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# Dynamics of the Nonlinear Rational Difference Equation $x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}$

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**Abstract.** In this article, we study the global stability and the asymptotic properties of the nonnegative solutions of the nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \left(px_n + x_{n-k}\right) / \left(q + x_{n-k}\right), \quad n = 0, 1, 2, .....$$

where the parameters $A, B, p, q$ and the initial conditions $x_{-k}, ..., x_{-1}, x_0$ are arbitrary nonnegative real numbers, while $k$ is a positive integer number. Some numerical examples will be given to illustrate our results.

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## 1. Introduction

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [2, 16, 19, 29]. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. This can be easily seen from the family $x_{n+1} = g_\mu(x_n)$, $\mu > 0$, $n \geq 0$. This behavior is ranging according to the value of $\mu$, from the existence of a bounded number of periodic solutions to chaos.

There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [1,7-14,21–31] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges

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to a period two solution. For other closely related results, (see [3-7,10,17,18]) and the references cited therein. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

Our goal in this article is to investigate some qualitative behavior of the solutions of the nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}, n = 0, 1, 2, \ldots. \tag{1}$$

where the parameters $A, B, p, q$ and the initial conditions $x_{-k}, \ldots x_{-1}, x_0$ are arbitrary nonnegative real numbers, while $k$ is a positive integer number. The global stability of Eq.(1) for $A = B = 0$ has been investigated in [29]. Kulenvic et al.[22] studied Eq.(1) when $A = B = 0$ and $k = 1$.

Our interest now is to study the behavior of solutions of Eq.(1) in the general case where $A \neq 0$, $B \neq 0$ and $k$ is a positive integer number. For the related work see [32-45]. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. Let us now recall some well know results [15] which will be useful in the sequel.

**Definition 1.** *A difference equation of order $(k+1)$ is of the form*

$$x_{n+1} = F(x_n, x_{n-k}), \quad n = 0, 1, 2, \ldots. \tag{2}$$

*where $F$ is a continuous function which maps some set $J^{k+1}$ into $J$ where $J$ is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ for all $n \geq -k$ is a solution of that equation.*

**Definition 2.** *Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (2). Then*

*(i) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $\left|x_{-k} - \widetilde{x}\right| + \ldots + \left|x_{-1} - \widetilde{x}\right| + \left|x_0 - \widetilde{x}\right| < \delta$, then $\left|x_n - \widetilde{x}\right| < \varepsilon$ for all $n \geq -k$.*

*(ii) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $\left|x_{-k} - \widetilde{x}\right| + \ldots + \left|x_{-1} - \widetilde{x}\right| + \left|x_0 - \widetilde{x}\right| < \gamma$, then*

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

*(iii) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called a global attractor if for every $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ we have*

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iv) *An equilibrium point $\widetilde{x}$ of the equation (2) is called globally asymptotically stable if it is locally stable and a global attractor.*

(v) *An equilibrium point $\widetilde{x}$ of the difference equation (2) is called unstable if it is not locally stable.*

**Definition 3.** *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period $p$ if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period $p$ if $p$ is the smallest positive integer having this property.*

**Definition 4.** *A positive semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \dots x_m\}$ all greater than or equal to $\widetilde{x}$ , with $l \geq -k$ and $m \leq \infty$ such that*

$$\text{either } l = -k \text{ or } +l > -k \text{ and } x_{l-1} < \widetilde{x},$$

*and*

$$\text{either } m = \infty \text{ or } +m < \infty and x_{m-1} < \widetilde{x},$$

*A negative semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \dots x_m\}$ all less than $\widetilde{x}$ , with $l \geq -k$ and $m \leq \infty$ such that*

$$\text{either } l = -k \text{ or } +l > -k \text{ and } x_{l-1} \geq \widetilde{x},$$

*and*

$$\text{either } m = \infty \text{ or } +m < \infty and x_{m-1} \geq \widetilde{x},$$

**Definition 5.** *Eq.(2) is said to be permanent if there exist positive real numbers $m$ and $M$ such that for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2) there exists a positive integer $N \geq -k$ which depends on the initial conditions, such that*

$$m \leq x_n \leq M, \text{ for all } n \geq N.$$

The linearized equation of the difference equation (2) about the equilibrium point $\widetilde{x}$ is the linear difference equation

$$z_{n+1} = \frac{\partial F(\widetilde{x}, \widetilde{x})}{\partial x_n} z_n + \frac{\partial F(\widetilde{x}, \widetilde{x})}{\partial x_{n-k}} z_{n-k}, \tag{3}$$

