EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 4, 2022, 1512-1520
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Pettis integrability in $L^1_{E'}[E]$ related to the truncation

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Abstract. We study the Pettis integrability in terms of truncation. We focus our study particularly on space $L^1_{E'}[E]$.

2020 Mathematics Subject Classifications: 28A20, 28A25, 40A05, 46G10
Key Words and Phrases: Convergence, Dual space, Gelfand integral, Pettis integral, Truncation

1. Introduction

Several authors studied the Pettis integrability of Banach space valued functions (see for example [1],[10],[11],[13],[14],[12],[18] and references therein) and especially of dual Banach space valued functions ([2],[17],[19]). Similarly, the study of Pettis integrability for multifunctions has been the focus of various papers (for example [9],[15] and [22]). In this note, we are interested in Pettis integrability for scalarly integrable functions of $L^1_{E'}[E]$. Our study is based on the truncation technique that has been adopted in ([5],[6]) to state some Komlós type theorems for Bochner integrable functions and in [16] to provide a Komlós type theorem in $L^1_{E'}[E]$. It is well known that a strongly measurable and scalarly integrable function $f : \Omega \to E$ is Pettis integrable if and only if the set $\{\langle x', f\rangle : \|x'\| \leq 1\}$ is uniformly integrable in $L^1_{\mathbb{R}}(\mu)$ ([14] Theorem 5.2). We give a characterization of Pettis integrability for scalarly integrable function (non-necessary strongly measurable) with norm measurable function (Proposition 1) and, when E is a separable Banach space, we establish that a function $f \in L^1_{E'}[E]$ is Pettis integrable if and only if its truncated function $1_{\{\|f\|\leq n\}}f$ is Pettis integrable for all $n \geq 1$ (Corollary 1). We also give some criteria that guarantee the Pettis integrability of the limit of a Pettis integrable $L^1_{E'}[E]$-convergent sequence. More precisely, we show that if a sequence of Pettis integrable functions bounded in $L^1_{E'}[E]$ converges weakly *a.e.* in E' (resp. converges pointwise in $L^\infty_{\mathbb{R}}(\mu) \bigotimes E''$) to a scalarly integrable function f, then f is Pettis integrable Theorem 2 (resp. Theorem 4). It is important to note that a bounded scalarly integrable function is not in general Pettis integrable, one can find some examples in [2],[19]. We note that the results in [16] will play an important role for the development of this work and a version of Theorem 4 in [16] with Pettis integrable functions is given (Theorem 6).

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DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4528

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2. Notations and Preliminaries

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, E a Banach space and E' its topological dual. The weak topology $\sigma(E, E')$ on E (resp. the weak* topology $\sigma(E', E)$ on E') will be referred to by the symbol "w" (resp. "w*"). A function $f : \Omega \to E$ (resp $f : \Omega \to E'$) is w-measurable (resp w*-measurable), if for any $x' \in E'$, (resp $x \in E$) the function $\langle f, x' \rangle : \omega \mapsto \langle f(\omega), x' \rangle$ (resp $\langle f, x \rangle : \omega \mapsto \langle f(\omega), x \rangle$) is measurable. Two functions $f, g : \Omega \to E$ (resp $f, g : \Omega \to E'$) are w-equivalent (resp w*-equivalent), if $\langle f, x' \rangle = \langle g, x' \rangle\ \mu - a.e.$ for every $x' \in E'$, (resp $\langle f, x \rangle = \langle g, x \rangle\ \mu - a.e.$ for every $x \in E$). A function $f : \Omega \to E$ (resp $f : \Omega \to E'$) is scalarly integrable (resp w*-scalarly integrable) if for every $x' \in E'$ the function $\langle f, x' \rangle$ (resp for every $x \in E$ the function $\langle f, x \rangle$) is μ-integrable. If $f : \Omega \to E$ is scalarly integrable, then ([7] Lemma 1. p. 52) for every $A \in \mathcal{F}$ there exists $x''_f(A)$ in E'' such that, for every $x' \in E'$

