



Discrete Wolbachia Diffusion in Mosquito Populations with Allee Effects

Unal Ufuktepe

*College of Engineering and Technology, American University of the Middle East,
Egaila 54200, Kuwait*

Abstract. We study stability analysis of a discrete-time dynamical system of Wolbachia diffusion in mosquito populations with Allee effects on the wild mosquito population. We analyze the competition between released mosquitoes and wild mosquitos. We show local and global stabilities of the fixed points, and type of bifurcations concerning parameters. The results are verified by numerical simulations.

2020 Mathematics Subject Classifications: 39A28, 39A30, 39A33, 39A60

Key Words and Phrases: Competition model, discrete dynamical systems, stability, bifurcation, fixed point, manifolds

1. Introduction

Malaria, dengue fever, West Nile virus, chikungunya, and Zika virus are well-known Mosquito-borne diseases. More than a billion people are at risk of these diseases all around the world. It has been estimated that 3.9 billion people are at risk of infection [1, 2]. The human viruses including dengue, Zika, chikungunya, and yellow fever are transmitted primarily by *Aedes aegypti* mosquitoes. Due to the lack of vaccines and efficient clinical cures [3], the only effective control strategy seems to be controlling the population of mosquitoes that transmit human viruses.

Since massive spraying of insecticides and elimination of mosquito breeding sites are not sustainable 4400 to reduce mosquito density and might also lead to serious environmental problems, a promising strategy is the Wolbachia approach: releasing male and female *Aedes aegypti* mosquitoes with Wolbachia so that these mosquitoes can breed with the wild mosquito population and pass Wolbachia to the entire mosquito population. On one hand, the ability to transmit viruses to humans for mosquitoes with Wolbachia is greatly reduced [4]. On the other hand, since the Wolbachia infection often induces cytoplasmic incompatibility (CI), which leads to early embryonic death when Wolbachia-infected males mate with uninfected females [5], the Wolbachia approach would greatly reduce the density

DOI: <https://doi.org/10.29020/nybg.ejpam.v15i4.4524>

Email address: unal.ufuktepe@aum.edu.kw (U. Ufuktepe)

of the mosquito population and can thus potentially eliminate the mosquito population and thus eradicate the mosquito-borne infectious diseases.

The sterile mosquitoes technique in which sterile mosquitoes are released to reduce or eradicate the wild mosquito population has been used in preventing malaria transmission. To study the impact of releasing sterile mosquitoes on malaria transmission, In [9], they formulate a simple SEIR (susceptible-exposed-infected-recovered) malaria transmission model as our baseline model, to derive a formula for the reproductive number of infections, and determine the existence of endemic equilibria. They then include sterile mosquitoes in the baseline model and consider the case of constant releases of sterile mosquitoes. They examine how the releases affect the reproductive numbers and endemic equilibria for the model with interactive mosquitoes and investigate how releasing sterile mosquitoes affects malaria transmission.

The use of the Wolbachia strategy to suppress vector populations is a novel approach, which has the potential to reduce mosquito populations and the risk of mosquito-borne disease transmission [1, 2]. This approach, commonly known as the Incompatible Insect Technique (IIT), is a species-specific and benign approach for controlling vector populations. Eggs produced from the successful mating between released male Wolbachia-carrying *Aedes aegypti* (Wolbachia-*Aedes*) mosquitoes and urban female *Aedes aegypti* mosquitoes in the environment (without Wolbachia) are non-viable, due to Cytoplasmic Incompatibility (CI), therefore suppression of mosquito populations could be achieved with regular releases over time.

Wolbachia technology is a novel vector control approach that can reduce mosquito populations and the risk of mosquito-borne diseases, which has recently gained popularity amongst countries. In 2016, Singapore embarked on a multi-phased field study named Project Wolbachia – Singapore, to evaluate the use of Wolbachia technology as an *Aedes aegypti* mosquito population suppression tool to fight dengue. Due to the novelty of this technology in Singapore, this study aims to understand the public's acceptance and sentiments towards the use of Wolbachia technology [10].

Threshold values for the releases of sterile mosquitoes are derived for all of the models that determine whether the wild mosquitoes are wiped out or coexist with the sterile mosquitoes. Numerical examples are given to demonstrate the dynamics of the models.

