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Quotient Pseudo Hyper GR-algebras

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Abstract. A pseudo hyper GR-algebra is an algebraic structure involving two distinct hyperoperations. Properties of this hyper algebra have been studied and given illustrations. This paper focuses on the quotient structure of pseudo hyper GR-algebras. From an equivalence relation on a pseudo hyper GR-algebra H, we can define a congruence relation on H that is used in the construction of the quotient structure H/I, where I is the congruence class of 0 under the congruence relation. Moreover, some isomorphism theorems of pseudo hyper GR-algebras are included in this paper.

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1. Introduction

Algebraic hyperstructures were introduced by a French mathematician, Marty [6], in 1934. They represent a natural extension of classical hyperstructures in which the composition of two elements of a given set is a set, instead of an element. Afterwards, this new idea was expanded rapidly and showed itself as a new view of sets.

The introduction of hyperstructure theory led to the study of several problems of noncommutative algebra. Algebraic hyperstructure theory has multiple applications to other fields such as: geometry, graphs and hypergraphs, binary relations, lattices, groups, relation algebras, artificial intelligence, probabilities, and so on.

In 1966, Y. Imai and K. Iséki [1] initiated the notion of BCK-algebra as a generalization of the concept of set-theoretic difference and propositional calculi. Furthermore, Y.B. Jun et al. [5] applied hyperstructure theory to BCK-algebras and introduced the notion of hyper BCK-algebras as a generalization of BCK-algebra.

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R.A. Indangan and G.C. Petalcorin [2] defined a new class of algebraic hyperstructure called hyper GR-algebra. In this algebra, they presented a helpful understanding on how this hyper algebra differs from the rest.

R.G. Manzano and G.C. Petalcorin [4] extended the study hyper GR-algebras by intorducing a new definition involving two hyperoperations. This gives birth to pseudo hyper GR-algebras.

2. Preliminaries

Let H be a nonempty set endowed with a hyperoperation " $*$ ", that is, " $*$ " is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\varnothing\}$. For two nonempty subsets A and B of H, $A * B = \bigcup_{a \in A, b \in B} a * b$. We shall use $x * y$ instead of $x * \{y\}, \{x\} * y$ or $\{x\} * \{y\}$. When A is a nonempty subset of H and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup_{a \in A} a * x$ and $x * A = \bigcup_{a \in A} x * a$. A set H endowed with a family Γ of hyperoperations is called a *hyperstructure*. If Γ is singleton, that is, $\Gamma = \{f\}$, then the hyperstructure is called a *hypergroupoid*.

Definition 2.1. [2] Let H be a nonempty set with "$\circledast$" a hyperoperation on H. Then $(H; \circledast, 0)$ is called a *hyper GR-algebra* if it contains a constant $0 \in H$ and for all $x, y, z \in H$, the following conditions are satisfied:

$[HGR_1]$ $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$;

$[HGR_2]$ $(x \circledast y) \circledast z = (x \circledast z) \circledast y$;

$[HGR_3]$ $x \ll x$;

$[HGR_4]$ $0 \circledast (0 \circledast x) \ll x$, for all $x \neq 0$; and

$[HGR_5]$ $(x \circledast y) \circledast z \ll y \circledast z$.

where $x \ll y$ if and only if $0 \in x \circledast y$, and for every $A, B \subseteq H$, $A \ll B$ means that for every $a \in A$, there exists $b \in B$ such that $a \ll b$.

Example 2.2. [2] Let $H = \{0, 1, 2\}$. Define the operation "$\circledast$" by the Cayley table shown below.

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0,1,2\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,1,2\}$	$\{0,2\}$

By routine calculations, $(H; \circledast, 0)$ is a hyper GR-algebra.

Definition 2.3. [2] A hyper GR-algebra H is *faithful* if for all $A, B \subseteq H$, $0 \in A \circledast B$ implies $A \ll B$.

Definition 2.4. [2] Let H be a hyper GR-algebra and S be a subset of H containing 0. If S is a hyper GR-algebra with respect to the hyperoperation $\circledast$ on H, then we say that S is a *hyper subGR-algebra* of H.

Theorem 2.5. [2] (Hyper SubGR-algebra Criterion)
Let H be a hyper GR-algebra and S be a nonempty subset of H. Then S is a *hyper subGR-algebra* of H if and only if $x \circledast y \subseteq S$, for all $x, y \in S$.