The characteristic equation associated with Eq.(3) is

$$p(\lambda) = \lambda^{k+1} - p_0 \lambda^k - p_1 = 0, \tag{4}$$

where

$$p_0 = \frac{\partial F(\widetilde{x}, \widetilde{x})}{\partial x_n}, \qquad p_1 = \frac{\partial F(\widetilde{x}, \widetilde{x})}{\partial x_{n-k}}.$$

**Theorem 1.** *([15]). The linearized stability theorem. Suppose $F$ is a continuously differentiable function defined on an open neighbourhood of the equilibrium $\widetilde{x}$. Then the following statements are true.*

(i) *If all the roots of the characteristic equation (4) of the linearized equation (3) have absolute value less than one, then the equilibrium point $\widetilde{x}$ of Eq.(2) is locally asymptotically stable.*

(ii) *If at least one root of Eq.(4) has absolute value greater than one, then the equilibrium point $\widetilde{x}$ of Eq.(2)*

(iii) *If all the roots of Eq.(4) have absolute value greater than one, then the equilibrium point $\widetilde{x}$ of Eq.(2) is a source.*

## 1.1. Equilibrium Points

In this section, we examine the nonnegative equilibrium points $\widetilde{x}$ of Eq.(1) and their local asymptotic behavior. The equilibrium points of Eq.(1) are the nonnegative solutions of the equation

$$\widetilde{x} = (A+B)\,\widetilde{x} + \frac{(p+1)\,\widetilde{x}}{q+\widetilde{x}}. \tag{5}$$

So, $\widetilde{x} = 0$ is always an equilibrium point of Eq.(1). If $0 < A+B < 1$, $p-q > -\left[1+q\,(A+B)\right]$ and $p > q$ then the positive equilibrium point is

$$\widetilde{x} = \frac{(p-q) + \left[1+q\,(A+B)\right]}{\left[1-(A+B)\right]}. \tag{6}$$

**Lemma 1.** *If $p > q$ and $0 < A+B < 1$, then the positive equilibrium point (6 satisfies the inequality $\widetilde{x} > \frac{q}{p}$.*

*Proof.* From (6 we deduce that

$$\begin{aligned}
\widetilde{x} &= \frac{p+1}{1-(A+B)} - q > \frac{q+1}{1-(A+B)} - q = \frac{1+q\,(A+B)}{1-(A+B)} \\
&= \left[1+q\,(A+B)\right]\left[1+(A+B)+(A+B)^2+......\right] \\
&> 1 > \frac{q}{p}.
\end{aligned}$$

The proof of Lemma 1 is now completed.

## 1.2. Linearization

In this section, we derive the linearized equation of Eq.(1). To this end, we introduce a continuous function $F : (0,\infty)^2 \to (0,\infty)$ which is defined by

$$F(u_0,u_1) = Au_0 + Bu_1 + \frac{pu_0+u_1}{q+u_1}. \tag{7}$$

Therefore,

$$\left\{\begin{aligned}
\frac{\partial F(u_0,u_1)}{\partial u_0} &= A + \frac{p}{q+u_1}, \\
\frac{\partial F(u_0,u_1)}{\partial u_1} &= B + \frac{q-pu_0}{(q+u_1)^2}.
\end{aligned}\right. \tag{8}$$

From (6 and (8) we have

$$\begin{cases} \frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_0} = A + \frac{p[1-(A+B)]}{p+1} = \rho_0, \\ \frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_1} = B - \frac{[1-(A+B)]\left[(p-q)+q(A+B)\right]}{p+1} = \rho_1. \end{cases} \tag{9}$$

The linearized equation of Eq.(1) about the zero equilibrium point $\widetilde{x} = 0$ is

$$z_{n+1} - \left(A + \frac{p}{q}\right) z_n - \left(B + \frac{1}{q}\right) z_{n-k} = 0, \tag{10}$$

and the linearized equation of Eq.(1) about the positive equilibrium point $\widetilde{x}$ is

$$z_{n+1} - \rho_0 z_n - \rho_1 z_{n-k} = 0, \tag{11}$$

where $\rho_0$ and $\rho_1$ are given by (9).