2. Stability Analysis of The Discrete Model

There are many population models about mosquitoes [7–9]. Below the discrete model of competition between two species is given by [6].

$$\begin{aligned}x_{t+1} &= \frac{b_1 x_t}{1 + \alpha(x_t + y_t)} + (1 - d_1)x_t, \\y_{t+1} &= \frac{b_2 y_t}{1 + \beta(x_t + y_t)} \frac{y_t}{(x_t + y_t)} + (1 - d_2)y_t,\end{aligned}\tag{1}$$

where x_t represents the number of mosquito population infected with Wolbachia, and y_t represents the number of uninfected mosquito population at time t . The parameters $b_1, b_2 > 0$ denote the birth rate of x_t, y_t , respectively. α and β denote the competition coefficients within or between species, respectively. $0 < d_2 < d_1 < 1$ are mortality rates of y_t and x_t .

We add the Allee effect to uninfected mosquitoes in this model.

$$\begin{aligned}x_{t+1} &= \frac{b_1 x_t}{1 + \alpha(x_t + y_t)} + (1 - d_1)x_t, \\y_{t+1} &= \frac{b_2 y_t}{1 + \beta(x_t + y_t)} \frac{y_t}{(x_t + y_t)} + (1 - d_2)y_t \frac{y_t}{c + y_t},\end{aligned}\tag{2}$$

where $c > 0$ is the Allee effect constant. The study of the dynamical properties of this map allows us to have information about the future behavior of mosquitoes populations.

The positive invariant region of our model is as follows. Let $a = 1 - d_2$:

$$y_{t+1} = \frac{b_2 y_t}{1 + \beta(x_t + y_t)} \frac{y_t}{(x_t + y_t)} + (1 - d_2)y_t \frac{y_t}{(c + y_t)}\tag{3}$$

We get the following inequality from this equation

$$y_{t+1} \leq \frac{b_2}{\beta} + (1 - d_2)y_t.\tag{4}$$

By t iteration of this recurrence inequality, we get

$$y_{t+1} - a^t y_0 \leq \frac{b_2}{\beta} (1 + a + a^2 + \dots + a^{t-1})\tag{5}$$

since $0 < a < 1$, then

$$\lim_{t \rightarrow \infty} y_t \leq \frac{b_2}{d_2 \beta}.$$

Similarly,

$$\lim_{t \rightarrow \infty} x_t \leq \frac{b_1}{d_1 \alpha}.$$

Therefore, there is a positive invariant for our mapping.

3. Fixed Points and their stability

In this section, We investigate the fixed points of the map (2) and their stability conditions[11, 12]. Let

$$\begin{aligned} x_{t+1} &= f(x_t, y_t) = \frac{b_1 x_t}{1 + \alpha(x_t + y_t)} + (1 - d_1)x_t, \\ y_{t+1} &= g(x_t, y_t) = \frac{b_2 y_t}{1 + \beta(x_t + y_t)} \frac{y_t}{(x_t + y_t)} + (1 - d_2)y_t \frac{y_t}{(c + y_t)}, \end{aligned} \tag{6}$$

where we assumed $g(0, 0) = 0$. Then the solution of the following system of equations gives us the fixed points:

$$\begin{aligned} x &= \frac{b_1 x}{1 + \alpha(x + y)} + (1 - d_1)x, \\ y &= \frac{b_2 y}{1 + \beta(x + y)} \frac{y}{(x + y)} + (1 - d_2)y \frac{y}{(c + y)}. \end{aligned}$$

The fixed points are $F_0(0, 0)$, $F_1(\frac{b_1 - d_1}{\alpha d_1}, 0)$, $F_2(0, y^*)$

where $y^* = \frac{\sqrt{(\beta c + b_2 c - d_2)^2 + 4c(b_2 - \beta d_2) - \beta c + b_2 c - d_2}}{2(\beta d_2 - b_2)}$ or $y^* = \frac{\sqrt{(-\beta c + b_2 c - d_2)^2 + 4c(b_2 - \beta d_2) + \beta c - b_2 c + d_2}}{2(b_2 - \beta d_2)}$, and $F_3(A, B)$