Definition 2.6. [4] Let H be a nonempty set with "$\circledast$" and "$\circ$" be the two hyperoperations on H. Then $(H; \circledast, \odot, 0)$ is called a *pseudo hyper GR-algebra*, if it contains a constant $0 \in H$ and for all $x, y, z \in H$, the following conditions are satisfied:

$[PHGR_1]$ $(x \odot z) \odot (y \odot z) \ll x \odot y$ and $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$;

$[PHGR_2]$ $(x \odot y) \circledast z = (x \circledast z) \odot y$;

$[PHGR_3]$ $0 \in x \circledast x$ and $0 \in x \odot x$;

$[PHGR_4]$ $0 \odot (0 \circledast x) \ll x$, for all $x \neq 0$; and

$[PHGR_5]$ $(x \circledast y) \circledast z \ll y \odot z$.

where $x \ll y$ if and only if $0 \in x \odot y$ and $0 \in x \circledast y$, and for every $A, B \subseteq H$, $A \ll B$ means that for every $a \in A$, there exists $b \in B$ such that $a \ll b$.

Example 2.7. [4] Let $H = \{0, 1, 2, 3\}$ and consider the following Cayley tables below.

$\circledast$	0	1	2	3
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,1,2\}$	$\{0,2\}$	$\{0,1,2\}$
3	$\{0,1,2\}$	$\{0,3\}$	$\{0,1,3\}$	$\{0,3\}$

$\odot$	0	1	2	3
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
3	$\{0,3\}$	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$

By routine calculations, we see that $(H; \circledast, \odot, 0)$ is a pseudo hyper GR-algebra.

Remark 2.8. [4] In a pseudo hyper GR-algebra H, the following are evident:

(i) $x \ll x$

(ii) $(x \odot y) \circledast z \ll (x \circledast z) \odot y$

(iii) $(A \odot B) \circledast C = (A \circledast C) \odot B$

(iv) $A \subseteq B$ implies $A \ll B$.

Example 2.9. [4] Let $H = \mathbb{N} \cup \{0\}$ be the set of all nonnegative integers and let the hyperoperations "$\circledast$" and "$\odot$" be defined on H as follows:

$$x \circledast y = \{0, x\} \text{ and } x \odot y = \{0, x, y\}.$$

Then H is a pseudo hyper GR-algebra.

Remark 2.10. [4] Note that if the two hyperoperations are equal , that is, $\circledast = \odot$, then a pseudo hyper-GR algebra H becomes a hyper GR-algebra.

Definition 2.11. [4] Let H be a pseudo hyper GR-algebra and S be a subset of H containing 0. If S itself is a pseudo hyper GR-algebra with respect to the hyperoperations $\circledast$ and $\odot$ on H, then S is called a *pseudo hyper subGR-algebra* of H.

Theorem 2.12. [4] (Pseudo Hyper SubGR-algebra Criterion)
Let S be a nonempty subset of a pseudo hyper GR-algebra H. Then S is a pseudo hyper subGR-algebra if and only if both $x \circledast y \subseteq S$ and $x \odot y \subseteq S$ for all $x, y \in S$.

Example 2.13. [4] For any pseudo hyper GR-algebra H, the set $S = \{0\}$ is a pseudo hyper subGR-algebra of H.

3. Quotient Pseudo Hyper GR-algebras

In this section, we construct the structure of the quotient pseudo hyper GR-algebra H/I from a pseudo hyper GR-algebra H via congruence relation.

All throughout, we denote a pseudo hyper GR-algebra $(H, \circledast, \odot, 0)$ simply by H, unless otherwise stated.

Definition 3.1. Let θ be an equivalence relation on a pseudo hyper GR-algebra H and A and B be nonempty subsets of H.

(i) $A\theta B$ if there exist $a \in A$ and $b \in B$ such that $a\theta b$;

(ii) $A\bar{\theta}B$ if for every $a \in A$, there exists $b \in B$ such that $a\theta b$ and for every $b \in B$, there exists $a \in A$ such that $a\theta b$;

(iii) θ is called a *right $\circledast$-congruence* (resp. *right $\odot$-congruence*) on H if $a\theta b$ implies $(a \circledast u)\bar{\theta}(b \circledast u)$ (resp. $(a \odot u)\bar{\theta}(b \odot u)$) for all u in H;

(iv) θ is called a *left $\circledast$-congruence* (resp. *left $\odot$-congruence*) on H if $a\theta b$ implies $(u \circledast a)\bar{\theta}(u \circledast b)$ (resp. $(u \odot a)\bar{\theta}(u \odot b)$) for all u in H;

