EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1444-1454
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Exact Solution of Burger's Equation Using Tensor Product Technique

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Abstract. In this paper a new technique using tensor product is presented which yields an exact solution to Burger's equation

$$u_t + \alpha u u_x = \upsilon u_{xx}$$

which is one of the very few nonlinear partial differential equations that can be solved analytically. More over we give an atomic solution for linear partial differential equations with and without variable coefficient terms.

2020 Mathematics Subject Classifications: 46M05, 35-xx, 46Bxx

Key Words and Phrases: Burger's equation, tensor product, atom solution

1. Introduction

One of the well known partial differential equations which governs a wide variety of mathematical models is the Burger's equation which provides the simplest nonlinear model of turbulence, and else occurring in various areas of applied mathematics such as fluid mechanics, gas dynamics, and traffic flow. This equation was first introduced by Harry Bateman in 1915, [1] and later studied by Johannes Martinus Burgers, [4] in 1948.

In this paper, we present a new way of solving the nonlinear (Burger equation) partial differential equation

$$u_t + \alpha u u_x = \upsilon u_{xx}$$

using tensor product technique.

General Burger's Equation:

Consider the one-dimensional quasi-linear Burger's equation with the following initial and boundary conditions:

$$\begin{aligned} u_t + \alpha u u_x &= \upsilon u_{xx} \\ u(x,0) &= f(x),\ 0 \le x \le l \end{aligned}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4468

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$$\begin{aligned} u(0,t) &= f_1(t) \\ u_x(0,t) &= f_2(t),\ t > 0, \end{aligned}$$

where $u = u(x,t)$ is unknown function in some domain and the nonlinear term coefficient α is an arbitrary constant and υ is the coefficient of the kinematics viscosity of fluids which is equal to $\frac{1}{R}$. Further, R is the Reynolds number, and when it is large the equation describes shock wave behavior, where uu_x is the nonlinear term.

This equation has been solved in different methods, such as Homotopy Perturbation Method [2], Linearized solution, and numerically like the least-squares quadratic B-spline finite element method [5, 6, 12–14], explicit and exact-explicit finite difference methods, variational iteration method (VIM) [3].

As well, tensor product used to solve one of the classical differential equations in Banach spaces is called Abstract Cauchy Problem by Ziqan, Al-Horani, and Khalil [15], and also Abdullah and Khalil [7] in a different conditions. Also, many of the non-homogeneous second order partial differential equations has been solved by finding an atomic solution $u = u_1 \otimes x$ [10, 11].

2. Tensor Product

Let X, Y be two Banach spaces and X^*, Y^* denote their respective duals. For $x \in X$ and $y \in Y$, define the linear operator

$$\begin{aligned} x \otimes y &: X^* \longrightarrow Y \\ x \otimes y(x^*) &= \langle x, x^* \rangle y, \end{aligned}$$

where $\langle x, x^* \rangle$ is the value of x^* at x, $x \otimes y$ is called an atom. It is easy to see that $x \otimes y$ is a bounded linear operator with norm $\|x \otimes y\| = \|x\|\,\|y\|$.

The tensor product $X \otimes Y := span\{x \otimes y : x \in X,\ y \in Y\}$, *where* $X \otimes Y \subseteq \mathcal{L}(X^*, Y)$; $\mathcal{L}(X^*, Y)$ is the space of bounded linear operators from X^* *into* Y, $X \otimes Y$ is a linear subspace of finite rank operators in $\mathcal{L}(X^*, Y)$ [8].

Lemma 1. [9] *For any $x, z \in X$, $y, t \in Y$ and scaler β, the following are valid:*

1- $\beta(x \otimes y) = \beta x \otimes y = x \otimes \beta y$.
2- $(x + z) \otimes y = x \otimes y + z \otimes y$.
3- $x \otimes (y + t) = x \otimes y + x \otimes t$.
4- $x \otimes 0 = 0 \otimes y = 0 \otimes 0$.
5- $||x \otimes y|| = ||x||\ ||y||$.
6- Any $T \in X \otimes Y$ can be written as $\sum_{i=1}^{n} \lambda_i (x_i \otimes y_i)$ with $||x_i|| = ||y_i|| = 1$.