where

$$A = \frac{1}{\alpha^2 b_2 d_1^2} (a + b)$$

($a = \alpha b_1 b_2 d_1 + \alpha b_1 d_1 - \alpha b_2 d_1^2 - \alpha b_1 d_1^2 + \beta b_1^2 - 2\beta b_1 d_1 + \beta d_1^2$, and $b = \alpha^2 b_2 c d_1^2 - \alpha b_1 d_1 d_2 + \alpha d_1^2 d_2 - \beta b_1^2 d_2 + 2\beta b_1 d_1 d_2 - \beta d_1^2 d_2$)

$$B = \frac{1}{\alpha^2 b_2 d_1^2} (m + n).$$

($m = -\alpha^2 b_2 c d_1^2 - \alpha b_1 d_1 + \alpha d_1^2 - \beta b_1^2 + 2\beta b_1 d_1 - \beta d_1^2$ and $n = \beta b_1^2 d_2 + \alpha b_1 d_1 d_2 - 2\beta b_1 d_1 d_2 - \alpha d_1^2 d_2 + \beta d_1^2 d_2$) The Jacobian matrix of the map (2) is:

$$J(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix},$$

where

$$f_x(x, y) = \frac{b_1}{(1 + \alpha(x + y))} - \frac{\alpha b_1 x}{(1 + \alpha(x + y))^2} + (1 - d_1)$$

,

$$f_y(x, y) = -\frac{ab_1x}{(a(x+y)+1)^2}$$

,

$$g_x(x, y) = -\frac{b_2y^2}{(x+y)^2(1+\beta(x+y))} - \frac{\beta b_2y^2}{(x+y)(1+\beta(x+y))^2} + \frac{2b_2y}{(x+y)(1+\beta(x+y))} - \frac{(1-d_2)y^2}{(c+y)^2} + \frac{2(1-d_2)y}{c+y},$$

and

$$g_y(x, y) = -\frac{b_2y^2}{(x+y)^2(1+\beta(x+y)+1)} - \frac{\beta b_2y^2}{(x+y)(\beta(x+y)+1)^2} + \frac{2b_2y}{(x+y)(\beta(x+y)+1)} - \frac{(1-d_2)y^2}{(c+y)^2} + \frac{2(1-d_2)y}{c+y},$$

For the fixed point $F_0(0, 0)$ the Jacobian matrix is ;

$$J(0, 0) = \begin{pmatrix} 1 - d_1 + b_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1 - d_1 + b_1$ and $\lambda_2 = 0$. Then if $|1 - d_1 + b_1| < 1$, then F_0 is asymptotically stable fixed point.

The Jacobian matrix for the fixed point $F_1(\frac{b_1-d_1}{\alpha d_1}, 0)$ is ;

$$J(F_1) = \begin{pmatrix} 1 - d_1 + \frac{d_1^2}{b_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1 - d_1 + \frac{d_1^2}{b_1}$ and $\lambda_2 = 0$. Then if $|1 - d_1 + \frac{d_1^2}{b_1}| < 1$ the F_1 is asymptotically stable fixed point otherwise unstable.

The Jacobian Matrix for the fixed point $F_2(0, y^*)$ is;

$$J(F_2) = \begin{pmatrix} 1 & 0 \\ g_x(F_2) & g_y(F_2) \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = g_y(F_2)$. We have non-hyperbolic fixed point. The Center Manifold Theorem must be applied. We explain stability of F_2 in the Numerical result section and gave the center manifold curves.

For the fixed point $F_3(A, B)$ = the Jacobian Matrix is;

$$J(F_2) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

where

$$B_{12} = \frac{b(b_1 - d_1)^2 - \alpha d_1 ((b_1(\alpha c d_1 - d_1 - 1) + b_1^2 + d_1))}{\alpha(b_1^2)},$$

$$B_{11} = \frac{b_1^2(\alpha + \beta) + d_1^2(\alpha(-\alpha b_1 c + b_1 - 1) + \beta) + b_1 d_1(\alpha - b_1) + \alpha - 2\beta}{\alpha b_1^2},$$

$$B_{21} = -\frac{b_2(\alpha d_1 + 2\beta(b_1 - d_1)) (\beta(b_1 - d_1)^2 - \alpha d_1(b_1(\alpha c d_1 - 1) + d_1))^2}{\alpha b_1^2 d_1 (b_1 - d_1)^2 (\alpha d_1 + \beta(b_1 - d_1))^2},$$