(v) θ is called a $\circledast$-*congruence* (resp. $\odot$-*congruence*) on H if it is a right and a left $\circledast$-congruence (resp. a right and left $\odot$-congruence);

(vi) θ is called a *left congruence* on H if it is a left $\circledast$-congruence and a left $\odot$-congruence on H;

(vii) θ is called a *right congruence* on H if it is a right $\circledast$-congruence and a right $\odot$-congruence on H;

(viii) θ is called a *congruence relation* on H if it is a $\circledast$-congruence and $\odot$-congruence on H; and

(ix) θ is called a *regular congruence relation* on H, if θ is a congruence relation on H and for any $x, y \in H$, whenever $(x \circledast y)\theta\{0\}$, $(y \circledast x)\theta\{0\}$, $(x \odot y)\theta\{0\}$, and $(y \odot x)\theta\{0\}$, we have $x\theta y$.

Example 3.2. Let $H = \{0, 1, 2\}$ and consider the following Cayley tables below.

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$

$\odot$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,2\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$
2	$\{0,2\}$	$\{0,1,2\}$	$\{0,2\}$

Then $(H; \circledast, \odot, 0)$ is a pseudo hyper GR-algebra.

Define θ on H by $\theta = \{(0,0), (0,1), (1,0), (1,1), (2,2)\}$. It can be easily verified that θ is an equivalence relation. We will show that θ is a congruence relation using Definition 3.1. Since $x\theta x$ for all $x \in H_1$, we have $(a \circledast x)\bar{\theta}(a \circledast x)$, $(x \circledast a)\bar{\theta}(x \circledast a)$, $(a \odot x)\bar{\theta}(a \odot x)$ and $(x \odot a)\bar{\theta}(x \odot a)$. Therefore, the remaining elements of θ that is left for verification are $(1, 0)$ and $(0, 1)$ which can be done simultaneously. Since $1\theta 0$, we will show that $(1 \circledast a)\bar{\theta}(0 \circledast a)$, $(a \circledast 1)\bar{\theta}(a \circledast 0)$, $(1 \odot a)\bar{\theta}(0 \odot a)$ and $(a \odot 1)\bar{\theta}(a \odot 0)$ for all $a \in H$.

For $a = 0$,
$1 \circledast 0 = \{0,1\}\bar{\theta}\{0\} = 0 \circledast 0$ and $0 \circledast 1 = \{0\}\bar{\theta}\{0\} = 0 \circledast 0$
$1 \odot 0 = \{0,1\}\bar{\theta}\{0\} = 0 \odot 0$ and $0 \odot 1 = \{0\}\bar{\theta}\{0\} = 0 \odot 0$.

For $a = 1$,
$1 \circledast 1 = \{0,1\}\bar{\theta}\{0\} = 0 \circledast 1$ and $1 \circledast 1 = \{0,1\}\bar{\theta}\{0,1\} = 1 \circledast 0$
$1 \odot 1 = \{0,1\}\bar{\theta}\{0,1\} = 0 \odot 1$ and $1 \odot 1 = \{0,1\}\bar{\theta}\{0,1\} = 1 \odot 0$.

For $a = 2$,
$1 \circledast 2 = \{0,1\}\bar{\theta}\{0\} = 0 \circledast 2$ and $2 \circledast 1 = \{0,2\}\bar{\theta}\{0,2\} = 2 \circledast 0$
$1 \odot 2 = \{0,1,2\}\bar{\theta}\{0,2\} = 0 \odot 2$ and $2 \odot 1 = \{0,1,2\}\bar{\theta}\{0,2\} = 2 \odot 0$.

Therefore, θ is a congruence relation.

Definition 3.3. Let θ be a congruence relation on a pseudo hyper GR-algebra H. The *congruence class* of x, denoted by $[x]_\theta$ or I_x is given by $[x]_\theta = I_x = \{y \in H \mid x\theta y\}$.

Lemma 3.4. Let θ be an equivalence relation on H and $A, B \subseteq H$. If $A\bar{\theta}B$ and $B\bar{\theta}C$, then $A\bar{\theta}C$.

Proof. Let $a \in A$. By Definition 3.1(ii), there exists $b \in B$ such that $a\theta b$. Also, there exists $c \in C$ such that $b\theta c$. By transitivity, $a\theta c$.