$$\langle x''_f(A), x' \rangle = \int_A \langle f, x' \rangle \, d\mu,$$

the element $x''_f(A)$ is called the Dunford integral of f over A and denoted by $(D) - \int_A f d\mu$. By definition, f is Pettis integrable if $(D) - \int_A f d\mu \in E$ for all $A \in \mathcal{F}$ and we write $(P) - \int_A f d\mu$ instead of $(D) - \int_A f d\mu$. Also, ([7] p. 53) if $f : \Omega \to E'$ is w*-scalarly integrable then for every $A \in \mathcal{F}$ there exists $x'_f(A)$ in E' such that, for every $x \in E$

$$\langle x'_f(A), x \rangle = \int_A \langle f, x \rangle \, d\mu,$$

the element $x'_f(A)$ is called the weak* integral (or Gelfand integral) of f over A and denoted by $(w^*) - \int_A f d\mu$. A sequence (f_n) of E-valued scalarly integrable functions converges pointwise on $L^\infty_{\mathbb{R}}(\mu) \bigotimes E'$ to an E-valued scalarly integrable function f if

$$\forall h \in L^\infty_{\mathbb{R}}(\mu), \forall x' \in E', \quad \int_\Omega h \langle f_n, x' \rangle \, d\mu \to \int_\Omega h \langle f, x' \rangle \, d\mu,$$

or equivalently ([8] Theorem 7. p. 291) for every $x' \in E'$, the sequence $(\langle f_n, x' \rangle)_n$ is bounded in $L^1_{\mathbb{R}}(\mu)$ and

$$\forall A \in \mathcal{F}, \quad \int_A \langle f_n, x' \rangle \, d\mu \to \int_A \langle f, x' \rangle \, d\mu.$$

Let $P^1_E(\mu)$ denote the (quotient) space of Pettis integrable E-valued functions. The weak topology on $P^1_E(\mu)$ is the weak topology induced by the duality $(P^1_E(\mu), L^\infty_{\mathbb{R}}(\mu) \bigotimes E')$. If E is separable and $f : \Omega \to E'$ is w*-measurable, the function $\|f(.)\|$ is measurable [20] however, this is not always the case if E is a general Banach space ([14] Example 3.3). With E being separable, the Banach space $(L^1_{E'}[E], \overline{N}_1)$ ([3],[21],[16]) is simply the (quotient) space of w*-scalarly integrable functions $f : \Omega \to E'$ such that $\|f(.)\|$ is μ-integrable, and

$$\overline{N}_1(f) = \int_\Omega \|f(\omega)\| \, d\mu(\omega), \qquad f \in L^1_{E'}[E].$$

Finally, we recall that a set H of $L^1_{\mathbb{R}}(\mu)$ is uniformly integrable (briefly UI) if it is bounded and

$$\lim_{\mu(A)\to 0} \sup_{f\in H} \int_A |f|\, d\mu = 0.$$

A set K of $L^1_{E'}[E]$ is UI [16] if the set $\{\|f(.)\| : f \in K\}$ is UI in $L^1_{\mathbb{R}}(\mu)$, and we say that a set H of E-valued scalarly integrable functions is scalarly uniformly integrable briefly SUI (resp w-scalarly uniformly integrable briefly WSUI), if the set $\{\langle x', f\rangle : \|x'\| \leq 1, f \in H\}$ (resp for each $x' \in E'$, the set $\{\langle x', f\rangle : f \in H\}$) is UI in $L^1_{\mathbb{R}}(\mu)$.

3. Pettis integrability and truncation

By ([10] p.82), if $f : \Omega \to E$ is Pettis integrable then $\{f\}$ is SUI and the converse remains true if f is strongly measurable ([14] Theorem 5.2). For the instance of $L^1_{E'}[E]$, we give some characterizations of the Pettis integrability by the mean of the associated truncated functions. Our work build on the following ([4], Theorem 3.1):

Theorem 1. *Let E be a Banach space, (f_n) a sequence of E-valued Pettis integrable functions and $f : \Omega \to E$ a scalarly integrable function satisfying:*

(i) *$\{f\}$ is SUI,*

(ii) *(f_n) converges pointwise on $L^\infty_{\mathbb{R}}(\mu) \bigotimes E'$ to f.*

Then f is Pettis integrable.

