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Ideals in BE-algebras based on Łukasiewicz fuzzy set

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Abstract. For the purpose of applying the concept of Łukasiewicz fuzzy set to ideals in BE-algebras, Łukasiewicz fuzzy ideal is introduced, and its properties are studied. The relationship between fuzzy ideal and Łukasiewicz fuzzy ideal is discussed. Conditions for the Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy ideal are provided, and characterizations of Łukasiewicz fuzzy ideal are displayed. Conditions in which three subsets, called $\in$-set, q-set and O-set, are ideals are explored.

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1. Introduction

In 1966, Y. Imai, K. Iséki and S. Tanaka introduced BCK-algebra and BCI-algebra as algebraic structures of universal algebra which describe fragments of propositional calculus related to implications known as BCK and BCI-logic. Various generalizations were then attempted, and BCC-algebra, BCH-algebra, BE-algebra, BH-algebra, and d-algebra etc. appeared. In 2008, S. S. Ahn and K. S. So studied ideal theory in BE-algebras (see [1]), and its fuzzy set theory is studied by Y. B. Jun, K. J. Lee and S. Z. Song (see [9]). Łukasiewicz logic, which is the logic of the Łukasiewicz t-norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic. Using the idea of Łukasiewicz t-norm, Y. B. Jun [3] constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras. S. S. Ahn et al. [8], and A. Rezaei and A. Borumand Saeid [7] studied fuzzy BE-algebras. G. Dymek and A. Walendziak [2] developed the theory of

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fuzzy filters in BE-algebras. Y. B. Jun and S. S. Ahn applied the Łukasiewicz fuzzy set to BE-filters and subalgebras (see [4]).

The purpose of this paper is to apply the Łukasiewicz fuzzy set to ideals in BE-algebras. We introduce the notion of Łukasiewicz fuzzy ideal, and investigate several properties. We discuss the characterization of Łukasiewicz fuzzy ideal. We consider the relationship between fuzzy ideal and Łukasiewicz fuzzy ideal. We provide conditions for the Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy ideal. We explore the conditions under which three subsets, called ∈-set, q-set and O-set, will become ideals.

2. Preliminaries

This section lists the known default content that will be used later.

Definition 1 ([5]). *A BE-algebra is defined to be a set X together with a binary operation "$*$" and a special element "1" satisfying the conditions:*

(BE1) $(\forall a \in X)\ (a * a = 1)$,
(BE2) $(\forall a \in X)\ (a * 1 = 1)$,
(BE3) $(\forall a \in X)\ (1 * a = a)$,
(BE4) $(\forall a, y, c \in X)\ (a * (y * c) = y * (a * c))$.

In the following, the BE-algebra is expressed as $(X, 1)_*$.

A relation "$\leq$" in $(X, 1)_*$ is defined as follows:

$$(\forall x, b \in X)(x \leq b \;\Leftrightarrow\; x * b = 1). \tag{1}$$

Definition 2. *A subset K of X is called an ideal of $(X, 1)_*$ (see [1]) if it satisfies:*

$$(\forall a, b \in X)\,(b \in K \;\Rightarrow\; a * b \in K)\,, \tag{2}$$

$$(\forall x, y, a \in X)\,(x, y \in K \;\Rightarrow\; (x * (y * a)) * a \in K)\,. \tag{3}$$

Lemma 1 ([9]). *A subset K of X is an ideal of $(X, 1)_*$ if and only if it satisfies:*

$$1 \in K, \tag{4}$$

$$(\forall a, b, c \in X)(a * (b * c) \in K,\, b \in K \;\Rightarrow\; a * c \in K). \tag{5}$$

Definition 3. *A fuzzy set ψ in X is called a fuzzy ideal of $(X, 1)_*$ (see [9]) if it satisfies:*

$$(\forall x, b \in X)\,(\psi(x * b) \geq \psi(b))\,, \tag{6}$$

$$(\forall x, b, c \in X)\,(\psi((b * (c * x)) * x) \geq \min\{\psi(b), \psi(c)\})\,. \tag{7}$$

A fuzzy set ψ in a set X of the form

$$\psi(b) := \begin{cases} t \in (0, 1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $\langle a/t \rangle$.

For a fuzzy set ψ in a set X, we say that a fuzzy point $\langle a/t \rangle$ is

(i) *contained* in ψ, denoted by $\langle a/t\rangle \in \psi$, ([6]) if $\psi(a) \geq t$.

(ii) *quasi-coincident* with ψ, denoted by $\langle a/t\rangle \, q \, \psi$, ([6]) if $\psi(a) + t > 1$.

Definition 4 ([3])**.** *Let ψ be a fuzzy set in a set X and let $\varepsilon \in (0,1)$. A function*

$$Ł_\psi^\varepsilon : X \to [0,1], \ x \mapsto \max\{0, \psi(x) + \varepsilon - 1\}$$

is called the Łukasiewicz fuzzy set (of ψ) in X.

For the Łukasiewicz fuzzy set $Ł_\psi^\varepsilon$ (of ψ) in X and $t \in (0,1]$, consider the sets

$$(Ł_\psi^\varepsilon, t)_\in := \{x \in X \mid \langle x/t\rangle \in Ł_\psi^\varepsilon\},$$

$$(Ł_\psi^\varepsilon, t)_q := \{x \in X \mid \langle x/t\rangle \, q \, Ł_\psi^\varepsilon\},$$

which are called the $\in$-*set* and q-*set*, respectively, of $Ł_\psi^\varepsilon$ (with value t). Also, consider a set:

$$O(Ł_\psi^\varepsilon) := \{x \in X \mid Ł_\psi^\varepsilon(x) > 0\} \tag{8}$$

which is called an O-*set* of $Ł_\psi^\varepsilon$. It is observed that

$$O(Ł_\psi^\varepsilon) = \{x \in X \mid \psi(x) + \varepsilon - 1 > 0\}.$$

3. Łukasiewicz fuzzy ideals

In this section, let ψ and ε be a fuzzy set in X and an element of $(0,1)$, respectively, unless otherwise specified.

Definition 5. *A Łukasiewicz fuzzy set $Ł_\psi^\varepsilon$ in X is called a Łukasiewicz fuzzy ideal of $(X,1)_*$ if it satisfies:*

$$(\forall x, y \in X)(\forall t \in (0,1]) \left(\langle y/t\rangle \in Ł_\psi^\varepsilon \ \Rightarrow \ \langle (x*y)/t\rangle \in Ł_\psi^\varepsilon\right), \tag{9}$$

$$(\forall x, y, z \in X)(\forall t_a, t_b \in (0,1]) \left(\begin{array}{l} \langle x/t_a\rangle \in Ł_\psi^\varepsilon, \ \langle y/t_b\rangle \in Ł_\psi^\varepsilon \\ \quad \Rightarrow \langle ((x*(y*z))*z)/\min\{t_a, t_b\}\rangle \in Ł_\psi^\varepsilon \end{array} \right). \tag{10}$$

Example 1. *Let $X = \{1, a, b, c, d, 0\}$ be a set with the binary operation "$*$" given by the following Cayley table:*

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X,1)_*$ is a BE-algebra (see [5]). Let ψ be a fuzzy set in X defined as follows.

$$\psi : X \to [0,1],\ x \mapsto \begin{cases} 0.57 & \text{if } x \in \{1,a,b\}, \\ 0.14 & \text{if } x = c, \\ 0.33 & \text{if } x = d, \\ 0.21 & \text{if } x = 0. \end{cases}$$

For $\varepsilon := 0.65$, the Łukasiewicz fuzzy set $Ł_\psi^\varepsilon$ of ψ in X is given as follows.

$$Ł_\psi^\varepsilon : X \to [0,1],\ y \mapsto \begin{cases} 0.22 & \text{if } y \in \{1,a,b\}, \\ 0.00 & \text{if } y \in \{c,d,0\}. \end{cases}$$

It is routine to verify that $Ł_\psi^\varepsilon$ is a Łukasiewicz fuzzy ideal of $(X,1)_*$.