**Theorem 2.** *[20] Assume that $\rho_0, \rho_1 \in R$ and $k \in \{1, 2, ...\}$. Then*

$$\left|\rho_0\right| + \left|\rho_1\right| < 1, \tag{12}$$

*is a sufficient condition for the asymptotic stability of the difference equation (2). Suppose in addition that one of the following two cases holds:*

*(i) $k$ is an odd integer and $\rho_1 > 0$.*

*(ii) $k$ is an even integer and $\rho_0\rho_1 > 0$.*

*Then (12) is also a necessary condition for the asymptotic stability of Eq.(2).*

**Theorem 3.** *[15] Consider the difference equation (2) where the function $F \in C\left(I^{k+1}, R\right)$ and $I$ is an open interval of real numbers. Let $\widetilde{x} \in I$ be an equilibrium point of Eq.(2). Suppose also that*

*(i) $F$ is a nondecreasing function in each of its arguments.*

*(ii) The function $F$ satisfies the negative feedback property*

$$[F(x, x) - x](x - \widetilde{x}) < 0 \text{ for all } x \in I - \{\widetilde{x}\}.$$

*Then the equilibrium point $\widetilde{x}$ of Eq.(2) is a global attractor for all solutions of Eq.(2) .*

## 2. Semi-Cycle Analysis

**Theorem 4.** *Assume that $F : (0,\infty)^2 \to (0,\infty)$ is a continuous function such that $F(x,y)$ is increasing in $x$ for fixed $y$, and $F(x,y)$ is increasing in $y$ for fixed $x$. Let $\widetilde{x}$ be a positive equilibrium of Eq.(1). Then, except possibly for the first semi-cycle, every oscillatory solution of Eq.(1) has semi-cycle of length at least $k$.*

*Proof.* We just give the proof of the theorem 4 for $k = 2$. The proof of the theorem 4 for $k \geq 3$, is similar and omitted here. Let $\{x_n\}$ be a solution of Eq.(1) with at least three semi-cycles. Then, there exists $N \geq 0$ such that either

$$x_{N+1} \geq x_{N-1} \geq \widetilde{x},$$

or

$$x_{N-1} \geq x_{N+1} \geq \widetilde{x}.$$

We first assume that

$$x_{N+1} \geq x_{N-1} \geq \widetilde{x}.$$

Since the function $F(x,y)$ given by (7) is increasing in $x$ for fixed $y$ and increasing in $y$ for fixed $x$, then we get

$$\begin{aligned} x_{N+2} &= F(x_{N+1}, x_{N-1}) = Ax_{N+1} + Bx_{N-1} + \frac{px_{N+1} + x_{N-1}}{q + x_{N-1}} \\ &\geq A\widetilde{x} + Bx_{N-1} + \frac{p\widetilde{x} + x_{N-1}}{q + x_{N-1}} = F(\widetilde{x}, x_{N-1}) \geq F(\widetilde{x}, \widetilde{x}) = \widetilde{x}, \end{aligned}$$

and

$$x_{N+3} = F(x_{N+2}, x_N) > F(\widetilde{x}, x_N) > F(\widetilde{x}, \widetilde{x}) = \widetilde{x} \quad for \quad x_N > \widetilde{x}.$$

Similarly, we can prove the theorem if $x_{N-1} \geq x_{N+1} \geq \widetilde{x}$ which is omitted. Now, the proof of Theorem 4 is completed.

## 3. Local Stability

In this section, we investigate the local stability of the positive solutions of Eq.(1). By using Theorems 1 and 3, we have the following result.

**Theorem 5.** *The zero equilibrium point $\widetilde{x} = 0$ is locally asymptotically stable if $p - q < -\left[1 + q(A+B)\right]$. In particular, if $p - q \geq -\left[1 + q(A+B)\right]$, then $\widetilde{x} = 0$ is unstable.*

*Proof.* First, suppose that $p - q < -\left[1 + q(A+B)\right]$. Then, from Eq.(10) we deduce that

$$\left|A + \frac{p}{q}\right| + \left|B + \frac{1}{q}\right| = (A+B) + \frac{p+1}{q} < (A+B) + \frac{q\left[1 - (A+B)\right]}{q} = 1.$$

Thus $\widetilde{x}=0$ is locally asymptotically stable. In particular, assume $p-q \geq -\left[1+q\left(A+B\right)\right]$, then we have

$$\left|A+\frac{p}{q}\right|+\left|B+\frac{1}{q}\right|=(A+B)+\frac{p+1}{q}\geq (A+B)+\frac{q\left[1-(A+B)\right]}{q}=1.$$

Thus $\widetilde{x}=0$ is unstable. The proof of Theorem 6 is now completed.