The next lemma is useful.

Lemma 1. *If $f : \Omega \to E$ is scalarly integrable and $\|f(.)\|$ is measurable, then the sequence $(1_{\{\|f\|\leq n\}} f)_n$ converges pointwise on $L^\infty_{\mathbb{R}}(\mu) \bigotimes E'$ to f.*

Proof. Let $h \in L^\infty_{\mathbb{R}}(\mu)$ and $x' \in E'$. We have

$$h(\omega)\langle 1_{\{\|f\|\leq n\}} f(\omega), x'\rangle \to h(\omega)\langle f(\omega), x'\rangle \qquad \forall \omega \in \Omega,$$

and

$$|h(\omega)\langle 1_{\{\|f\|\leq n\}} f(\omega), x'\rangle| \leq \|h\|_\infty |\langle f(\omega), x'\rangle| \qquad a.e.,$$

then by the Lebesgue dominated convergence theorem

$$\int_\Omega |\langle h(\omega) 1_{\{\|f\|\leq n\}} f(\omega) - f(\omega), x'\rangle| d\mu(\omega) \to 0,$$

and therefore

$$\int_\Omega h(\omega)\langle 1_{\{\|f\|\leq n\}} f(\omega), x'\rangle d\mu(\omega) \to \int_\Omega h(\omega)\langle f(\omega), x'\rangle d\mu(\omega).$$

Proposition 1. *If $f : \Omega \to E$ is scalarly integrable and $\|f(.)\|$ is measurable, then f is Pettis integrable if and only if*

(i) *$\{f\}$ is SUI, and*

(ii) *$1_{\{\|f\| \leq n\}} f$ is Pettis integrable for all $n \geq 1$.*

Proof. If f is Pettis integrable then $\{f\}$ is SUI and $1_{\{\|f\| \leq n\}} f$ is Pettis integrable $\forall n \geq 1$. The converse follows from Theorem 1 and Lemma 1.

The above result gives a characterization of Pettis integrability for scalarly integrable function with measurable norm function (compare with Theorem 5.2 in [14]) and it can be seen as a generalization for the case of strongly measurable functions since, if $f : \Omega \to E$ is strongly measurable then $\|f(.)\|$ is measurable and hence $1_{\{\|f\| \leq n\}} f$ is Bochner then Pettis integrable. We obtain the following characterization of Pettis integrability in $L^1_{E'}[E]$.

Corollary 1. *Let E be a separable Banach space and $f \in L^1_{E'}[E]$. Then f is Pettis integrable iff $1_{\{\|f\| \leq n\}} f$ is Pettis integrable for all $n \geq 1$.*

Proof. As E is separable then $\|f(.)\|$ is measurable. The direct implication is immediate we show the converse. For every $x'' \in E''$ the function $\langle f(.), x'' \rangle$ is measurable a simple limit of $(\langle 1_{\{\|f\| \leq n\}} f(.), x' \rangle)_n$. For all $\omega \in \Omega$ and $x'' \in B_{E''}$, we have $|\langle f(\omega), x'' \rangle| \leq \|f(\omega)\|$. As $\|f(.)\| \in L^1_{\mathbb{R}}(\mu)$ then $\{f\}$ is SUI. Therefore we apply Proposition 1.

From now, we suppose that E is separable. If (f_n) is a convergent sequence of Pettis integrable functions of $L^1_{E'}[E]$, when does (f_n) have a Pettis integrable limit? Here the convergence is taken in the sense of weak convergence *a.e.* or the pointwise convergence on $L^\infty_{\mathbb{R}}(\mu) \bigotimes E''$. The following result is an analogue of Vitali's convergence theorem for Pettis integrable functions.