Theorem 1. *A Łukasiewicz fuzzy set $Ł_\psi^\varepsilon$ in X is a Łukasiewicz fuzzy ideal of $(X,1)_*$ if and only if it satisfies:*

$$(\forall x,y \in X)\left(Ł_\psi^\varepsilon(x*y) \geq Ł_\psi^\varepsilon(y)\right). \tag{11}$$

$$(\forall x,y,z \in X)\left(Ł_\psi^\varepsilon((x*(y*z))*z) \geq \min\{Ł_\psi^\varepsilon(x), Ł_\psi^\varepsilon(y)\}\right). \tag{12}$$

Proof. Assume that $Ł_\psi^\varepsilon$ is a Łukasiewicz fuzzy ideal of $(X,1)_*$. Let $x,y \in X$. Since $\langle y/Ł_\psi^\varepsilon(y)\rangle \in Ł_\psi^\varepsilon$, we have $\langle (x*y)/Ł_\psi^\varepsilon(y)\rangle \in Ł_\psi^\varepsilon$ by (9), and so $Ł_\psi^\varepsilon(x*y) \geq Ł_\psi^\varepsilon(y)$. Note that $\langle x/Ł_\psi^\varepsilon(x)\rangle \in Ł_\psi^\varepsilon$ and $\langle y/Ł_\psi^\varepsilon(y)\rangle \in Ł_\psi^\varepsilon$ for all $x,y \in X$. It follows from (10) that $\langle ((x*(y*z))*z)/\min\{Ł_\psi^\varepsilon(x), Ł_\psi^\varepsilon(y)\}\rangle \in Ł_\psi^\varepsilon$, that is, $Ł_\psi^\varepsilon((x*(y*z))*z) \geq \min\{Ł_\psi^\varepsilon(x), Ł_\psi^\varepsilon(y)\}$ for all $x,y,z \in X$.

Conversely, let $Ł_\psi^\varepsilon$ be a Łukasiewicz fuzzy set satisfying (11) and (12). If $\langle y/t\rangle \in Ł_\psi^\varepsilon$ for all $y \in X$ and $t \in (0,1]$, then $Ł_\psi^\varepsilon(x*y) \geq Ł_\psi^\varepsilon(y) \geq t$ for all $x \in X$ by (11). Hence $\langle (x*y)/t\rangle \in Ł_\psi^\varepsilon$. Let $x,y,z \in X$ and $t_a, t_b \in (0,1]$ be such that $\langle x/t_a\rangle \in Ł_\psi^\varepsilon$ and $\langle y/t_b\rangle \in Ł_\psi^\varepsilon$. Then $Ł_\psi^\varepsilon(x) \geq t_a$ and $Ł_\psi^\varepsilon(y) \geq t_b$. It follows from (12) that

$$Ł_\psi^\varepsilon((x*(y*z))*z) \geq \min\{Ł_\psi^\varepsilon(x), Ł_\psi^\varepsilon(y)\} \geq \min\{t_a, t_b\}.$$

Hence $\langle ((x*(y*z))*z)/\min\{t_a,\ t_b\}\rangle \in Ł_\psi^\varepsilon$, and therefore $Ł_\psi^\varepsilon$ is a Łukasiewicz fuzzy ideal of $(X,1)_*$.

Proposition 1. *Every Łukasiewicz fuzzy ideal $Ł_\psi^\varepsilon$ of $(X,1)_*$ satisfies:*

$$(\forall x \in X)(\forall t \in (0,1])\left(\langle x/t\rangle \in Ł_\psi^\varepsilon \ \Rightarrow\ \langle 1/t\rangle \in Ł_\psi^\varepsilon\right). \tag{13}$$

$$(\forall x,y \in X)(\forall t \in (0,1])\left(\langle x/t\rangle \in Ł_\psi^\varepsilon \ \Rightarrow\ \langle ((x*y)*y)/t\rangle \in Ł_\psi^\varepsilon\right). \tag{14}$$

$$(\forall x,y \in X)(\forall t \in (0,1])\left(x \leq y,\ \langle x/t\rangle \in Ł_\psi^\varepsilon \ \Rightarrow\ \langle y/t\rangle \in Ł_\psi^\varepsilon\right). \tag{15}$$

$$(\forall x,y \in X)(\forall t_a, t_b \in (0,1])\left(\begin{array}{c} \langle (x*y)/t_b\rangle \in Ł_\psi^\varepsilon,\ \langle x/t_a\rangle \in Ł_\psi^\varepsilon \\ \Rightarrow\ \langle y/\min\{t_a,t_b\}\rangle \in Ł_\psi^\varepsilon. \end{array}\right). \tag{16}$$

$$(\forall x,y,z \in X)(\forall t_a, t_b \in (0,1])\left(\begin{array}{c} \langle (x*(y*z))/t_a\rangle \in Ł_\psi^\varepsilon,\ \langle y/t_b\rangle \in Ł_\psi^\varepsilon \\ \Rightarrow\ \langle (x*z)/\min\{t_a,t_b\}\rangle \in Ł_\psi^\varepsilon. \end{array}\right). \tag{17}$$

Proof. The condition (13) is derived from the combination of (BE1) and (9). Let $x \in X$ and $t \in (0, 1]$ be such that $\langle x/t \rangle \in Ł^{\varepsilon}_{\psi}$. Then

$$\langle((x * y) * y)/t\rangle = \langle((x * (1 * y)) * y)/t\rangle = \langle((x * (1 * y)) * y)/\min\{t, t\}\rangle \in Ł^{\varepsilon}_{\psi}$$

by (BE3), (10) and (13). The combination of (BE3), (1) and (14) induces (15). Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $\langle(x * y)/t_b\rangle \in Ł^{\varepsilon}_{\psi}$ and $\langle x/t_a\rangle \in Ł^{\varepsilon}_{\psi}$. Then

$$\langle y/\min\{t_a, t_b\}\rangle = \langle(1 * y)/\min\{t_a, t_b\}\rangle = \langle(((x * y) * (x * y)) * y)/\min\{t_a, t_b\}\rangle \in Ł^{\varepsilon}_{\psi}$$

by (BE1), (BE3) and (10), which proves (16). The condition (17) is derived from the combination of (BE4) and (16).

We provide conditions for the Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy ideal.

Theorem 2. *If a Łukasiewicz fuzzy set* $Ł^{\varepsilon}_{\psi}$ *in* X *satisfies conditions* (13) *and* (17)*, then it is a Łukasiewicz fuzzy ideal of* $(X, 1)_*$.

Proof. Assume that $Ł^{\varepsilon}_{\psi}$ satisfies conditions (13) and (17). Let $y \in X$ and $t \in (0, 1]$ be such that $\langle y/t\rangle \in Ł^{\varepsilon}_{\psi}$. Then $\langle(x * (y * y))/t\rangle = \langle(x * 1)/t\rangle = \langle 1/t\rangle \in Ł^{\varepsilon}_{\psi}$ for all $x \in X$ by (BE1), (BE2) and (13). It follows from (17) that $\langle(x * y)/t\rangle \in Ł^{\varepsilon}_{\psi}$. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $\langle x/t_a\rangle \in Ł^{\varepsilon}_{\psi}$ and $\langle y/t_b\rangle \in Ł^{\varepsilon}_{\psi}$. Then $\langle((x * z) * (x * z))/t_b\rangle = \langle 1/t_b\rangle \in Ł^{\varepsilon}_{\psi}$ and so $\langle((x * z) * z)/\min\{t_a, t_b\}\rangle \in Ł^{\varepsilon}_{\psi}$ for all $z \in X$ by (17). In particular, $\langle(((x * (y * z)) * (y * z))/\min\{t_a, t_b\}\rangle \in Ł^{\varepsilon}_{\psi}$, which implies from (17) that $\langle((x * (y * z)) * z)/\min\{t_a, t_b\}\rangle \in Ł^{\varepsilon}_{\psi}$ for all $z \in X$. Hence $Ł^{\varepsilon}_{\psi}$ is a Łukasiewicz fuzzy ideal of $(X, 1)_*$.

Corollary 1. *If a Łukasiewicz fuzzy set* $Ł^{\varepsilon}_{\psi}$ *in* X *satisfies* (13) *and* (17)*, then it satisfies the conditions* (14)*,* (15) *and* (16)*.*

We discuss the relationship between fuzzy ideal and Łukasiewicz fuzzy ideal.

Theorem 3. *If* ψ *is a fuzzy ideal of* $(X, 1)_*$*, then* $Ł^{\varepsilon}_{\psi}$ *is a Łukasiewicz fuzzy ideal of* $(X, 1)_*$.