Let $c \in C$. Then there exists $b \in B$ such that $b\theta c$. But there exists $a \in A$ such that $a\theta b$. So, by transitivity, $a\theta c$. Therefore, $A\bar{\theta}C$. □

Lemma 3.5. Let θ be an equivalence relation on H such that $x \circledast 0 = x$ and $x \odot 0 = x$, for all $x \in H$. Then the following hold:

(i) If θ is a left $\circledast$-congruence (left $\odot$-congruence) on H, then $[0]_\theta$ is a pseudo hyper GR-ideal of type 8.

(ii) If θ is a left congruence on H, then $[0]_\theta$ is a pseudo hyper GR-ideal of type 4.

Proof. (i) Suppose that θ is a left $\circledast$-congruence. Let $y \in [0]_\theta$ and $x \in [0]^{\ll}_{\theta_{\circledast},y}$. Then $x \circledast y \ll [0]_\theta$. Then for all $a \in x \circledast y$, there exists $b \in [0]_\theta$ such that $0 \in a \circledast b$. Since θ is a left $\circledast$-congruence, $b\theta 0$ implies that $(a \circledast b)\bar{\theta}(a \circledast 0)$ and so $(a \circledast b)\bar{\theta}a$. Now, $0 \in a \circledast b$ and $(a \circledast b)\bar{\theta}a$ would imply that $0\theta a$ and so $x \circledast y \subseteq [0]_\theta$. Since $y\theta 0$ and θ is a left $\circledast$-congruence, $(x \circledast y)\bar{\theta}(x \circledast 0)$. Thus, for all $z \in x \circledast y$, we have $z\theta x$. Since $x \circledast y \subseteq [0]_\theta$, $z\theta 0$. By commutativity and transitivity, $z\theta 0$ and $z\theta x$ means that $x\theta 0$. Hence, $x \in [0]_\theta$. Therefore, $[0]_\theta$ is a pseudo hyper GR-ideal of type 8.

Similarly, it is easy to show that $[0]_\theta$ is a pseudo hyper GR-algebra of type 8 for the case of left $\odot$-congruence.

(ii) Suppose that θ is a left congruence on H. Let $y \in [0]_\theta$ and $x \in [0]^{\ll}_{\theta_{\circledast},y}$. Then $x \circledast y \subseteq [0]_\theta$. This means that $x \circledast y \ll [0]_\theta$. Then for all $a \in x \circledast y$, there exists $b \in [0]_\theta$ such that $0 \in a \circledast b$. Since θ is a left $\circledast$-congruence, $b\theta 0$ implies that $(a \circledast b)\bar{\theta}(a \circledast 0)$ and so $(a \circledast b)\bar{\theta}a$. Now, $0 \in a \circledast b$ and $(a \circledast b)\bar{\theta}a$ would imply that $0\theta a$ and so $x \circledast y \subseteq [0]_\theta$. Since $y\theta 0$ and θ is a left $\circledast$-congruence, $(x \circledast y)\bar{\theta}(x \circledast 0)$ or equivalently $(x \circledast y)\bar{\theta}\{x\}$. Thus, for all $z \in x \circledast y$, we have $z\theta x$. Since $x \circledast y \subseteq [0]_\theta$, $z\theta 0$. Now, $z\theta 0$ and $z\theta x$ mean that $x\theta 0$. Hence, $x \in [0]_\theta$. Similarly, if $y \in [0]_\theta$ and $x \in [0]^{\ll}_{\theta_{\odot},y}$, then $x \in [0]_\theta$. Therefore, $[0]_\theta$ is a pseudo hyper GR-ideal of type 4. □

The next example will give us the idea on how the quotient structure on a pseudo hyper GR-algebra is constructed.

Example 3.6. Consider the pseudo hyper GR-algebra $H = \{0, 1, 2\}$ in Example 3.2. The relation θ defined on H is a congruence relation as shown. Moreover, $I = [0]_\theta = \{0, 1\}$ and $I_2 = \{2\}$. Let H/I denote the set of all congruence classes on H, that is, $H/I = \{I_x \mid x \in H\}$. In our case, $H/I = \{I, I_2\}$. Define hyperoperations

$\otimes$ and $\circledcirc$ on H/I by $I_x \otimes I_y = \{I_z \,|\, z \in x \circledast y\}$ and $I_x \circledcirc I_y = \{I_z \,|\, z \in x \odot y\}$ and $I_x \ll I_y \iff I_0 \in I_x \circledast I_y$ and $I_0 \in I_x \odot I_y$. Thus, for our case, we have the following Cayley tables:

$\otimes$	I	I_2
I	$\{I\}$	$\{I\}$
I_2	$\{I, I_2\}$	$\{I, I_2\}$

$\circledcirc$	I	I_2
I	$\{I\}$	$\{I, I_2\}$
I_2	$\{I, I_2\}$	$\{I, I_2\}$

By routine calculations, $(H/I; \otimes, \circledcirc, I)$ is a pseudo hyper GR-algebra.