Lemma 2. *Let $f \in L^1_{E'}[E]$ be a scalarly integrable function. Suppose that there exists a sequence of Pettis integrable functions (f_n) such that*

(i) *(f_n) is WSUI, and*

(ii) *for each $x'' \in E''$, $\lim_{n \to \infty} \langle f_n, x'' \rangle = \langle f, x'' \rangle$ a.e.*

Then f is Pettis integrable and (f_n) converges weakly to f in $P^1_{E'}(\mu)$.

Proof. As $\|f(.)\|$ is integrable then $\{f\}$ is SUI. We apply Theorem 1 and Vitali's theorem in $L^1_{\mathbb{R}}(\mu)$.

Theorem 2. *Let $(f_n)_{n \in \mathbb{N}}$ a bounded sequence in $L^1_{E'}[E]$.*
If (f_n) w^*-converges a.e. to a function $f : \Omega \to E'$ then $f \in L^1_{E'}[E]$.
If f_n is Pettis integrable for all n and (f_n) w-converges a.e. to f, then f is Pettis integrable.

Proof. As $(f_n(\omega))_n$ w*-converges *a.e.* to $f(\omega)$ we have

$$\|f(\omega)\| \leq \liminf_n \|f_n(\omega)\| \quad a.e.$$

By Fatou's lemma and the boundedness of (f_n) in $L^1_{E'}[E]$ we get

$$\int_\Omega \|f\| d\mu \leq \liminf_n \int_\Omega \|f_n\| d\mu < \infty,$$

thus $f \in L^1_{E'}[E]$.
Now suppose that f_n is Pettis integrable for all n and (f_n) w-converges *a.e.* to f. Then f is w-measurable with $\|f(.)\|$ is integrable, so that f is scalarly integrable. By Lemma 2 in [16] there exists a subsequence (g_n) of (f_n) such that $(1_{\{\|g_n\|<n\}} g_n)$ is UI and $(g_n - 1_{\{\|g_n\|<n\}} g_n)$ converges *a.e.* to 0 in E', hence $(1_{\{\|g_n\|<n\}} g_n)$ is WSUI and weakly converges *a.e.* to f. It remains to use Lemma 2 to conclude that f is Pettis integrable.

Now we give a criterion of the $\sigma(L^1_{E'}[E], L^\infty_{\mathbb{R}}(\mu) \bigotimes E)$-compactness for $L^1_{E'}[E]$-bounded subsets.

Theorem 3. *Let H a bounded subset of $L^1_{E'}[E]$. Then H is $\sigma(L^1_{E'}[E], L^\infty_{\mathbb{R}}(\mu) \bigotimes E)$-sequentially relatively compact if and only if for each $x \in E$ the set $H_x = \{\langle f, x\rangle : f \in H\}$ is UI in $L^1_{\mathbb{R}}(\mu)$.*

Proof. If H is $\sigma(L^1_{E'}[E], L^\infty_{\mathbb{R}}(\mu) \bigotimes E)$-sequentially relatively compact then for each $x \in E$, H_x is $\sigma(L^1_{E'}[E], L^\infty_{\mathbb{R}}(\mu))$-sequentially relatively compact and equivalently is UI in $L^1_{\mathbb{R}}(\mu)$. Conversely, let (f_n) a sequence of H. By Theorem 2 in [16] there exists a subsequence (f'_n) of (f_n) and a function $f \in L^1_{E'}[E]$ such that, for every subsequence (h_n) of (f'_n)

$$(\langle \frac{1}{n}\sum_{i=1}^n h_i(\omega)) \text{ w*-converges } a.e. \text{ to } f(\omega).$$

For every $x \in E$, the sequence $(\langle \frac{1}{n}\sum_{i=1}^n h_i, x\rangle)_n$ is UI in $L^1_{\mathbb{R}}(\mu)$ since H_x it is, then by the Vitali's theorem in $L^1_{\mathbb{R}}(\mu)$

$$\forall A \in \mathcal{F}, \quad \int_A \langle \frac{1}{n}\sum_{i=1}^n h_i, x\rangle d\mu \to \int_A \langle f, x\rangle \, d\mu. \tag{1}$$