Proof. Let $y \in X$ and $t \in (0, 1]$ be such that $\langle y/t\rangle \in Ł^{\varepsilon}_{\psi}$. Then $Ł^{\varepsilon}_{\psi}(y) \geq t$, and so

$$Ł^{\varepsilon}_{\psi}(x * y) = \max\{0, \psi(x * y) + \varepsilon - 1\} \geq \max\{0, \psi(y) + \varepsilon - 1\} = Ł^{\varepsilon}_{\psi}(y) \geq t$$

for all $x \in X$. Hence $\langle(x * y)/t\rangle \in Ł^{\varepsilon}_{\psi}$ for all $x \in X$. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $\langle x/t_a\rangle \in Ł^{\varepsilon}_{\psi}$ and $\langle y/t_b\rangle \in Ł^{\varepsilon}_{\psi}$. Then $Ł^{\varepsilon}_{\psi}(x) \geq t_a$ and $Ł^{\varepsilon}_{\psi}(y) \geq t_b$. It follows that

$$\begin{aligned}
Ł^{\varepsilon}_{\psi}((x * (y * z)) * z) &= \max\{0, \psi((x * (y * z)) * z) + \varepsilon - 1\} \\
&\geq \max\{0, \min\{\psi(x), \psi(y)\} + \varepsilon - 1\} \\
&= \max\{0, \min\{\psi(x) + \varepsilon - 1, \psi(y) + \varepsilon - 1\}\} \\
&= \min\{\max\{0, \psi(x) + \varepsilon - 1\}, \max\{0, \psi(y) + \varepsilon - 1\}\}
\end{aligned}$$

$$= \min\{Ł^{\varepsilon}_{\psi}(x), Ł^{\varepsilon}_{\psi}(y)\} \geq \min\{t_a, t_b\}$$

for all $z \in X$. Thus $\langle((x * (y * z)) * z)/\min\{t_a, t_b\}\rangle \in Ł^{\varepsilon}_{\psi}$ for all $z \in X$. Therefore $Ł^{\varepsilon}_{\psi}$ is a Łukasiewicz fuzzy ideal of $(X, 1)_*$.

In Example 1, $Ł^{\varepsilon}_{\psi}$ is a Łukasiewicz fuzzy ideal of $(X, 1)_*$. But ψ is not a fuzzy ideal of $(X, 1)_*$ since $\psi(b * 0) = \psi(c) = 0.14 \ngeq 0.21 = \psi(0)$. Therefore, the converse of Theorem 3 may not be true. In the sense of Theorem 3, we can say that Łukasiewicz fuzzy ideal is a generalization of fuzzy ideal.

We explore the conditions under which $\in$-set and q-set of the Łukasiewicz fuzzy set can be ideal.

Theorem 4. *Let $Ł^{\varepsilon}_{\psi}$ be a Łukasiewicz fuzzy set in X. Then the $\in$-set $(Ł^{\varepsilon}_{\psi}, t)_{\in}$ of $Ł^{\varepsilon}_{\psi}$ with value $t \in (0.5, 1]$ is an ideal of $(X, 1)_*$ if and only if $Ł^{\varepsilon}_{\psi}$ satisfies:*

$$(\forall x, y \in X)\left(Ł^{\varepsilon}_{\psi}(y) \leq \max\{Ł^{\varepsilon}_{\psi}(x * y), 0.5\}\right), \tag{18}$$

$$(\forall x, y, z \in X)\left(\min\{Ł^{\varepsilon}_{\psi}(x), Ł^{\varepsilon}_{\psi}(y)\} \leq \max\{Ł^{\varepsilon}_{\psi}((x * (y * z)) * z), 0.5\}\right). \tag{19}$$

Proof. Assume that $(Ł^{\varepsilon}_{\psi}, t)_{\in}$ is an ideal of $(X, 1)_*$ for $t \in (0.5, 1]$. If there exist $a, b \in X$ such that $Ł^{\varepsilon}_{\psi}(b) > \max\{Ł^{\varepsilon}_{\psi}(a * b), 0.5\}$, then $Ł^{\varepsilon}_{\psi}(b) \in (0.5, 1]$ and $Ł^{\varepsilon}_{\psi}(a * b) < Ł^{\varepsilon}_{\psi}(b)$. Hence $\langle b/Ł^{\varepsilon}_{\psi}(b)\rangle \in Ł^{\varepsilon}_{\psi}$, and so $b \in (Ł^{\varepsilon}_{\psi}, Ł^{\varepsilon}_{\psi}(b))_{\in}$, but $a * b \notin (Ł^{\varepsilon}_{\psi}, Ł^{\varepsilon}_{\psi}(b))_{\in}$. This is a contradiction, and thus $Ł^{\varepsilon}_{\psi}(y) \leq \max\{Ł^{\varepsilon}_{\psi}(x * y), 0.5\}$ for all $x, y \in X$. If the condition (19) is not valid, then there exist $a, b, c \in X$ such that

$$\min\{Ł^{\varepsilon}_{\psi}(a), Ł^{\varepsilon}_{\psi}(b)\} > \max\{Ł^{\varepsilon}_{\psi}((a * (b * c)) * c), 0.5\}.$$

If we take $t := \min\{Ł^{\varepsilon}_{\psi}(a), Ł^{\varepsilon}_{\psi}(b)\}$, then $t \in (0.5, 1]$, $\langle a/t\rangle \in Ł^{\varepsilon}_{\psi}$ and $\langle b/t\rangle \in Ł^{\varepsilon}_{\psi}$, but $\langle((a*(b*c))*c)/t\rangle \,\overline{\in}\, Ł^{\varepsilon}_{\psi}$, that is, $a \in (Ł^{\varepsilon}_{\psi}, t)_{\in}$ and $b \in (Ł^{\varepsilon}_{\psi}, t)_{\in}$, but $(a*(b*c))*c \notin (Ł^{\varepsilon}_{\psi}, t)_{\in}$. This is a contradiction, and thus (19) is valid.

Conversely, suppose that $Ł^{\varepsilon}_{\psi}$ satisfies (18) and (19), and let $y \in (Ł^{\varepsilon}_{\psi}, t)_{\in}$ for $t \in (0.5, 1]$. Then $t \leq Ł^{\varepsilon}_{\psi}(y) \leq \max\{Ł^{\varepsilon}_{\psi}(x*y), 0.5\}$ by (18). Hence $Ł^{\varepsilon}_{\psi}(x*y) \geq t$, and so $x*y \in (Ł^{\varepsilon}_{\psi}, t)_{\in}$. Let $x, y \in X$ and $t \in (0.5, 1]$ be such that $x \in (Ł^{\varepsilon}_{\psi}, t)_{\in}$ and $y \in (Ł^{\varepsilon}_{\psi}, t)_{\in}$. Then $Ł^{\varepsilon}_{\psi}(x) \geq t$ and $Ł^{\varepsilon}_{\psi}(y) \geq t$, which imply from (19) that

$$0.5 < t \leq \min\{Ł^{\varepsilon}_{\psi}(x), Ł^{\varepsilon}_{\psi}(y)\} \leq \max\{Ł^{\varepsilon}_{\psi}((x * (y * z)) * z), 0.5\}$$

for all $z \in X$. Hence $\langle((x*(y*z))*z)/t\rangle \in Ł^{\varepsilon}_{\psi}$, that is, $(x*(y*z))*z \in (Ł^{\varepsilon}_{\psi}, t)_{\in}$. Therefore $(Ł^{\varepsilon}_{\psi}, t)_{\in}$ is an ideal of $(X, 1)_*$ for $t \in (0.5, 1]$.

Theorem 5. *Let $Ł^{\varepsilon}_{\psi}$ be a Łukasiewicz fuzzy set in X. Then the $\in$-set $(Ł^{\varepsilon}_{\psi}, t)_{\in}$ of $Ł^{\varepsilon}_{\psi}$ with value $t \in (0.5, 1]$ is an ideal of $(X, 1)_*$ if and only if $Ł^{\varepsilon}_{\psi}$ satisfies:*

$$(\forall x \in X)\left(Ł^{\varepsilon}_{\psi}(x) \leq \max\{Ł^{\varepsilon}_{\psi}(1), 0.5\}\right), \tag{20}$$

$$(\forall x, y, z \in X)\left(\min\{Ł^{\varepsilon}_{\psi}(x * (y * z)), Ł^{\varepsilon}_{\psi}(y)\} \leq \max\{Ł^{\varepsilon}_{\psi}(x * z), 0.5\}\right). \tag{21}$$