Definition 1. [9] *Let $T = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, define the* ***injective norm*** *on $X \otimes Y$ as*

$$||T||_\vee = \sup \left\{ \sum_{i=1}^{n} |\langle x, x^* \rangle \langle y, y^* \rangle|,\ x^* \otimes y^* \in X^* \otimes Y^*,\ ||x^*|| = ||y^*|| = 1 \right\}.$$

The space $(X \otimes Y, ||.||_{\vee})$ need not be complete. We let $X \overset{\vee}{\otimes} Y$ denote the completion of $X \otimes Y$ in $\mathcal{L}(X^*, Y)$ with respect to the injective norm.

Theorem 1. *[9]For any compact Hausdorff space I and a Banach space X, we have $C(I, X)$ is isometrically isomorphic to $C(I) \overset{\vee}{\otimes} X$.*

For more on tensor product we refer the reader to [9].

3. Atomic Solution of linear partial differential equations

In this section, we solve two kinds of partial differential equations by using tensor product technique. We start by the following lemma:

Lemma 2. *Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be two non zero atoms in $X \overset{\vee}{\otimes} Y$. Then the following are equivalent:*
(i) $x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3$ a non zero atom.
(ii) x_1, x_2 or y_1, y_2 are linearly dependent.

Proof. $(i) \to (ii)$ If $x_3 \otimes y_3 = 0$, we are done. Assume $x_3 \otimes y_3 \neq 0$. Then there exists $t_0 \in I$ and $y^* \in Y^*$ such that $x_3(t_0) \neq 0$ and $y^*(y_3) \neq 0$. using (i) we have

$$\begin{aligned} x_3 &= \frac{y^*(y_1)}{y^*(y_3)} x_1 + \frac{y^*(y_2)}{y^*(y_3)} x_2 = c_1 x_1 + c_2 x_2, \\ y_3 &= \frac{x_1(t_0)}{x_3(t_0)} y_1 + \frac{x_2(t_0)}{x_3(t_0)} y_2 = b_1 y_1 + b_2 y_2. \end{aligned}$$

Consequently

$$\begin{aligned} x_3 \otimes y_3 &= c_1 b_1 x_1 \otimes y_1 + c_1 b_2 x_1 \otimes y_2 + c_2 b_1 x_2 \otimes y_1 + b_2 c_2 x_2 \otimes y_2 \\ &= x_1 \otimes y_1 + x_2 \otimes y_2. \end{aligned}$$

Hence

$$x_1 \otimes y_1 (1 - c_1 b_1) + x_2 \otimes y_2 (1 - c_2 b_2) + c_1 b_2 x_1 \otimes y_2 + c_2 b_1 x_2 \otimes y_1 = 0.$$

If x_1, x_2 and y_1, y_2 are linearly independent, it follows that

$$1 - b_1 c_1 = 1 - b_2 c_2 = b_2 c_1 = b_1 c_2 = 0,$$

which turns out to a contradiction $1 = b_1 c_1$, $1 - b_2 c_2, b_2 c_1 = 0$, $b_1 c_2 = 0$. Hence the result.

$(ii) \to (i)$ If x_1, x_2 are linearly dependent, then $x_1 = \lambda x_2$. Using (ii)

$$x_3 \otimes y_3 = \lambda x_2 \otimes y_1 + x_2 \otimes y_2 = x_2 \otimes (\lambda y_1 + y_2)$$

which completes the proof.

Theorem 2. *Let $u(x,t) \in C(I \times J)$, where $I,\ J = [0,1]$ or $[0,\infty)$. If u has continuous second partial derivatives, then the linear differential equation*

$$u_t + u_x = u_{xx} \tag{1}$$

has an atomic solution.

Proof. Let $u(x,t) = \Phi \otimes \Psi$, where Φ is a function of x and Ψ is a function of t with $\Phi(0) = 1$, $\Phi'(0) = 1$ and $\Psi(0) = 1$. Then $u_x = \Phi' \otimes \Psi$, $u_t = \Phi \otimes \Psi'$ and $u_{xx} = \Phi'' \otimes \Psi$. This implies that

$$\Phi \otimes \Psi' + \Phi' \otimes \Psi = \Phi'' \otimes \Psi. \tag{2}$$

By using Lemma (2) either $\Phi' = \lambda\Phi$ or $\Psi' = \mu\Psi$. Without loss of generality we can assume $\lambda = \mu = 1$.