and

$$B_{22} = \frac{(\alpha d_1(b_1(\alpha c d_1 - b_2\beta(b_1 - d_1)^2) G}{\alpha b_1^2 d_1 (b_1 - d_1)^2 (\alpha d_1 + \beta(b_1 - d_1))^2}$$

where

$$G = (\alpha^2 d_1^2 (b_1^3(d_2 - 1)(\alpha c d_1 + 1) + b_1 b_2(d_1(2 - \alpha c) + 1) + b_1^2(-2b_2 - d_1 d_2 + d_1) - b_2 d_1) +$$

$$\alpha \beta d_1 (b_1 - d_1) (b_1 b_2(d_1(2 - 2\alpha c) + 3) + b_1^3(d_2 - 1) + b_1^2(-2b_2 - d_1 d_2 + d_1) - 3b_2 d_1) + 2\beta^2 b_2 (b_1 - d_1)^3)$$

By the trace-determinant (Jury Condition), if

$$|trJ| - 1 < detJ < 1$$

The positive fixed point is stable.

4. Numerical Results

In this section, we verify the theoretical results of our model by numerical simulations. We use Mathematica and Sage software for these simulations. In order to investigate the impact their interaction and the Allee effect on we changed the values of $b_1, b_2, c, \alpha, \beta$, and the mortality rates (d_1, d_2) . We fix the intraspecific competition coefficients $(c_{ij} = 1)$. The most important result we get, F_1 is globally asymptotically stable, this means that whatever the initial value of Wolbachia-infected mosquito population x_t , the population of uninfected mosquito population y_t will extinct but x_t will persist with respect to Allee effect.

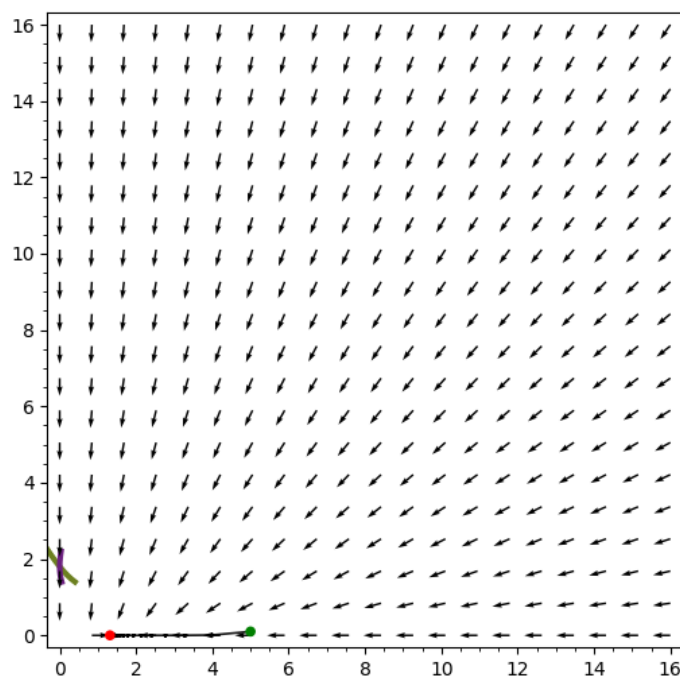


Figure 1: Phase portrait for the model (2) for F_1 fixed point, $\alpha = \beta = 1.8$, $c = 8$, $d_1 = 0.3$, $d_2 = 0.3$ and $b_1 = b_2 = 1$, $(x_0, y_0) = (5, 0.1)$.

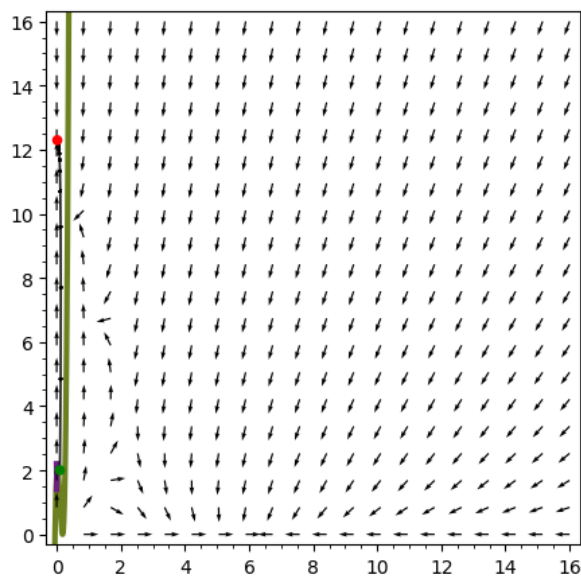


Figure 2: Phase portrait for the model (2) for F_2 fixed point, $\alpha = \beta = 0.5$, $c = 8$, $d_1 = 0.25$, $d_2 = 0.5$ and $b_1 = 1$, $b_2 = 5$, $(x_0, y_0) = (0.1, 2)$.