Proof. Assume that $(\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$ is an ideal of $(X,1)_*$ for $t \in (0.5, 1]$. If there exist $a \in X$ such that $\text{Ł}^{\varepsilon}_{\psi}(a) > \max\{\text{Ł}^{\varepsilon}_{\psi}(1), 0.5\}$, then $\text{Ł}^{\varepsilon}_{\psi}(a) \in (0.5, 1]$ and $\text{Ł}^{\varepsilon}_{\psi}(1) < \text{Ł}^{\varepsilon}_{\psi}(a)$. Hence $\langle a/\text{Ł}^{\varepsilon}_{\psi}(a)\rangle \in \text{Ł}^{\varepsilon}_{\psi}$, and so $a \in (\text{Ł}^{\varepsilon}_{\psi}, \text{Ł}^{\varepsilon}_{\psi}(a))_{\in}$, but $1 \notin (\text{Ł}^{\varepsilon}_{\psi}, \text{Ł}^{\varepsilon}_{\psi}(a))_{\in}$. This is a contradiction, and thus $\text{Ł}^{\varepsilon}_{\psi}(x) \leq \max\{\text{Ł}^{\varepsilon}_{\psi}(1), 0.5\}$ for all $x \in X$. If the condition (21) is not valid, then there exist $a, b, c \in X$ such that

$$\min\{\text{Ł}^{\varepsilon}_{\psi}(a * (b * c)), \text{Ł}^{\varepsilon}_{\psi}(b)\} > \max\{\text{Ł}^{\varepsilon}_{\psi}(a * c), 0.5\}.$$

If we take $t := \min\{\text{Ł}^{\varepsilon}_{\psi}(a * (b * c)), \text{Ł}^{\varepsilon}_{\psi}(b)\}$, then $t \in (0.5, 1]$, $\langle (a * (b * c))/t\rangle \in \text{Ł}^{\varepsilon}_{\psi}$ and $\langle b/t\rangle \in \text{Ł}^{\varepsilon}_{\psi}$, but $\langle (a * c)/t\rangle \,\overline{\in}\, \text{Ł}^{\varepsilon}_{\psi}$, that is, $a * (b * c) \in (\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$ and $b \in (\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$, but $a * c \notin (\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$. This is a contradiction, and thus (21) is valid.

Conversely, suppose that $\text{Ł}^{\varepsilon}_{\psi}$ satisfies (20) and (21), and let $t \in (0.5, 1]$. For every $x \in (\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$, we have $t \leq \text{Ł}^{\varepsilon}_{\psi}(x) \leq \max\{\text{Ł}^{\varepsilon}_{\psi}(1), 0.5\}$ by (20). Hence $\text{Ł}^{\varepsilon}_{\psi}(1) \geq t$, and so $1 \in (\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$. Let $x, y, z \in X$ and $t \in (0.5, 1]$ be such that $x * (y * z) \in (\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$ and $y \in (\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$. Then $\text{Ł}^{\varepsilon}_{\psi}(x * (y * z)) \geq t$ and $\text{Ł}^{\varepsilon}_{\psi}(y) \geq t$, which imply from (21) that

$$0.5 < t \leq \min\{\text{Ł}^{\varepsilon}_{\psi}(x * (y * z)), \text{Ł}^{\varepsilon}_{\psi}(y)\} \leq \max\{\text{Ł}^{\varepsilon}_{\psi}(x * z), 0.5\}.$$

Hence $\langle (x * z)/t\rangle \in \text{Ł}^{\varepsilon}_{\psi}$, that is, $x * z \in (\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$. Therefore $(\text{Ł}^{\varepsilon}_{\psi}, t)_{\in}$ is an ideal of $(X,1)_*$ for $t \in (0.5, 1]$ by Lemma 1.

Remark 1. *In Theorems 4 and 5, if* $t \notin (0.5, 1]$*, that is, there exists at least one* $t \leq 0.5$*, then Theorems 4 and 5 are incorrect as shown in the following example.*

Example 2. *Consider the BE-algebra* $(X,1)_*$ *in Example 1 and let* ψ *be a fuzzy set in* X *defined as follows.*

$$\psi : X \to [0,1],\ x \mapsto \begin{cases} 0.92 & \text{if } x = 1, \\ 0.66 & \text{if } x = a, \\ 0.66 & \text{if } x = b, \\ 0.77 & \text{if } x = c, \\ 0.81 & \text{if } x = d, \\ 0.95 & \text{if } x = 0. \end{cases}$$

For $\varepsilon := 0.61$*, the Łukasiewicz fuzzy set* $\text{Ł}^{\varepsilon}_{\psi}$ *of* ψ *in* X *is given as follows.*

$$\text{Ł}^{\varepsilon}_{\psi} : X \to [0,1],\ y \mapsto \begin{cases} 0.53 & \text{if } y = 1, \\ 0.27 & \text{if } y \in \{a, b\}, \\ 0.38 & \text{if } y = c, \\ 0.42 & \text{if } y = d, \\ 0.56 & \text{if } y = 0. \end{cases}$$

Then $(\text{Ł}^{\varepsilon}_{\psi}, 0.41)_{\in} = \{1, d, 0\}$ is not an ideal of $(X,1)_*$ because of $b * 0 = c \notin (\text{Ł}^{\varepsilon}_{\psi}, 0.41)_{\in}$. In this case, we know that $\text{Ł}^{\varepsilon}_{\psi}(0) = 0.56 \nleq 0.5 = \max\{\text{Ł}^{\varepsilon}_{\psi}(b * 0), 0.5\}$ and $\text{Ł}^{\varepsilon}_{\psi}(0) = 0.56 \nleq 0.53 = \max\{\text{Ł}^{\varepsilon}_{\psi}(1), 0.5\}$.

Theorem 6. *If a Łukasiewicz fuzzy set $Ł^{\varepsilon}_{\psi}$ in X satisfies:*

$$(\forall x \in X)(\forall t \in (0.5, 1]) \left(\langle x/t \rangle \, q \, Ł^{\varepsilon}_{\psi} \;\Rightarrow\; \langle 1/t \rangle \in Ł^{\varepsilon}_{\psi}\right), \tag{22}$$

$$(\forall x, y, z \in X)(\forall t_a, t_b \in (0.5, 1]) \begin{pmatrix} \langle (x*(y*z))/t_a \rangle \, q \, Ł^{\varepsilon}_{\psi}, \, \langle y/t_b \rangle \, q \, Ł^{\varepsilon}_{\psi} \\ \Rightarrow \langle (x*z)/\max\{t_a, t_b\} \rangle \in Ł^{\varepsilon}_{\psi} \end{pmatrix}, \tag{23}$$

then the non-empty $\in$-set $(Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X, 1)_*$ for all $t_a, t_b \in (0.5, 1]$.

Proof. Let $t_a, t_b \in (0.5, 1]$ and assume that the $\in$-set $(Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ of $Ł^{\varepsilon}_{\psi}$ is non-empty. Then there exists $x \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$, and so $Ł^{\varepsilon}_{\psi}(x) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, i.e., $\langle x/\max\{t_a, t_b\} \rangle \, q \, Ł^{\varepsilon}_{\psi}$. Hence $\langle 1/\max\{t_a, t_b\} \rangle \in Ł^{\varepsilon}_{\psi}$ by (22), and thus $1 \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$. Let $x, y, z \in X$ be such that $x*(y*z) \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ and $y \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$. Then $Ł^{\varepsilon}_{\psi}(x*(y*z)) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$ and $Ł^{\varepsilon}_{\psi}(y) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, that is, $\langle (x*(y*z))/\max\{t_a, t_b\} \rangle \, q \, Ł^{\varepsilon}_{\psi}$ and $\langle y/\max\{t_a, t_b\} \rangle \, q \, Ł^{\varepsilon}_{\psi}$. It follows from (23) that $\langle (x*z)/\max\{t_a, t_b\} \rangle \in Ł^{\varepsilon}_{\psi}$. Hence $x*z \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$, and therefore $(Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ is an ideal of $(X, 1)_*$ for all $t_a, t_b \in (0.5, 1]$ by Lemma 1.

Theorem 7. *If a Łukasiewicz fuzzy set $Ł^{\varepsilon}_{\psi}$ in X satisfies* (22) *and*

$$(\forall x, y, z \in X)(\forall t_a, t_b \in (0.5, 1]) \begin{pmatrix} \langle (x*(y*z))/t_a \rangle \, q \, Ł^{\varepsilon}_{\psi}, \, \langle y/t_b \rangle \, q \, Ł^{\varepsilon}_{\psi} \\ \Rightarrow \langle (x*z)/\min\{t_a, t_b\} \rangle \in Ł^{\varepsilon}_{\psi} \end{pmatrix}, \tag{24}$$

then the non-empty $\in$-set $(Ł^{\varepsilon}_{\psi}, \min\{t_a, t_b\})_{\in}$ of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X, 1)_*$ for all $t_a, t_b \in (0.5, 1]$.

Proof. It can be verified through a process similar to the proof in Theorem 6.