Case (1) If $\Phi' = \Phi = \Phi''$, then

$$\frac{\Phi'}{\Phi} = 1.$$

Integrating both sides,we get

$$\begin{aligned} \int \frac{d\Phi}{\Phi} &= \int dx \\ \ln|\Phi| &= x + c \\ \Phi &= ce^x \end{aligned}$$

since $\Phi(0) = 1 \Longrightarrow$

$$\Phi = e^x.$$

Now, since the first and the second derivatives of Φ are equal, then equation (2) becomes

$$\begin{aligned} \Phi \otimes \Psi' + \Phi \otimes \Psi &= \Phi \otimes \Psi. \\ (\Psi' + \Psi - \Psi) \otimes \Phi &= 0. \\ \Psi' \otimes \Phi &= 0. \end{aligned}$$

Here $\Phi = 0$ or $\Psi' = 0$. If $\Phi = 0$, then we have a contradiction since $\Phi \neq 0$. So $\Psi' = 0$, this implies $\Psi = K$, where K is a constant.

To verify equation (2), set $u = \Phi \otimes \Psi$, $u_x = \Phi' \otimes \Psi = \Phi \otimes \Psi$, $u_{xx} = \Phi'' \otimes \Psi = \Phi \otimes \Psi$, $u_t = \Phi \otimes \Psi' = \Phi \otimes 0 = 0$.

Then

$$\begin{aligned} u_t + u_x &= \Phi \otimes 0 + \Phi \otimes \Psi \\ &= 0 + \Phi \otimes \Psi \\ &= \Phi \otimes \Psi \\ &= u_{xx}. \end{aligned}$$

This implies $u = \Phi \otimes \Psi$, where $\Phi = e^x$ and $\Psi = K$.
Case (2) If $\Psi' = \Psi$, then

$$\frac{\Psi'}{\Psi} = 1.$$

Integrating both sides, we get

$$\begin{aligned} \int \frac{d\Psi}{\Psi} &= \int dt \\ \ln|\Psi| &= t + a \\ \Psi &= ae^t. \end{aligned}$$

since $\Psi(0) = 1 \Longrightarrow$

$$\Psi = e^t.$$

Now, $\Psi' = e^t = \Psi$, then equation (2) becomes

$$\begin{aligned} \Phi \otimes \Psi + \Phi' \otimes \Psi &= \Phi'' \otimes \Psi \\ [\Phi + \Phi' - \Phi''] \otimes \Psi &= 0. \end{aligned}$$

Using Lemma (1) $\Phi'' - \Phi' - \Phi = 0$ or $\Psi = 0$. If $\Psi = 0$, then we have a contradiction since $\Psi \neq 0$. So

$$\Phi'' - \Phi' - \Phi = 0. \tag{3}$$

The characteristic equation of equation (3) is

$$\lambda^2 - \lambda - 1 = 0,$$

with roots $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Hence

$$\Phi = Ge^{\lambda_1 x} + Fe^{\lambda_2 x},$$

since $\Phi(0) = 1$ and $\Phi'(0) = 1$, then $F = \frac{\lambda_1 - 1}{\lambda_1 - \lambda_2}$ and $G = \frac{\lambda_2 - 1}{\lambda_2 - \lambda_1}$.

To verify equation (2), set $u = \Phi \otimes \Psi$, $u_x = \Phi' \otimes \Psi$, $u_{xx} = \Phi'' \otimes \Psi$, $u_t = \Phi \otimes \Psi'$, where $\Phi = \frac{\lambda_2-1}{\lambda_2-\lambda_1}e^{\lambda_1 x} + \frac{\lambda_1-1}{\lambda_1-\lambda_2}e^{\lambda_2 x}$, $\Phi' = \lambda_1 \frac{\lambda_2-1}{\lambda_2-\lambda_1}e^{\lambda_1 x} + \lambda_2 \frac{\lambda_1-1}{\lambda_1-\lambda_2}e^{\lambda_2 x}$, $\Phi'' = \lambda_1^2 \frac{\lambda_2-1}{\lambda_2-\lambda_1}e^{\lambda_1 x} + \lambda_2^2 \frac{\lambda_1-1}{\lambda_1-\lambda_2}e^{\lambda_2 x}$, $\Psi = e^t$, and $\Psi' = e^t$.