We take $\alpha = \beta = .5$, $b_1 = 1$, $b_2 = 5$, $c = 8$, $d_1 = .25$, $d_2 = .5$, we get the fixed point $F_2(0, 12.4)$, and $\lambda_1 = 0.89$, $\lambda_2 = 0.51$ eigenvalues. The center manifold for these parameters

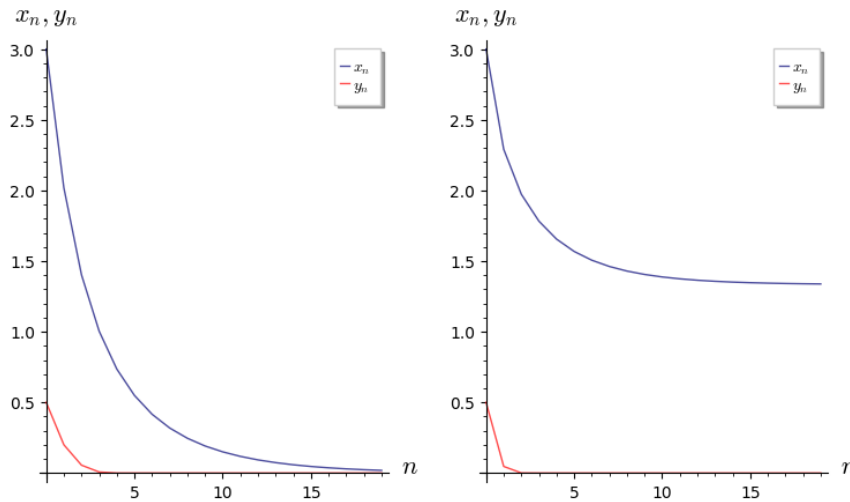


Figure 3: a) for F_1 fixed point, $\alpha = \beta = 1.8$, $c = 8$, $d_1 = 0.3$, $d_2 = 0.3$ and $b_1 = b_2 = 1$, $(x_0, y_0) = (5, 0.1)$.
 b) Time series of (x, y) for the model for F_1 fixed point, $\alpha = \beta = 1.8$, $c = 8$, $d_1 = 0.5$, $d_2 = 0.2$ and $b_1 = b_2 = 1$, $(x_0, y_0) = (5, 0.1)$.