Theorem 8. *If a Łukasiewicz fuzzy set $Ł^{\varepsilon}_{\psi}$ in X satisfies:*

$$(\forall x, y \in X)(\forall t \in (0.5, 1]) \left(\langle y/t \rangle \, q \, Ł^{\varepsilon}_{\psi} \;\Rightarrow\; \langle (x*y)/t \rangle \in Ł^{\varepsilon}_{\psi}\right), \tag{25}$$

and

$$\langle x/t_a \rangle \, q \, Ł^{\varepsilon}_{\psi}, \, \langle y/t_b \rangle \, q \, Ł^{\varepsilon}_{\psi} \;\Rightarrow\; \langle ((x*(y*z))*z)/\max\{t_a, t_b\} \rangle \in Ł^{\varepsilon}_{\psi}, \tag{26}$$

for all $x, y, z \in X$ and $t_a, t_b \in (0.5, 1]$, then the non-empty $\in$-set $(Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X, 1)_*$ for all $t_a, t_b \in (0.5, 1]$.

Proof. Let $y \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ for $t_a, t_b \in (0.5, 1]$. Then $Ł^{\varepsilon}_{\psi}(y) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, and so $\langle y/\max\{t_a, t_b\} \rangle \, q \, Ł^{\varepsilon}_{\psi}$. Hence $\langle (x*y)/\max\{t_a, t_b\} \rangle \in Ł^{\varepsilon}_{\psi}$ for all $x \in X$ by (25), which implies that $x*y \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ for all $x \in X$. Let $x, y \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ for $t_a, t_b \in (0.5, 1]$. Then $Ł^{\varepsilon}_{\psi}(x) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$ and $Ł^{\varepsilon}_{\psi}(y) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, that is, $\langle x/\max\{t_a, t_b\} \rangle \, q \, Ł^{\varepsilon}_{\psi}$ and $\langle y/\max\{t_a, t_b\} \rangle \, q \, Ł^{\varepsilon}_{\psi}$. It follows from (26) that $\langle ((x*(y*z))*z)/\max\{t_a, t_b\} \rangle \in Ł^{\varepsilon}_{\psi}$ for all $z \in X$. Hence $(x*(y*z))*z \in (Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ for all $z \in X$. Therefore $(Ł^{\varepsilon}_{\psi}, \max\{t_a, t_b\})_{\in}$ of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X, 1)_*$ for all $t_a, t_b \in (0.5, 1]$.

Lemma 2. *Every Łukasiewicz fuzzy ideal* $Ł_\psi^\varepsilon$ *of* $(X,1)_*$ *satisfies:*

$$(\forall x, y, z \in X)\left(Ł_\psi^\varepsilon(x*z) \geq \max\{Ł_\psi^\varepsilon(x*(y*z)), Ł_\psi^\varepsilon(y)\}\right).$$

Proof. Note that $\langle (x*(y*z))/Ł_\psi^\varepsilon(x*(y*z))\rangle \in Ł_\psi^\varepsilon$ and $\langle y/Ł_\psi^\varepsilon(y)\rangle \in Ł_\psi^\varepsilon$ for all $x, y, z \in X$. It follows from (17) that $\langle (x*z)/\min\{Ł_\psi^\varepsilon(x*(y*z)), Ł_\psi^\varepsilon(y)\}\rangle \in Ł_\psi^\varepsilon$, that is, $Ł_\psi^\varepsilon(x*z) \geq \min\{Ł_\psi^\varepsilon(x*(y*z)), Ł_\psi^\varepsilon(y)\}$ for all $x, y, z \in X$.

Theorem 9. *If* $Ł_\psi^\varepsilon$ *is a Łukasiewicz fuzzy ideal of* $(X,1)_*$*, then its q-set* $(Ł_\psi^\varepsilon, t)_q$ *is an ideal of* $(X,1)_*$ *for all* $t \in (0,1]$*.*

Proof. Let $Ł_\psi^\varepsilon$ be a Łukasiewicz fuzzy ideal of $(X,1)_*$ and let $t \in (0,1]$. If $1 \notin (Ł_\psi^\varepsilon, t)_q$, then $\langle 1/t\rangle\, \overline{q}\, Ł_\psi^\varepsilon$, i.e., $Ł_\psi^\varepsilon(1) + t \leq 1$. Since $\langle x/Ł_\psi^\varepsilon(x)\rangle \in Ł_\psi^\varepsilon$ for all $x \in X$, we get $\langle 1/Ł_\psi^\varepsilon(x)\rangle \in Ł_\psi^\varepsilon$ for all $x \in X$ by (13). Hence $Ł_\psi^\varepsilon(1) \geq Ł_\psi^\varepsilon(x)$ for $x \in (Ł_\psi^\varepsilon, t)_q$, and so $1 - t \geq Ł_\psi^\varepsilon(1) \geq Ł_\psi^\varepsilon(x)$. This shows that $\langle x/t\rangle\, \overline{q}\, Ł_\psi^\varepsilon$, that is, $x \notin (Ł_\psi^\varepsilon, t)_q$, a contradiction. Thus $1 \in (Ł_\psi^\varepsilon, t)_q$. Let $x, y, z \in X$ be such that $x*(y*z) \in (Ł_\psi^\varepsilon, t)_q$ and $y \in (Ł_\psi^\varepsilon, t)_q$. Then $\langle (x*(y*z))/t\rangle\, q\, Ł_\psi^\varepsilon$ and $\langle y/t\rangle\, q\, Ł_\psi^\varepsilon$, that is, $Ł_\psi^\varepsilon(x*(y*z)) > 1 - t$ and $Ł_\psi^\varepsilon(y) > 1 - t$. It follows from Lemma 2 that

$$Ł_\psi^\varepsilon(x*z) \geq \max\{Ł_\psi^\varepsilon(x*(y*z)), Ł_\psi^\varepsilon(y)\} > 1 - t.$$

Hence $\langle (x*z)/t\rangle\, q\, Ł_\psi^\varepsilon$, and so $x*z \in (Ł_\psi^\varepsilon, t)_q$. Therefore $(Ł_\psi^\varepsilon, t)_q$ is an ideal of $(X,1)_*$ by Lemma 1.

Corollary 2. *If* ψ *is a fuzzy ideal of* $(X,1)_*$*, then the q-set of* $Ł_\psi^\varepsilon$ *is an ideal of* $(X,1)_*$*.*

Proposition 2. *For the Łukasiewicz fuzzy set* $Ł_\psi^\varepsilon$ *in* X*, if the q-set of* $Ł_\psi^\varepsilon$ *is an ideal of* $(X,1)_*$*, then the following arguments are satisfied.*

$$1 \in (Ł_\psi^\varepsilon, t)_\in, \tag{27}$$

$$\langle x/t_a\rangle\, q\, Ł_\psi^\varepsilon,\ \langle y/t_b\rangle\, q\, Ł_\psi^\varepsilon \ \Rightarrow\ (x*(y*z))*z \in (Ł_\psi^\varepsilon, \max\{t_a, t_b\})_\in, \tag{28}$$

$$\langle (x*(y*z))/t_a\rangle\, q\, Ł_\psi^\varepsilon,\ \langle y/t_b\rangle\, q\, Ł_\psi^\varepsilon \ \Rightarrow\ x*z \in (Ł_\psi^\varepsilon, \max\{t_a, t_b\})_\in \tag{29}$$

for all $x, y, z \in X$ *and* $t, t_a, t_b \in (0, 0.5]$*.*

Proof. Assume that the q-set $(Ł_\psi^\varepsilon, t)_q$ of $Ł_\psi^\varepsilon$ is an ideal of $(X,1)_*$. Then $1 \in (Ł_\psi^\varepsilon, t)_q$ by Lemma 1. If $1 \notin (Ł_\psi^\varepsilon, t)_\in$ for some $t \in (0, 0.5]$, then $\langle 1/t\rangle\, \overline{\in}\, Ł_\psi^\varepsilon$. Hence $Ł_\psi^\varepsilon(1) < t \leq 1 - t$ since $t \in (0, 0.5]$, and so $\langle 1/t\rangle\, \overline{q}\, Ł_\psi^\varepsilon$, i.e., $1 \notin (Ł_\psi^\varepsilon, t)_q$. This is a conradiction, and thus $1 \in (Ł_\psi^\varepsilon, t)_\in$. Let $x, y \in X$ and $t_a, t_b \in (0, 0.5]$ be such that $\langle x/t_a\rangle\, q\, Ł_\psi^\varepsilon$ and $\langle y/t_b\rangle\, q\, Ł_\psi^\varepsilon$. Then $x \in (Ł_\psi^\varepsilon, t_a)_q \subseteq (Ł_\psi^\varepsilon, \max\{t_a, t_b\})_q$ and