Then

$$\begin{aligned} u_t + u_x &= \Phi \otimes \Psi' + \Phi' \otimes \Psi \\ &= \Phi \otimes \Psi + \Phi' \otimes \Psi \\ &= [\Phi + \Phi'] \otimes \Psi, \end{aligned}$$

and so

$$\Phi + \Phi' = \left(\frac{\lambda_2 - 1}{\lambda_2 - \lambda_1}e^{\lambda_1 x} + \frac{\lambda_1 - 1}{\lambda_1 - \lambda_2}e^{\lambda_2 x} + \lambda_1 \frac{\lambda_2 - 1}{\lambda_2 - \lambda_1}e^{\lambda_1 x} + \lambda_2 \frac{\lambda_1 - 1}{\lambda_1 - \lambda_2}e^{\lambda_2 x}\right)$$

$$= \quad (1+\lambda_1)(\frac{\lambda_2 - 1}{\lambda_2 - \lambda_1})e^{\lambda_1 x} + (1+\lambda_2)(\frac{\lambda_1 - 1}{\lambda_1 - \lambda_2})e^{\lambda_2 x}.$$

Now, $1+\lambda_1 = 1 + \frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$ and $1+\lambda_2 = 1 + \frac{1-\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$, but $\lambda_1^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2^2 = (\frac{1-\sqrt{5}}{2})^2 = \frac{3-\sqrt{5}}{2}$ so $1+\lambda_1 = \lambda_1^2$ and $1+\lambda_2 = \lambda_2^2$, then

$$\begin{aligned} \Phi + \Phi' &= (1+\lambda_1)e^{\lambda_1 x} + (1+\lambda_2)e^{\lambda_2 x} \\ &= \lambda_1^2 e^{\lambda_1 x} + \lambda_2^2 e^{\lambda_2 x} \\ &= \Phi''. \end{aligned}$$

$$\begin{aligned} u_t + u_x &= [\Phi + \Phi'] \otimes \Psi \\ &= \Phi'' \otimes \Psi \\ &= u_{xx}. \end{aligned}$$

This implies that $u = \Phi \otimes \Psi$ is a solution of equation (1), where $\Phi = (\frac{\lambda_2-1}{\lambda_2-\lambda_1})e^{\lambda_1 x} + (\frac{\lambda_1-1}{\lambda_1-\lambda_2})e^{\lambda_2 x}$ and $\Psi = e^t$.

In the following Theorem we use Tensor product technique to find an exact solution of a general form of equation (1).

Theorem 3. *Let $u(x,t) \in C(I \times J)$, where $I,\ J = [0,1]$ or $[0,\infty)$. If u has continuous second partial derivatives and f any continuous function of t, then the differential equation*

$$u_t + f u_x = u_{xx} \tag{4}$$

can be solved by tensor product.

Proof. Put $u = \Phi \otimes \Psi$, with $\Phi(0) = 1$ and $\Psi(0) = 1$

$$\Phi \otimes \Psi' + \Phi' \otimes f\Psi = \Phi'' \otimes \Psi. \tag{5}$$

Since the sum of two atoms is an atom using Lemma (2), we have $\Phi' = \Phi$ or $\Psi' = f\Psi$.
Case (1) if $\Psi' = f\Psi$, then

$$\frac{\Psi'}{\Psi} = f.$$

Integrating both sides, we get

$$\begin{aligned} \int_0^t \frac{d\Psi}{\Psi} &= \int_0^t f du \\ \ln \Psi \Big|_0^t &= \int_0^t f du \\ \ln \Psi - \ln \Psi(0) &= \int_0^t f du \end{aligned}$$

$$\begin{aligned}
\ln\Psi - \ln 1 &= \int_0^t f du\\
\ln\Psi &= \int_0^t f du\\
\Psi &= e^{\int_0^t f du}.
\end{aligned}$$