As (1) is valid for every subsequence (h_n) of (f'_n), by an elementary property of Cesàro convergence in $\mathbb{R}$ we get

$$\forall A \in \mathcal{F}, \forall x \in E \quad \int_A \langle f'_n, x\rangle d\mu \to \int_A \langle f, x\rangle \, d\mu.$$

It follows by the boundedness of (f_n) in $L^1_{E'}[E]$ that

$$\forall h \in L^\infty_{\mathbb{R}}(\mu), \forall x \in E, \quad \int_\Omega h\langle f'_n, x\rangle d\mu \to \int_\Omega h\langle f, x\rangle \, d\mu.$$

Thus (f_n) is $\sigma(L^1_{E'}[E], L^\infty_{\mathbb{R}}(\mu)\bigotimes E)$-sequentially relatively compact.

The next result show that if a sequence of Pettis integrable functions bounded in $L^1_{E'}[E]$ converges pointwise in $L^\infty_{\mathbb{R}}(\mu)\bigotimes E''$ to a scalarly integrable function f, then f is Pettis integrable.

Theorem 4. *Let $f : \Omega \to E'$ and (f_n) a bounded sequence of $L^1_{E'}[E]$.*

(1) *If f is w^*-scalarly integrable and*

$$\forall A \in \mathcal{F}, \forall x \in E, \quad \int_A \langle f_n, x\rangle \, d\mu \to \int_A \langle f, x\rangle \, d\mu, \tag{2}$$

then $f \in L^1_{E'}[E]$.

(2) *If f is scalarly integrable, f_n is Pettis integrable for all n and*

$$\forall A \in \mathcal{F}, \forall x'' \in E'', \quad \int_A \langle f_n, x''\rangle \, d\mu \to \int_A \langle f, x''\rangle \, d\mu, \tag{3}$$

then f is Pettis integrable.

Proof. **(1)** By the Vitali-Hahn-Saks theorem ([7] Corollary I.4.10) for each $x \in E$ the sequence $(\langle f_n, x\rangle)_n$ is UI in $L^1_{\mathbb{R}}(\mu)$. Applying Theorem 3 to (f_n) we have that (f_n) is $\sigma(L^1_{E'}[E], L^\infty_{\mathbb{R}}(\mu)\bigotimes E)$-sequentially relatively compact, so there exists a subsequence (f'_n) converging $\sigma(L^1_{E'}[E], L^\infty_{\mathbb{R}}(\mu)\bigotimes E)$ to a $g \in L^1_{E'}[E]$. Then we have

$$\forall A \in \mathcal{F}, \forall x \in E, \quad \int_A \langle f'_n, x\rangle \, d\mu \to \int_A \langle g, x\rangle \, d\mu. \tag{4}$$

By (2) and (4) we get

$$\forall A \in \mathcal{F}, \forall x \in E, \quad \int_A \langle g, x\rangle \, d\mu = \int_A \langle f, x\rangle \, d\mu,$$

hence

$$\forall x \in E, \quad \langle g, x\rangle = \langle f, x\rangle. \quad a.e.$$

It follows by the separability of E that $\|g\| = \|f\|$ *a.e.* and therefore $f \in L^1_{E'}[E]$.
(2) Now suppose that f is scalarly integrable, f_n is Pettis integrable for each n and (3) is satisfied and let us prove that f is Pettis integrable. By Theorem 1 it is enough to check that $\{f\}$ is SUI, which is the case since $\|f(.)\|$ is integrable.

The next result is an immediate application of the above theorem.

Corollary 2. *The subset of $L^1_{E'}[E]$ of Pettis integrable functions is norm closed.*

Proof. Let (f_n) a norm convergent sequence of Pettis integrable functions of $L^1_{E'}[E]$ and f its limit in $L^1_{E'}[E]$, there exists a subsequence (f'_n) of (f_n) such that

$$\lim_n \|f'_n(\omega) - f(\omega)\| = 0 \qquad a.e.$$

So f is w-measurable and

$$\forall A \in \mathcal{F}, \forall x'' \in E'', \quad \int_A \langle f_n, x'' \rangle \, d\mu \to \int_A \langle f, x'' \rangle \, d\mu.$$

By Theorem 4 (2) f is Pettis integrable.

