14] S. Weerakoon and T.G.I. Fernando. A variant of newton's method with accelerated third-order convergence. *Applied Mathematics Letters*, 13(8):87–93, 2000.

[15] Ya-xiang Yuan. A modified bfgs algorithm for unconstrained optimization. *IMA Journal of Numerical Analysis*, 11(3):325–332, 1991.