Since $\Psi' = f\Psi$, then equation (5) becomes

$$\begin{aligned}
\Phi \otimes f\Psi + \Phi' \otimes f\Psi &= \Phi'' \otimes \Psi\\
[\Phi + \Phi'] \otimes f\Psi &= \Phi'' \otimes \Psi\\
[\Phi + \Phi'] \otimes [f\Psi] - \Phi'' \otimes \Psi &= 0\\
([\Phi + \Phi']f - \Phi'') \otimes \Psi &= 0.
\end{aligned}$$

Using Lemma (1), we have $[\Phi + \Phi']f - \Phi'' = 0$ or $\Psi = 0$. If $\Psi = 0$, then we have a contradiction since $\Psi \neq 0$. So

$$\begin{aligned}
[\Phi + \Phi']f - \Phi'' &= 0\\
\frac{\Phi''}{\Phi' + \Phi} &= f.
\end{aligned}$$

Contradiction, since Φ is a function depends only on x, which means that this case does not hold.
Case(2) if $\Phi' = \Phi = \Phi''$, then

$$\frac{\Phi'}{\Phi} = 1.$$

Integrating both sides, we get:

$$\begin{aligned}
\int \frac{d\Phi}{\Phi} &= \int dx\\
\ln|\Phi| &= x + w\\
\Phi &= w_1 e^x,
\end{aligned}$$

since $\Phi(0) = 1 \Longrightarrow$

$$\Phi = e^x.$$

Now, since the first and the second derivative of Φ are equal then equation (5) become

$$\begin{aligned}
\Phi \otimes \Psi^{'} + \Phi \otimes f\Psi &= \Phi \otimes \Psi\\
\Phi \otimes [\Psi^{'} + f\Psi - \Psi] &= 0\\
\Phi \otimes [\Psi^{'} + (f-1)\Psi] &= 0.
\end{aligned}$$

Thus using Lemma (1) either $\Phi = 0$ or $\Psi^{'} + (f-1)\Psi = 0$. If $\Phi = 0$, then we have a contradiction since $\Phi \neq 0$. So

$$\Psi^{'} + (f-1)\Psi = 0$$

$$\begin{aligned}
\frac{\Psi'}{\Psi} &= 1-f \\
\int \frac{d\Psi}{\Psi} &= \int (1-f)dt \\
\ln|\Psi| &= \int (1-f)dt \\
\Psi &= e^{\int (1-f)dt}.
\end{aligned}$$

To verify equation (5), set $\Phi = e^x = \Phi' = \Phi''$ and $\Psi = e^{\int(1-f)dt}$ and $\Psi' = (1-f)e^{\int(1-f)dt} = (1-f)\Psi$, then $u = \Phi \otimes \Psi$, $u_x = \Phi' \otimes \Psi = \Phi \otimes \Psi$, $u_{xx} = \Phi'' \otimes \Psi = \Phi \otimes \Psi$, and $u_t = \Phi \otimes \Psi' = \Phi \otimes (1-f)\Psi$.

Then

$$\begin{aligned}
u_t + fu_x &= \Phi \otimes (1-f)\Psi + f(\Phi \otimes \Psi) \\
&= (1-f)(\Phi \otimes \Psi) + f(\Phi \otimes \Psi) \\
&= (\Phi \otimes \Psi) - f(\Phi \otimes \Psi) + f(\Phi \otimes \Psi) \\
&= \Phi \otimes \Psi \\
&= u_{xx},
\end{aligned}$$

this implies $u = \Phi \otimes \Psi$ is a solution of equation (4), where $\Phi = e^x$ and $\Psi = e^{\int(1-f)dt}$.

Now, if $f(t) = t^2$, then equation (4) becomes

$$u_t + t^2 u_x = u_{xx},$$

which has the solution $u = \Phi \otimes \Psi$, where $\Phi = e^x$ and $\Psi = e^{\int(1-t^2)dt} = e^{t-\frac{t^3}{3}}$.

4. Atomic Solution of Burger equation

Theorem 4. *Let $u(x,t) \in C(I \times J)$, where I, $J = [0,1]$ or $[0,\infty)$. If u has continuous second partial derivatives, then the differential equation*

$$u_t + \alpha u u_x = \upsilon u_{xx} \tag{6}$$

has an atomic solution.