**Theorem 6.** *If* $\left(p-q\right)>-\left[1+q\left(A+B\right)\right]$, $0<A+B<1$, $p>q$ *and*

$$B>\frac{\left[1-(A+B)\right]\left[\left(p-q\right)+q\left(A+B\right)\right]}{\left(p+1\right)\left(q+1\right)}.$$

*Then, the positive equilibrium point $\widetilde{x}$ is locally asymptotically stable. Furthermore, the condition (12) can be considered as a necessary and sufficient condition for the asymptotically stability of Eq.(1).*

*Proof.* Under these assumptions we deduce from (9) that

$$\begin{aligned}
\left|\rho_0\right|+\left|\rho_1\right| &= \left|A+\frac{p\left[1-(A+B)\right]}{p+1}\right|+\left|B-\frac{\left[1-(A+B)\right]\left[\left(p-q\right)+q\left(A+B\right)\right]}{p+1}\right| \\
&= A+\frac{p\left[1-(A+B)\right]}{p+1}+B-\frac{\left[1-(A+B)\right]\left[\left(p-q\right)+q\left(A+B\right)\right]}{p+1} \\
&< \frac{(A+B)\left(p+1\right)+\left(p+1\right)\left[1-(A+B)\right]}{p+1}=1.
\end{aligned}$$

This proves that the positive equilibrium point $\widetilde{x}$ of Eq.(1) is locally asymptotically stable. Thus, the condition (12) is sufficient for the asymptotic stability of Eq.(1). In addition to that condition, we see that if $k$ is an odd positive integer and

$$\rho_1=B-\frac{\left[1-(A+B)\right]\left[\left(p-q\right)+q\left(A+B\right)\right]}{p+1}>0,$$

or if $k$ is an even positive integer and

$$\rho_0\rho_1=\left(A+\frac{p\left[1-(A+B)\right]}{p+1}\right)\left(B-\frac{\left[1-(A+B)\right]\left[\left(p-q\right)+q\left(A+B\right)\right]}{p+1}\right)>0,$$

then the condition (12) is also necessary for the asymptotic stability of Eq.(1). According to Theorem 2, the proof of Theorem 7 is now completed.

## 4. Periodic Solutions

In this section, we investigate the periodic character of the positive solutions of Eq.(1).

**Theorem 7.** *If $k$ is an even positive integer, then Eq.(1) has no positive solutions of prime period two for all $A, B, p, q \in (0, \infty)$.*

*Proof.* Assume for the sake of contradiction that there exists distinctive positive real numbers $\Phi$ and $\Psi$, such that

$$\ldots \Phi, \Psi, \Phi, \Psi, \ldots$$

is a prime period two solution of Eq.(1). If $k$ is even, then $x_n = x_{n-k}$. It follows from the difference equation (1) that

$$\Phi = (A+B)\,\Psi + \frac{(p+1)\,\Psi}{q+\Psi} \text{ and } \Psi = (A+B)\,\Phi + \frac{(p+1)\,\Phi}{q+\Phi}.$$

Consequently, we obtain

$$q\Phi + \Phi\Psi = qA\Psi + A\Psi^2 + qB\Psi + B\Psi^2 + p\Psi + \Psi,$$

and

$$q\Psi + \Phi\Psi = qA\Phi + A\Phi^2 + qB\Phi + B\Phi^2 + p\Phi + \Phi.$$

By subtracting, we deduce that

$$(\Phi - \Psi)\left\{q\,(A+B+1) + (\Phi+\Psi)(A+B) + p + 1\right\} = 0.$$

This implies $\Phi = \Psi$. This contradicts the hypothesis $\Phi \neq \Psi$. Thus, the proof of Theorem 7 is completed.