By combining Theorem 2 and Theorem 4 we have the following.

Theorem 5. *Let (f_n) a bounded sequence of $L^1_{E'}[E]$. Suppose that the following hold:*

(1) *f_n w*-converges a.e. to a function f,*

(2) *f_n is Pettis integrable for each n,*

(3) *for each $k \in \mathbb{N}^*$ there is a scalarly integrable function v_k such that*

$$\forall A \in \mathcal{F}, \forall x'' \in E'', \quad \int_A \langle 1_{\{\|f_n\| \le k\}} f_n, x'' \rangle \, d\mu \to \int_A \langle v_k, x'' \rangle \, d\mu. \tag{5}$$

Then f is Pettis integrable.

Proof. By (1) and Theorem 2 we get $f \in L^1_{E'}[E]$, so by Corollary 1 we have to prove that $1_{\{\|f\| \le k\}} f$ is Pettis integrable for every $k \in \mathbb{N}^*$. Fix $k \in \mathbb{N}^*$ and applying Theorem 4 (2) to v_k and $(1_{\{\|f_n\| \le k\}} f_n)_n$ we get that v_k is Pettis integrable. As $(1_{\{\|f_n\| \le k\}} f_n)_n$ is WSUI and by (1) is w*-converges *a.e.* to $1_{\{\|f\| \le k\}} f$, it follows by the Vitali's theorem in $L^1_{\mathbb{R}}(\mu)$ that

$$\forall A \in \mathcal{F}, \forall x \in E, \quad \int_A \langle 1_{\{\|f_n\| \le k\}} f_n, x \rangle \, d\mu \to \int_A \langle 1_{\{\|f\| \le k\}} f, x \rangle \, d\mu. \tag{6}$$

By (5) and (6) we get

$$\forall x \in E, \quad \langle 1_{\{\|f\| \le k\}}, x \rangle = \langle v_k, x \rangle \quad a.e.$$

Being E separable, it follows that $v_k = 1_{\{\|f\| \le k\}} f$ *a.e.* and therefore $1_{\{\|f\| \le k\}} f$ is Pettis integrable.

We finish this work by the following version of Theorem 4 in [16] with Pettis integrable functions. Recall that $\mathcal{R}wc(E')$ denoted the set of nonempty convex ball weakly compact subsets of E'.

Theorem 6. *Let (f_n) be a bounded sequence in $L^1_{E'}[E]$. Suppose that f_n is Pettis integrable for all $n \in \mathbb{N}$ and there exist a $\mathcal{R}wc(E')$-valued multifunction Γ such that $f_n(\omega) \in \Gamma(\omega)$ for a.e. $\omega \in \Omega$ and for all $n \in \mathbb{N}$. Then there exists a Pettis integrable function $f \in L^1_{E'}[E]$ and a subsequence (g_n) of (f_n) such for every subsequence (h_n) of (g_n) the following holds*

(j) $(\frac{1}{n}\sum_{i=1}^{n} h_i)$ *w-converges a.e. to* f.

(jj) $(1_{\{\|h_n\|<n\}}h_n)$ *converges* $\sigma(L^1_{E'}[E], (L^1_{E'}[E])')$ *(weakly) to* f *in* $L^1_{E'}[E]$ *and* $(h_n - 1_{\{\|h_n\|<n\}}h_n)$ *converges a.e. to* 0 *in* E'.

Proof. By Theorem 4 in [16] there exists a function $f \in L^1_{E'}[E]$ and a subsequence (g_n) of (f_n) such that (j) and (jj) hold. Now since $(\frac{1}{n}\sum_{i=1}^{n} h_i)$ is bounded in $L^1_{E'}[E]$ and weak converges *a.e.* to f, it follows by Theorem 2 that f is Pettis integrable.

Acknowledgements

The authors wish to thank the referees for their constructive critique of the first draft.

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