$$y \in (Ł_\psi^\varepsilon, t_b)_q \subseteq (Ł_\psi^\varepsilon, \max\{t_a, t_b\})_q,$$

from which $(x*(y*z))*z \in (Ł_\psi^\varepsilon, \max\{t_a, t_b\})_q$ is derived. Hence

$$Ł_\psi^\varepsilon((x*(y*z))*z) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\},$$

i.e., $\langle((x*(y*z))*z)/\max\{t_a,t_b\}\rangle \in Ł_\psi^\varepsilon$. Hence $(x*(y*z))*z \in (Ł_\psi^\varepsilon, \max\{t_a,t_b\})_\in$. Let $x,y,z \in X$ and $t_a,t_b \in (0,0.5]$ be such that $\langle(x*(y*z))/t_a\rangle\, q\, Ł_\psi^\varepsilon$ and $\langle y/t_b\rangle\, q\, Ł_\psi^\varepsilon$. Then $x*(y*z) \in (Ł_\psi^\varepsilon, t_a)_q \subseteq (Ł_\psi^\varepsilon, \max\{t_a,t_b\})_q$ and

$$y \in (Ł_\psi^\varepsilon, t_b)_q \subseteq (Ł_\psi^\varepsilon, \max\{t_a,t_b\})_q,$$

from which $x*z \in (Ł_\psi^\varepsilon, \max\{t_a,t_b\})_q$ is derived by Lemma 1. Hence

$$Ł_\psi^\varepsilon(x*z) > 1-\max\{t_a,t_b\} \geq \max\{t_a,t_b\},$$

i.e., $\langle(x*z)/\max\{t_a,t_b\}\rangle \in Ł_\psi^\varepsilon$. Therefore $x*z \in (Ł_\psi^\varepsilon, \max\{t_a,t_b\})_\in$.

Theorem 10. *If a Łukasiewicz fuzzy set $Ł_\psi^\varepsilon$ in X satisfies*

$$(\forall x,y \in X)(\forall t \in (0,1])\left(\langle y/t\rangle \in Ł_\psi^\varepsilon \;\Rightarrow\; \langle(x*y)/t\rangle\, q\, Ł_\psi^\varepsilon\right), \tag{30}$$

and

$$\langle x/t_a\rangle \in Ł_\psi^\varepsilon,\ \langle y/t_b\rangle \in Ł_\psi^\varepsilon \;\Rightarrow\; \langle((x*(y*z))*z)/\min\{t_a,t_b\}\rangle\, q\, Ł_\psi^\varepsilon \tag{31}$$

for all $x,y,z \in X$ and $t_a,t_b \in (0,1]$, then the q-set $(Ł_\psi^\varepsilon, \min\{t_a,t_b\})_q$ of $Ł_\psi^\varepsilon$ is an ideal of $(X,1)_*$ for all $t_a,t_b \in (0,0.5]$.

Proof. Let $t := \min\{t_a,t_b\}$ for all $t_a,t_b \in (0,0.5]$. If $y \in (Ł_\psi^\varepsilon, t)_q$, then $Ł_\psi^\varepsilon(y) > 1-t \geq t$ since $t \leq 0.5$, and so $\langle y/t\rangle \in Ł_\psi^\varepsilon$. Thus $\langle(x*y)/t\rangle\, q\, Ł_\psi^\varepsilon$ by (30), that is, $x*y \in (Ł_\psi^\varepsilon, t)_q = (Ł_\psi^\varepsilon, \min\{t_a,t_b\})_q$ for all $x \in X$. Let $x,y \in X$ be such that $x,y \in (Ł_\psi^\varepsilon, \min\{t_a,t_b\})_q$. Then $Ł_\psi^\varepsilon(x)+t_a \geq Ł_\psi^\varepsilon(x)+\min\{t_a,t_b\} > 1$ and $Ł_\psi^\varepsilon(y)+t_b \geq Ł_\psi^\varepsilon(y)+\min\{t_a,t_b\} > 1$, which implies that $Ł_\psi^\varepsilon(x) > 1-t_a \geq t_a$ and $Ł_\psi^\varepsilon(y) > 1-t_b \geq t_b$, that is, $\langle x/t_a\rangle \in Ł_\psi^\varepsilon$ and $\langle y/t_b\rangle \in Ł_\psi^\varepsilon$. It follows from (31) that

$$\langle((x*(y*z))*z)/\min\{t_a,t_b\}\rangle\, q\, Ł_\psi^\varepsilon$$

for all $z \in X$. Hence $(x*(y*z))*z \in (Ł_\psi^\varepsilon, \min\{t_a,t_b\})_q$ for all $z \in X$. Therefore $(Ł_\psi^\varepsilon, \min\{t_a,t_b\})_q$ is an ideal of $(X,1)_*$ for all $t_a,t_b \in (0,0.5]$.

Theorem 11. *If a Łukasiewicz fuzzy set $Ł_\psi^\varepsilon$ in X satisfies:*

$$(\forall x \in X)(\forall t \in (0,1])\left(\langle x/t\rangle \in Ł_\psi^\varepsilon \;\Rightarrow\; \langle 1/t\rangle\, q\, Ł_\psi^\varepsilon\right), \tag{32}$$

and

$$\langle(x*(y*z))/t_a\rangle \in Ł_\psi^\varepsilon,\ \langle y/t_b\rangle \in Ł_\psi^\varepsilon \;\Rightarrow\; \langle(x*z)/\min\{t_a,t_b\}\rangle\, q\, Ł_\psi^\varepsilon \tag{33}$$

for all $x,y,z \in X$ and $t_a,t_b \in (0,1]$, then the non-empty q-set $(Ł_\psi^\varepsilon, \min\{t_a,t_b\})_q$ of $Ł_\psi^\varepsilon$ is an ideal of $(X,1)_*$ for all $t_a,t_b \in (0,0.5]$.

Proof. Let $t_a, t_b \in (0, 0.5]$. If $(Ł^{\varepsilon}_{\psi}, \min\{t_a, t_b\})_q$ is non-empty, then there exists $x \in (Ł^{\varepsilon}_{\psi}, \min\{t_a, t_b\})_q$. Hence $Ł^{\varepsilon}_{\psi}(x) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$, which shows that $\langle x/\min\{t_a, t_b\}\rangle \in Ł^{\varepsilon}_{\psi}$. It follows from (32) that $\langle 1/\min\{t_a, t_b\}\rangle \, q \, Ł^{\varepsilon}_{\psi}$. Thus $1 \in (Ł^{\varepsilon}_{\psi}, \min\{t_a, t_b\})_q$. Let $x, y, z \in X$ be such that $x * (y * z) \in (Ł^{\varepsilon}_{\psi}, \min\{t_a, t_b\})_q$ and $y \in (Ł^{\varepsilon}_{\psi}, \min\{t_a, t_b\})_q$. Then $Ł^{\varepsilon}_{\psi}(x * (y * z)) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$ and $Ł^{\varepsilon}_{\psi}(y) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$. Thus $\langle (x*(y*z))/\min\{t_a, t_b\}\rangle \in Ł^{\varepsilon}_{\psi}$ and $\langle y/\min\{t_a, t_b\}\rangle \in Ł^{\varepsilon}_{\psi}$. It follows from (33) that $\langle (x*z)/\min\{t_a, t_b\}\rangle \, q \, Ł^{\varepsilon}_{\psi}$, i.e., $x*z \in (Ł^{\varepsilon}_{\psi}, \min\{t_a, t_b\})_q$. Therefore $(Ł^{\varepsilon}_{\psi}, \min\{t_a, t_b\})_q$ is an ideal of $(X, 1)_*$ by Lemma 1.

Theorem 12. *If a Łukasiewicz fuzzy set $Ł^{\varepsilon}_{\psi}$ in X satisfies* (27) *and* (29) *for all $x, y, z \in X$ and $t, t_a, t_b \in (0.5, 1]$, then the q-set $(Ł^{\varepsilon}_{\psi}, t)_q$ of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X, 1)_*$ for all $t \in (0.5, 1]$.*

Proof. Assume that $Ł^{\varepsilon}_{\psi}$ satisfies (27) and (29) for all $x, y, z \in X$ and $t, t_a, t_b \in (0.5, 1]$. The condition (27) induces $Ł_{\psi}(1) + t \geq 2t > 1$, i.e., $\langle 1/t\rangle \, q \, Ł^{\varepsilon}_{\psi}$. Hence $1 \in (Ł^{\varepsilon}_{\psi}, t)_q$. Let $x, y, z \in X$ be such that $x * (y * z) \in (Ł^{\varepsilon}_{\psi}, t)_q$ and $y \in (Ł^{\varepsilon}_{\psi}, t)_q$. Then $\langle (x * (y * z))/t\rangle \, q \, Ł^{\varepsilon}_{\psi}$ and $\langle y/t\rangle \, q \, Ł^{\varepsilon}_{\psi}$. It follows from (29) that $x * z \in (Ł^{\varepsilon}_{\psi}, \min\{t, t\})_{\in} = (Ł^{\varepsilon}_{\psi}, t)_{\in}$. Hence $Ł^{\varepsilon}_{\psi}(x * z) \geq t > 1 - t$, that is, $x * z \in (Ł^{\varepsilon}_{\psi}, t)_q$. Therefore $(Ł^{\varepsilon}_{\psi}, t)_q$ is an ideal of $(X, 1)_*$ for all $t \in (0.5, 1]$ by Lemma 1.