**Theorem 8.** *If $k$ is an odd positive integer then for all $A, B, p, q \in (0, \infty)$ Eq.(1) has no prime period two solutions if $A - B + 1 > 0$.*

*Proof.* Assume for the sake of contradiction that there exists distinctive positive real numbers $\Phi$ and $\Psi$, such that

$$\ldots \Phi, \Psi, \Phi, \Psi, \ldots$$

is a prime period two solution of Eq.(1). If $k$ is odd, then $y_{n+1} = y_{n-k}$. It follows from Eq.(1) that

$$\Phi = A\Psi + B\Phi + \frac{p\Psi + \Phi}{q+\Phi} \text{ and } \Psi = A\Phi + B\Psi + \frac{p\Phi + \Psi}{q+\Psi}.$$

Consequently, we obtain

$$q\Phi + \Phi^2 = qA\Psi + A\Phi\Psi + qB\Phi + B\Phi^2 + p\Psi + \Phi, \tag{13}$$

and

$$q\Psi + \Psi^2 = qA\Phi + A\Phi\Psi + qB\Psi + B\Psi^2 + p\Phi + \Psi. \tag{14}$$

By subtracting (13) from (14), we deduce that

$$\Phi + \Psi = \frac{\left[(p+q) - \left\{q\,(B-A) + 1\right\}\right]}{B-1}, \tag{15}$$

while, by adding (13), (14) and using (15) we get

$$\Phi\Psi = \frac{[qA+p]\,[(p+q)-\{q(B-A)+1\}]}{(B-1)\,[B-(A+1)]}. \tag{16}$$

From (15) and (16) we have

$$\Phi\Psi(\Phi+\Psi) = \frac{-\,(qA+p)}{(1+A-B)}\left\{\frac{(p+q)-[1+q(B-A)]}{B-1}\right\}^2 < 0. \tag{17}$$

This contradicts the hypothesis that both $\Phi, \Psi$ are positive. Thus, the proof of Theorem 8 is now completed.

## 5. Boundedness Character

In this section, we investigate the boundedness character of the positive solutions of Eq.(1).

**Theorem 9.** *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1). Then the following statements are true:*

*(i) Suppose $p < q$ and for some $N \geq 0$, the initial conditions*

$$x_{N-k+1}, \ldots x_{N-1}, x_N \in \left[\frac{p}{q}, 1\right],$$

*then*

$$x_n \in \left[\frac{p}{q}\left(A+B+\frac{p+1}{q+1}\right), \frac{q}{p}(A+B+1)\right], \textit{ for all } n \geq N.$$

*(ii) Suppose $p > q$ and for some $N \geq 0$, the initial conditions*

$$x_{N-k+1}, \ldots x_{N-1}, x_N \in \left[1, \frac{p}{q}\right],$$

*then*

$$x_n \in \left[\frac{q}{p}(A+B+1), \frac{p}{q}\left(A+B+\frac{p+1}{q+1}\right)\right], \textit{ for all } n \geq N.$$

*Proof.* First of all, if for some $N \geq 0$ and $\frac{p}{q} \leq x_N \leq 1$ and $p < q$, then

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}} \leq Ax_n + Bx_{n-k} + \frac{qx_n + x_{n-k}}{q + x_{n-k}} \\ &\leq A+B+1 \leq \frac{q}{p}(A+B+1), \end{aligned}$$

and

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}} \geq \frac{p}{q}\left(A+B+\frac{p+1}{q+1}\right).$$

Thus, the proof of part (i) is completed. Secondly, if for some $N \geq 0$ and $1 \leq x_N \leq \frac{p}{q}$ and $p > q$, then

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}} \leq \frac{p}{q}\left(A + B + \frac{p+1}{q+1}\right),$$

and

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}} \geq Ax_n + Bx_{n-k} + \frac{qx_n + x_{n-k}}{q + x_{n-k}} \geq A + B + 1 \\ &\geq \frac{q}{p}(A + B + 1). \end{aligned}$$

Thus, the proof of part (ii) is completed. The proof of Theorem 9 is now finished.

## 6. Global Stability

In this section, we investigate the global stability of the positive solutions of Eq.(1).

**Theorem 10.** *If $p - q < -\left[1 + q(A + B)\right]$, then, the zero equilibrium point $\widetilde{x} = 0$ of Eq.(1) is globally asymptotically stable.*

*Proof.* Under this condition, we have shown in Theorem 5 that $\widetilde{x} = 0$ is locally asymptotically stable. It remains to prove that $\widetilde{x} = 0$ is a global attractor. To this end, we consider the function

$$F(x, y) = Ax + By + \frac{px + y}{q + y}. \tag{18}$$

We note that the function (18) is continuous and satisfying the following conditions:

(i) $F(x, y)$ is nondecreasing in $x \in [\frac{q}{p}, \infty)$ for fixed $y > -q$.