Proof. Put $u = \Phi \otimes \Psi$ where $\Phi(0) = 0$ and $\Psi(1) = 1$, to get

$$\Phi \otimes \Psi' + \alpha[\Phi \otimes \Psi][\Phi' \otimes \Psi] = \upsilon \Phi'' \otimes \Psi. \tag{7}$$

The product of two atoms is one atom $(\Phi \otimes \Psi)(\Phi' \otimes \Psi) = (\Phi\Phi' \otimes \Psi\Psi)$, so that equation (6) becomes

$$\Phi \otimes \Psi' + \alpha \Phi\Phi' \otimes \Psi^2 = \upsilon \Phi'' \otimes \Psi, \tag{8}$$

and the sum of two atoms is an atom using Lemma 1, we have $\alpha\Phi'\Phi = \Phi$ or $\Psi' = \Psi^2$.
Case (1) If $\Psi' = \Psi^2$, then

$$\frac{\Psi'}{\Psi^2} = 1.$$

Integrating both sides, we get

$$\begin{aligned} \int \frac{d\Psi}{\Psi^2} &= \int dt \\ \frac{1}{\Psi} &= -t + g \\ \Psi &= -\frac{1}{t} + g_1, \end{aligned}$$

since $\Psi(1) = 1 \Longrightarrow$

$$\Psi = -\frac{1}{t} + 2.$$

Now, $\Psi^2 = \frac{1}{t^2} - 4(\frac{1}{t} - 1) \neq \frac{1}{t^2} = \Psi'$, which means this case dose not hold.
But, if we take $g = 0$ to hold this case we get

$$\Psi = -\frac{1}{t},$$

so $\Psi' = \frac{1}{t^2} = \Psi^2$, then equation (8) becomes

$$\begin{aligned} \Phi \otimes \Psi^2 + \alpha\Phi\Phi' \otimes \Psi^2 &= \upsilon\Phi'' \otimes \Psi \\ [\Phi + \alpha\Phi\Phi'] \otimes \Psi^2 &= \upsilon\Phi'' \otimes \Psi. \end{aligned}$$

Thus $\Phi + \alpha\Phi\Phi' = \upsilon\Phi''$ and $\Psi^2 = \Psi$, when $\Psi^2 = \Psi$ implies $\Psi = 1$. Contradiction, since $\Psi = -\frac{1}{t}$, which means that this case does not hold.
Case (2) If $\alpha\Phi\Phi' = \Phi$, then

$$\begin{aligned} \Phi' &= \frac{1}{\alpha} \\ \int d\Phi &= \int \frac{1}{\alpha} dx \\ \Phi &= \frac{x}{\alpha} + q, \end{aligned}$$

since $\Phi(0) = 0 \Longrightarrow$

$$\Phi = \frac{x}{\alpha}.$$

Now, $\Phi' = \frac{1}{\alpha}$ and $\Phi'' = 0$, then equation (8) becomes

$$\begin{aligned} \Phi \otimes \Psi^{'} + \alpha\frac{1}{\alpha}\Phi \otimes \Psi^2 &= 0 \otimes \upsilon\Psi \\ [\Psi^{'} + \Psi^2] \otimes \Phi &= 0. \end{aligned}$$

So, $\Phi = 0$ or $\Psi^{'} + \Psi^2 = 0$. If $\Phi = 0$, it is a contradiction since $\Phi \neq 0$. Hence

$$\begin{aligned} \Psi^{'} + \Psi^2 &= 0 \\ \int -\frac{d\Psi}{\Psi^2} &= \int dt \\ \frac{1}{\Psi} &= t + c, \end{aligned}$$

since $\Psi(1) = 1 \Longrightarrow$

$$\Psi = \frac{1}{t}.$$

To verify equation (6), set $\Phi = \frac{x}{\alpha}$, $\Phi' = \frac{1}{\alpha}$, $\Phi'' = 0$, and $\Psi = \frac{1}{t}$, $\Psi' = \frac{-1}{t^2} = -\ \Psi^2$, then $u = \Phi \otimes \Psi$, $u_x = \Phi' \otimes \Psi$, $\upsilon u_{xx} = \upsilon(\Phi'' \otimes \Psi) = \upsilon(0 \otimes \Psi) = \upsilon 0 \otimes \Psi = 0 \otimes \Psi = 0$, $u_t = \Phi \otimes \Psi'$.