Theorem 13. *If a Łukasiewicz fuzzy set $Ł^{\varepsilon}_{\psi}$ in X satisfies* (28) *for all $x, y, z \in X$ and $t_a, t_b \in (0.5, 1]$, and*

$$(\forall x, y \in X)(\forall t \in (0.5, 1]) \left(\langle y/t\rangle \, q \, Ł^{\varepsilon}_{\psi} \;\Rightarrow\; \langle (x * y)/t\rangle \in Ł^{\varepsilon}_{\psi}\right), \tag{34}$$

then the q-set $(Ł^{\varepsilon}_{\psi}, t)_q$ of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X, 1)_*$ for all $t \in (0.5, 1]$.

Proof. Let $x, y \in X$ and $t \in (0.5, 1]$ be such that $y \in (Ł^{\varepsilon}_{\psi}, t)_q$. Then $\langle y/t\rangle \, q \, Ł^{\varepsilon}_{\psi}$, and so $\langle (x * y)/t\rangle \in Ł^{\varepsilon}_{\psi}$ by (34). Thus $Ł^{\varepsilon}_{\psi}(x * y) \geq t > 1 - t$, that is, $\langle (x * y)/t\rangle \, q \, Ł^{\varepsilon}_{\psi}$. Hence $x * y \in (Ł^{\varepsilon}_{\psi}, t)_q$. Let $x, y \in X$ and $t \in (0.5, 1]$ be such that $x \in (Ł^{\varepsilon}_{\psi}, t)_q$ and $y \in (Ł^{\varepsilon}_{\psi}, t)_q$. Then $Ł^{\varepsilon}_{\psi}(x) \geq t > 1 - t$ and $Ł^{\varepsilon}_{\psi}(y) \geq t > 1 - t$, i.e., $\langle x/t\rangle \, q \, Ł^{\varepsilon}_{\psi}$ and $\langle y/t\rangle \, q \, Ł^{\varepsilon}_{\psi}$. It follows from (28) that $\langle ((x * (y * z)) * z)/t\rangle = \langle ((x * (y * z)) * z)/\min\{t, t\}\rangle \, q \, Ł^{\varepsilon}_{\psi}$. This shows that $(x * (y * z)) * z \in (Ł^{\varepsilon}_{\psi}, t)_q$. Therefore the q-set $(Ł^{\varepsilon}_{\psi}, t)_q$ of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X, 1)_*$ for all $t \in (0.5, 1]$.

Theorem 14. *If ψ is a fuzzy ideal of $(X, 1)_*$, then the non-empty O-set of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X, 1)_*$.*

Proof. If ψ is a fuzzy ideal of $(X, 1)_*$, then $Ł^{\varepsilon}_{\psi}$ is a Łukasiewicz fuzzy ideal of $(X, 1)_*$ (see Theorem 3). It is clear that $1 \in O(Ł^{\varepsilon}_{\psi})$. Let $x, y, z \in X$ be such that $y \in O(Ł^{\varepsilon}_{\psi})$ and $x * (y * z) \in O(Ł^{\varepsilon}_{\psi})$. Then $Ł^{\varepsilon}_{\psi}(x * (y * z)) > 0$ and $Ł^{\varepsilon}_{\psi}(y) > 0$. Since $\langle (x * (y * z))/Ł^{\varepsilon}_{\psi}(x * (y * z))\rangle \in Ł^{\varepsilon}_{\psi}$ and $\langle y/Ł^{\varepsilon}_{\psi}(y)\rangle \in Ł^{\varepsilon}_{\psi}$, we have

$$\langle (x * z)/\min\left\{Ł^{\varepsilon}_{\psi}(x * (y * z)), Ł^{\varepsilon}_{\psi}(y)\right\}\rangle \in Ł^{\varepsilon}_{\psi}$$

by (17). It follows that

$$Ł^{\varepsilon}_{\psi}(x * z) \geq \min\left\{Ł^{\varepsilon}_{\psi}(x * (y * z)), Ł^{\varepsilon}_{\psi}(y)\right\} > 0.$$

Hence $x * z \in O(Ł_\psi^\varepsilon)$, and therefore $O(Ł_\psi^\varepsilon)$ is an ideal of $(X, 1)_*$ by Lemma 1.

Theorem 15. *If a Łukasiewicz fuzzy set $Ł_\psi^\varepsilon$ in X satisfies* (13) *and*

$$(\forall x, y, z \in X)(\forall t_a, t_b \in (0, 1]) \left(\begin{array}{c} \langle (x * (y * z))/t_a \rangle \in Ł_\psi^\varepsilon,\ \langle y/t_b \rangle \in Ł_\psi^\varepsilon \\ \Rightarrow\ \langle (x * z)/\max\{t_a, t_b\} \rangle\, q\, Ł_\psi^\varepsilon \end{array} \right). \tag{35}$$

then the non-empty O-set of $Ł_\psi^\varepsilon$ is an ideal of $(X, 1)_*$.

Proof. Let $O(Ł_\psi^\varepsilon)$ be a non-empty O-set of $Ł_\psi^\varepsilon$. Then there exists $x \in O(Ł_\psi^\varepsilon)$, and so $t := Ł_\psi^\varepsilon(x) > 0$, i.e., $\langle x/t \rangle \in Ł_\psi^\varepsilon$ for $t > 0$. Hence $\langle 1/t \rangle \in Ł_\psi^\varepsilon$ by (13), and thus $Ł_\psi^\varepsilon(1) \geq t > 0$. Thus $1 \in O(Ł_\psi^\varepsilon)$. Let $x, y, z \in X$ be such that $x * (y * z) \in O(Ł_\psi^\varepsilon)$ and $y \in O(Ł_\psi^\varepsilon)$. Then $\psi(x * (y * z)) + \varepsilon > 1$ and $\psi(y) + \varepsilon > 1$. Since $\langle (x * (y * z))/Ł_\psi^\varepsilon(x * (y * z)) \rangle \in Ł_\psi^\varepsilon$ and $\langle y/Ł_\psi^\varepsilon(y) \rangle \in Ł_\psi^\varepsilon$, it follows from (35) that

$$\langle (x * z)/\max\{Ł_\psi^\varepsilon(x * (y * z)),\ Ł_\psi^\varepsilon(y)\} \rangle\, q\, Ł_\psi^\varepsilon.$$

If $x * z \notin O(Ł_\psi^\varepsilon)$, then $Ł_\psi^\varepsilon(x * z) = 0$, and so

$$\begin{aligned} & Ł_\psi^\varepsilon(x * z) + \max\{Ł_\psi^\varepsilon(x * (y * z)),\ Ł_\psi^\varepsilon(y)\} = \max\{Ł_\psi^\varepsilon(x * (y * z)),\ Ł_\psi^\varepsilon(y)\} \\ & = \max\{\max\{0, \psi(x * (y * z)) + \varepsilon - 1\},\ \max\{0, \psi(y) + \varepsilon - 1\}\} \\ & = \max\{\psi(x * (y * z)) + \varepsilon - 1,\ \psi(y) + \varepsilon - 1\} \\ & = \max\{\psi(x * (y * z)),\ \psi(y)\} + \varepsilon - 1 \\ & \leq 1 + \varepsilon - 1 \leq 1. \end{aligned}$$

Hence $\langle (x * z)/\max\{Ł_\psi^\varepsilon(x * (y * z)),\ Ł_\psi^\varepsilon(y)\} \rangle\, \overline{q}\, Ł_\psi^\varepsilon$, a contradiction. Thus $x * z \in O(Ł_\psi^\varepsilon)$, and therefore $O(Ł_\psi^\varepsilon)$ is an ideal of $(X, 1)_*$ by Lemma 1.