(ii) $F(x, y)$ satisfies the inequality

$$[F(x, x) - x][x - \widetilde{x}] < 0 \text{ for } \widetilde{x} = 0.$$

Let us now prove (ii) as follows:

$$[F(x, x) - x][x - 0] = \left[(A + B)x + \frac{(p+1)x}{q + x} - x\right]x.$$

Since $x \in [\frac{q}{p}, \infty)$, then $\frac{p+1}{q+x} < \frac{p}{q}$ and we have

$$\begin{aligned} [F(x, x) - x][x - 0] &< \left[\frac{(A + B)q + (p - q)}{q}\right]x^2 \\ &< \left[\frac{(A + B)q - \{1 + q(A + B)\}}{q}\right]x^2 = -\frac{x^2}{q} < 0. \end{aligned}$$

According to Theorem 4, the zero equilibrium point $\widetilde{x} = 0$ is a global attractor. The proof of Theorem 11 is now completed.

**Theorem 11.** *Assume that* $p - q > -\left[1 + q(A+B)\right]$, $p > q$, $0 < A + B < 1$ *and* $B > \frac{[1-(A+B)]\left[(p-q)+q(A+B)\right]}{p+1}$, *then, the positive equilibrium point* $\widetilde{x}$ *of Eq.(1) is globally asymptotically stable.*

*Proof.* Under these assumptions, we have shown in Theorem 6 that the positive equilibrium point $\widetilde{x}$ of Eq.(1) is locally asymptotically stable. It remains to prove that the positive equilibrium point $\widetilde{x}$ is a global attractor. To this end, we consider the function $F\left(x, y\right)$ given by (18) which satisfies the following conditions:

(i) $F(x, y)$ is nondecreasing in $x \in [\frac{q}{p}, \infty)$ for fixed $y > -q$.

(ii) $F(x, y)$ satisfies the inequality

$$[F(x,x) - x]\,[x - \widetilde{x}] < 0,$$

where $\widetilde{x}$ given by (6. Let us now prove the inequality (ii) using Lemma 1 as follows:

$$\begin{aligned}
[F(x,x) - x]\,[x - \widetilde{x}] &= \left[(A+B) + \frac{(p+1)}{q+x} - 1\right]\left[x^2 - x\widetilde{x}\right] \\
&< \left[(A+B) + \frac{(p-q)}{q}\right]\left[\frac{q^2}{p^2} - \frac{q}{p}\widetilde{x}\right] \\
&= \frac{-q}{p}\left[(A+B) + \frac{(p-q)}{q}\right]\left[\widetilde{x} - \frac{q}{p}\right] < 0.
\end{aligned}$$

According to Theorem 3, the positive equilibrium point $\widetilde{x}$ is a global attractor. The proof of Theorem 11 is now completed.

## 7. Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the nonlinear difference equation (1).

**Example 1.** *Figure 1 shows that the solution of Eq.(1) is global stability if* $k = 1$, $x_{-1} = 1$, $x_0 = 2$, $A = 0.25$, $B = 0.3$, $p = 2$, $q = 1$, $(p > q)$.

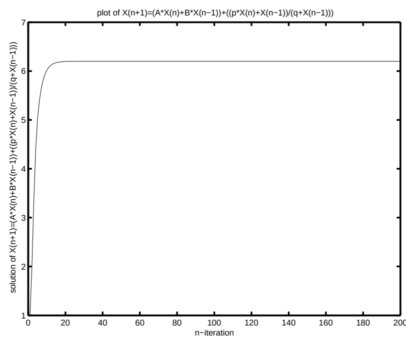


Figure 1: $x_{n+1} = 0.25x_n + 0.3x_{n-1} + \frac{2x_n + x_{n-1}}{1 + x_{n-1}}$

**Example 2.** *Figure 2 shows that the solution of Eq.(1) is global stability if* $k = 2$, $x_{-2} = 1$, $x_{-1} = 2$, $x_0 = 3$, $A = 0.25$, $B = 0.3$, $p = 20$, $q = 5$, $(p > q)$.

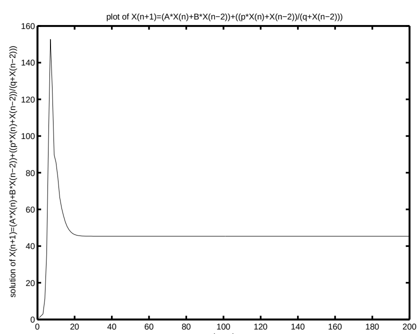


Figure 2: $x_{n+1} = 0.25x_n + 0.3x_{n-2} + \frac{20x_n + x_{n-2}}{5 + x_{n-2}}$

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