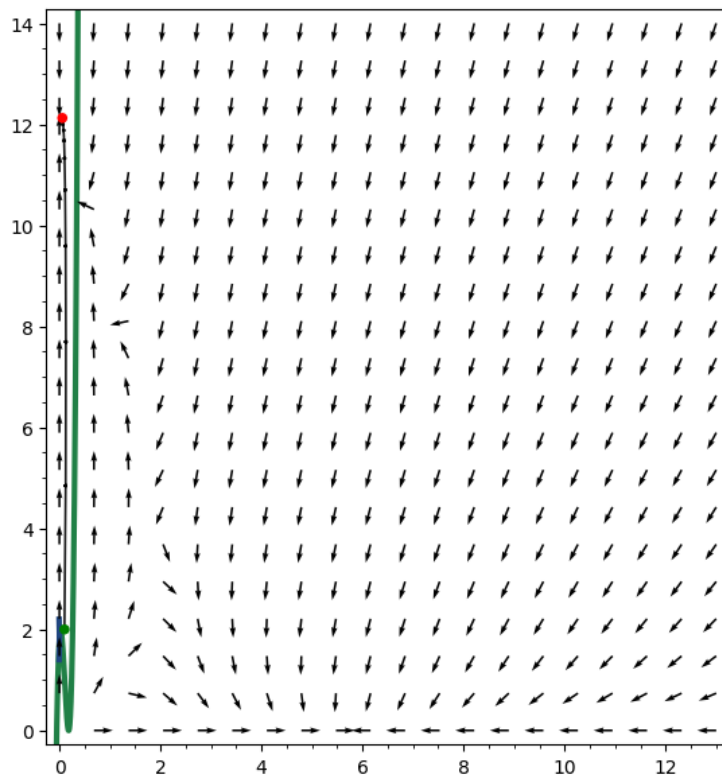


Figure 4: Phase diagram with isoclines for the model for F_1 fixed point, $\alpha = \beta = 0.8$, $c = 1.8$, $d_1 = d_2 = 0.5$, and $b_1 = b_2 = 1$, $(x_0, y_0) = (2, 3)$.

are $h(x) = 20x^3 - 316x^2 + 17x$, and $(y) = 0.0001y^3 - 0.04y^2$ (with green and blue colors in Figure 4). In Figure 1. and Figure 2. , we give the phase diagrams for the fixed points and In Figure 3. we give the time series of the model.

if we let $c = 0$, that means there is no Allee Effect, and keeping the other parameters the same, we see that only the positive fixed point is stable.

5. Conclusion

An innovative and effective method to control mosquitoes is to employ Wolbachia, which has led to a growing number of researchers building models to study the dynamics of Wolbachia transmission. Considering that the collection data of mosquitoes in the wild are discrete, we established a discrete competition model to study the conditions for Wolbachia to successful spread in mosquitoes.

Numerical simulations are also provided to demonstrate these theoretical results. We mainly showed that the simulation results are consistent with the theoretical results. In particular, since we cannot determine the exact position of the stable manifold, we show its approximate position through simulations by Sage.

References

- [1] O. J. Brady, P. W. Gething, S. Bhatt, et al., Refining the global spatial limits of dengue virus transmission by evidence-based consensus, *PLoS Negl. Trop. Dis.*, 6 (2012), e1760.
- [2] Dengue Situation Update 453, World Health Organization, (2014), Available from: http://www.wpro.who.int/emerging_diseases/denguebiweekly_02dec2014.pdf.
- [3] L. M. Schwartz, M. E. Halloran, A. P. Durbin, et al., The dengue vaccine pipeline: Implications for the future of dengue control, *Vaccine*, 33 (2015), 3293–3298.
- [4] H. Dutra, M. Rocha, F. Dias, et al., Wolbachia blocks currently circulating Zika virus isolates *Aedes aegypti* mosquitoes, *Cell Host and Microbe*, 19 (2016), 771–774.
- [5] Z. Xi, C. C. Khoo and S. I. Dobson, Wolbachia establishment and invasion in an *Aedes aegypti* laboratory population, *Science*, 310 (2005), 326–328.
- [6] Y. Li., Z. Guo, and Y.Xing, Modeling Wolbachia Diffusion in Mosquito Populations by Discrete Competition Model, *Hindawi Discrete Dynamics in Nature and Society*, V 2020, 11p, doi.org/10.1155/2020/8987490, 2020.
- [7] U.A.Rozikov, Z.S. Boxonov, A Discrete-Time Dynamical System of Wild Mosquito Population with Allee Effects, *arXiv:2102.0848v1 [math.DS]*, 2021.
- [8] Brett, A., Kulenović, M.R. Two species competitive model with the Allee effect. *Adv Differ Equ* 2014, 307 (2014). <https://doi.org/10.1186/1687-1847-2014-307>.

- [9] Hongyan Yin, Cuihong Yang, Xin'an Zhang, Jia Li. The impact of releasing sterile mosquitoes on malaria transmission. *Discrete and Continuous Dynamical Systems - B*, 2018, 23 (9) : 3837-3853. doi: 10.3934/dcdsb.2018113
- [10] Liew, C., Soh, L.T., Chen, I. et al. Public sentiments towards the use of Wolbachia-Aedes technology in Singapore. *BMC Public Health* 21, 1417 (2021). <https://doi.org/10.1186/s12889-021-11380-w>.
- [11] P. Debnath et al., *Metric Fixed Point Theory, Applications in Science, Engineering and Behavioural Sciences*, Forum for Interdisciplinary Mathematics, Springer 2021.
- [12] V.Todorcevic, *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*, Springer Nature Switzerland AG 2019.