Theorem 16. *If a Łukasiewicz fuzzy set $Ł_\psi^\varepsilon$ in X satisfies*

$$(\forall x, y \in X)(\forall t \in (0, 1]) \left(\langle y/t \rangle \in \psi\ \Rightarrow\ \langle (x * y)/t \rangle\, q\, Ł_\psi^\varepsilon \right), \tag{36}$$

and

$$\langle x/t_a \rangle \in \psi,\ \langle y/t_b \rangle \in \psi\ \Rightarrow\ \langle ((x * (y * z)) * z)/\max\{t_a, t_b\} \rangle\, q\, Ł_\psi^\varepsilon \tag{37}$$

for all $x, y, z \in X$ and $t_a, t_b \in (0, 1]$, then the O-set of $Ł_\psi^\varepsilon$ is an ideal of $(X, 1)_*$.

Proof. If $y \in O(Ł_\psi^\varepsilon)$, then $\psi(y) > 1 - \varepsilon$, i.e., $\langle y/(1 - \varepsilon) \rangle \in \psi$. Hence $\langle (x * y)/(1 - \varepsilon) \rangle\, q\, Ł_\psi^\varepsilon$ for all $x \in X$ by (36), and thus $Ł_\psi^\varepsilon(x * y) + 1 - \varepsilon > 1$. Thus $Ł_\psi^\varepsilon(x * y) > \varepsilon > 0$, which shows that $x * y \in O(Ł_\psi^\varepsilon)$ for all $x \in X$. Let $x, y, z \in X$ be such that $x, y \in O(Ł_\psi^\varepsilon)$. Then $\psi(x) > 1 - \varepsilon$ and $\psi(y) > 1 - \varepsilon$, that is, $\langle x/(1 - \varepsilon) \rangle \in \psi$ and $\langle y/(1 - \varepsilon) \rangle \in \psi$. It follows from (37) that

$$\langle ((x * (y * z)) * z)/(1 - \varepsilon) \rangle = \langle ((x * (y * z)) * z)/\max\{1 - \varepsilon, 1 - \varepsilon\} \rangle\, q\, Ł_\psi^\varepsilon.$$

Thus $Ł_\psi^\varepsilon((x * (y * z)) * z) + 1 - \varepsilon > 1$, and so $Ł_\psi^\varepsilon((x * (y * z)) * z) > \varepsilon > 0$. Hence $(x * (y * z)) * z \in O(Ł_\psi^\varepsilon)$, and therefore $O(Ł_\psi^\varepsilon)$ is an ideal of $(X, 1)_*$.

Theorem 17. *Let $Ł^{\varepsilon}_{\psi}$ be a Łukasiewicz fuzzy set in X that satisfies $\langle 1/\varepsilon\rangle\, q\, \psi$ and*

$$(\forall x, y, z \in X)\left(\begin{array}{l} \langle (x*(y*z))/\varepsilon\rangle\, q\,\psi,\ \langle y/\varepsilon\rangle\, q\,\psi \\ \ \Rightarrow\ \langle (x*z)/\varepsilon\rangle \in Ł^{\varepsilon}_{\psi} \end{array}\right). \tag{38}$$

Then the O-set of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X,1)_*$.

Proof. Let $O(Ł^{\varepsilon}_{\psi})$ be the O-set of $Ł^{\varepsilon}_{\psi}$. If $\langle 1/\varepsilon\rangle\, q\,\psi$, then $\psi(1)+\varepsilon > 1$ and so $Ł^{\varepsilon}_{\psi}(1) = \max\{0, \psi(1)+\varepsilon-1\} = \psi(1)+\varepsilon-1 > 0$. Hence $1 \in O(Ł^{\varepsilon}_{\psi})$. Let $x, y, z \in X$ be such that $x*(y*z) \in O(Ł^{\varepsilon}_{\psi})$ and $y \in O(Ł^{\varepsilon}_{\psi})$. Then $\psi(x*(y*z))+\varepsilon > 1$ and $\psi(y)+\varepsilon > 1$, i.e., $\langle (x*(y*z))/\varepsilon\rangle\, q\,\psi$ and $\langle y/\varepsilon\rangle\, q\,\psi$. It follows from (38) that $\langle (x*z)/\varepsilon\rangle \in Ł^{\varepsilon}_{\psi}$, which shows $Ł^{\varepsilon}_{\psi}(x*z) \geq \varepsilon > 0$. Hence $x*z \in O(Ł^{\varepsilon}_{\psi})$, and therefore $O(Ł^{\varepsilon}_{\psi})$ is an ideal of $(X,1)_*$ by Lemma 1.

Theorem 18. *Let $Ł^{\varepsilon}_{\psi}$ be a Łukasiewicz fuzzy set in X that satisfies:*

$$(\forall x, y \in X)(\forall t \in [\varepsilon, 1])\left(\langle y/t\rangle\, q\,\psi\ \Rightarrow\ \langle (x*y)/\varepsilon\rangle \in Ł^{\varepsilon}_{\psi}\right), \tag{39}$$

$$(\forall x, y, z \in X)(\forall t_a, t_b \in [\varepsilon, 1])\left(\begin{array}{l} \langle x/t_a\rangle\, q\,\psi,\ \langle y/t_b\rangle\, q\,\psi \\ \ \Rightarrow (x*(y*z))*z \in (Ł^{\varepsilon}_{\psi}, \varepsilon)_{\in} \end{array}\right). \tag{40}$$

Then the O-set of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X,1)_*$.

Proof. Let $t \in [\varepsilon, 1]$, $x \in X$ and $y \in O(Ł^{\varepsilon}_{\psi})$. Then $\psi(y)+t \geq \psi(y)+\varepsilon > 1$, and so $\langle y/t\rangle\, q\,\psi$, which implies that $\langle (x*y)/\varepsilon\rangle \in Ł^{\varepsilon}_{\psi}$ by (39). Hence $Ł^{\varepsilon}_{\psi}(x*y) \geq \varepsilon > 0$, i.e., $x*y \in O(Ł^{\varepsilon}_{\psi})$. Let $t_a, t_b \in [\varepsilon, 1]$ and $x, y, z \in X$ be such that $x \in O(Ł^{\varepsilon}_{\psi})$ and $y \in O(Ł^{\varepsilon}_{\psi})$. Then $\psi(x)+t_a \geq \psi(x)+\varepsilon > 1$ and $\psi(y)+t_b \geq \psi(y)+\varepsilon > 1$. Thus $\langle x/t_a\rangle\, q\,\psi$ and $\langle y/t_b\rangle\, q\,\psi$ Using (40) leads to $(x*(y*z))*z \in (Ł^{\varepsilon}_{\psi}, \varepsilon)_{\in}$. Hence $Ł^{\varepsilon}_{\psi}((x*(y*z))*z) \geq \varepsilon > 0$, and so $(x*(y*z))*z \in O(Ł^{\varepsilon}_{\psi})$. Consequently, $O(Ł^{\varepsilon}_{\psi})$ is an ideal of $(X,1)_*$.

Corollary 3. *Let $Ł^{\varepsilon}_{\psi}$ be a Łukasiewicz fuzzy set in X that satisfies:*

$$(\forall x, y \in X)\left(\langle y/\varepsilon\rangle\, q\,\psi\ \Rightarrow\ \langle (x*y)/\varepsilon\rangle \in Ł^{\varepsilon}_{\psi}\right), \tag{41}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} \langle x/\varepsilon\rangle\, q\,\psi,\ \langle y/\varepsilon\rangle\, q\,\psi \\ \ \Rightarrow (x*(y*z))*z \in (Ł^{\varepsilon}_{\psi}, \varepsilon)_{\in} \end{array}\right). \tag{42}$$

Then the O-set of $Ł^{\varepsilon}_{\psi}$ is an ideal of $(X,1)_*$.

4. Conclusions and future work

The concept of Łukasiewicz fuzzy sets using Łukasiewicz t-norm was introduced by Y. B. Jun. In this paper, Łukasiewicz fuzzy set has been applied to the ideal in BE-algebra, and introducing the concept of Łukasiewicz fuzzy ideal and examining several properties. We discussed the characterization of Łukasiewicz fuzzy ideal and considered the relationship between fuzzy ideal and Łukasiewicz fuzzy ideal. We provided conditions under which Łukasiewicz fuzzy set can be Łukasiewicz fuzzy ideal, and further explored conditions under which three subsets, $\in$-set, q-set, and O-set, will be ideal

The ideas and results obtained in this paper will be applied to the relevant algebraic systems in the future, further examining their usability as a mathematical tool applicable to decision theory, medical diagnosis systems, and automation systems etc.

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