$$\begin{aligned} u_t + \alpha u u_x &= \Phi \otimes \Psi' + \alpha(\Phi \otimes \Psi)(\Phi' \otimes \Psi) \\ &= \Phi \otimes -\Psi^2 + \alpha(\Phi \otimes \Psi)(\Phi' \otimes \Psi) \\ &= \Phi \otimes -\Psi^2 + \alpha(\Phi\Phi' \otimes \Psi^2) \\ &= -\Phi \otimes \Psi^2 + \alpha(\Phi\Phi' \otimes \Psi^2) \\ &= (-\Phi + \alpha\Phi\Phi') \otimes \Psi^2. \end{aligned}$$

Consequently, $-\Phi + \alpha\Phi\Phi' = -\frac{x}{\alpha} + \alpha\frac{x}{\alpha}\frac{1}{\alpha} = -\frac{x}{\alpha} + \frac{x}{\alpha} = 0$. So

$$\begin{aligned} u_t + \alpha u u_x &= 0 \otimes \Psi^2 \\ &= 0 \\ &= \upsilon u_{xx}. \end{aligned}$$

This implies $u = \Phi \otimes \Psi$ is a solution of equation (6), where $\Phi = \frac{x}{\alpha}$ and $\Psi = \frac{1}{t}$.

Conclusion

In this paper we find an exact solution for secand order partial diffrential equation of linear type. Further, exact solution using tensor product technique of Burger equation is presented.

References

[1] H. Bateman. Some recent researches on the motion of fluids. *Monthly Weather Review*, 43:163–170, 1915.

[2] J. Biazar and H. Ghazvini. Exact solutions for nonlinear Burger's equation by homotopy perturbation method. *Numerical Methods for Partial Differential Equations*, 25:833 – 842, 2009.

[3] J. Biazar and A. Hossein. Exact and numerical solutions for non-linear Burger's equation by VIM. *Mathematical and Computer Modelling*, 49:1394–1400, 2009.

[4] J. M. Burgers. A mathematical model illustrating the theory of turbulence. *In Advances in applied mechanics*, 1:171–199, 1947.

[5] M. Darvishi and M. Javidi. A numerical solution of Burger's equation by pseudospectral method and Darvishi's preconditioning. *Applied Mathematics and Computation*, pages 421–429, 2006.

[6] M. Kapoor and V. Joshi. Numerical approximation of coupled 2D Burgers equation by employing non-uniform algebraic hyperbolic (NUAH) B-spline based differential quadrature method. *Mathematics in Engineering, Science and Aerospace*, 13(2):171–199, 2022.

[7] R. Khalil and L.Abdullah. Atomic solution of certain inverse problems. *European Journal of Pure and Applied Mathematics*, 3:725–729, 2012.

[8] Ch. Kuo and S. Lee. A new exact solution of Burger's equation with linearized solution. *Mathematical Problems in Engineering*, 2015:1–7, 2015.

[9] W. Light and E. Cheney. *Approximation theory in tensor product spaces.* Springer Verlag, Berlin, New York, 1985.

[10] R. Lynch, J. Rice, and D. Thomas. Tensor product analysis of partial difference equations. *Bulletin of the American Mathematical Society*, 70:378–384, 1964.

[11] R. Lynch, J. Rice, and D. Thomas. Direct solution of partial difference equations by tensor product methods. *Numerische Mathematik*, 6:185–199, 1964.

[12] K. Selçuk, E. Alaattin, and D. Idris. Numerical solution of the Burger's equation by the least-squares quadratic B-spline finite element method. *Journal of Computational and Applied Mathematics*, 167:21–33, 2004.

[13] K. Selçuk, A.R. Bahadir, and A. Özdeş. Numerical solution of one-dimensional Burger's equation: Explicit and exact-explicit finite difference methods. *Journal of Computational and Applied Mathematics*, 103:251–261, 1999.

[14] S. Xie, S. Heo, S. Kim, G. Woo, and S. Yi. Numerical solution one-dimensional Burger's equation using reproducing kernel function. *Journal of Computational and Applied Mathematics*, 214:417–434, 1999.

[15] A. Ziqan, M. Al Horani, and R. Khalil. Tensor product technique and the degenerate nonhomogeneous abstract cauchy problem. *Journal of Applied Functional Analysis*, 5:121–138, 2010.