Lemma 3.7. Let θ be a congruence relation on a pseudo hyper GR-algebra H such that $x\theta x'$ and $y\theta y'$. Then $(x \circledast y)\theta(x' \circledast y')$ and $(x \odot y)\theta(x' \odot y')$.

Proof. Suppose that θ is a congruence relation such that $x\theta x'$ and $y\theta y'$. Then $I_x = I_{x'}$ and $I_y = I_{y'}$. Let $z \in x \circledast y$. Then $I_z \in I_x \otimes I_y = I_{x'} \otimes I_{y'}$. Thus, $I_z \in I_{x'} \otimes I_{y'}$. This means that $z \in x' \circledast y'$. Hence, $(x \circledast y)\theta(x' \circledast y')$.

Similarly, we can show that $(x \odot y)\theta(x' \odot y')$. □

We will now show in general that using congruence relation, the quotient structure obtained is a pseudo hyper GR-algebra.

Theorem 3.8. Let θ be a congruence relation on a pseudo hyper GR-algebra H such that $I = [0]_\theta$ and $H/I = \{I_x \,|\, x \in H\}$, where $I_x = [x]_\theta$ for all $x \in H$. Then H/I with hyperoperations $\otimes$ and $\circledcirc$, and hyperorder $\ll$ which are defined as follows:

$$I_x \otimes I_y = \{I_z \,|\, z \in x \circledast y\} \text{ and } I_x \circledcirc I_y = \{I_z \,|\, z \in x \odot y\}, \text{ and}$$
$$I_x \ll I_y \iff I_0 \in I_x \otimes I_y \text{ and } I_0 \in I_x \circledcirc I_y.$$

is a pseudo hyper GR-algebra which we call the *quotient pseudo hyper GR-algebra.*

Proof. Let us show first that the hyperoperations $\otimes$ and $\circledcirc$ on H/I are well-defined. Suppose that $x, y, x', y' \in H$ such that $I_x = I_{x'}$ and $I_y = I_{y'}$. Let $I_z \in I_x \otimes I_y$. Then there exists $u \in x \circledast y$ such that $I_u = I_z$. Since $x\theta x'$ and $y\theta y'$, and θ is a congruence on H, by Lemma 3.7, $(x \circledast y)\theta(x'\theta y')$. Hence, there exists $z' \in x' \circledast y'$ such that $u\theta z'$ and thus, $I_{z'} = I_u$. Since $I_{z'} \in I_{x'} \otimes I_{y'}$ and $I_z = I_u = I_{z'}$, we have $I_z \in I_{x'} \otimes I_{y'}$. Thus, $I_x \otimes I_y \subseteq I_{x'} \otimes I_{y'}$. Similarly, we can show that $I_{x'} \otimes I_{y'} \subseteq I_x \otimes I_y$. Hence, $I_x \otimes I_y = I_{x'} \otimes I_{y'}$. Therefore, the hyperoperation "$\otimes$" is well-defined.

Similarly, we can show that $I_x \circledcirc I_y = I_{x'} \circledcirc I_{y'}$ so that "$\circledcirc$" is also well-defined.

Now, since H is a pseudo hyper GR-algebra, $0 \in H$ and so, $I_0 = [0]_\theta = I \in H/I$. Hence, H/I is nonempty and $I \in H/I$. It remains to show that H/I satisfies all the axioms of a pseudo hyper GR-algebra.

$[PHGR_1]$ Let $I_w \in (I_x \circledcirc I_z) \circledcirc (I_y \circledcirc I_z)$, for some $I_x, I_y, I_z \in H/I$. Then there are $I_u \in I_x \circledcirc I_z$ and $I_v \in I_y \circledcirc I_z$ such that $I_w \in I_u \circledcirc I_v$. Hence, there are $u' \in x \odot z$, $v' \in y \odot z$ and $w' \in u \odot v$ such that $I_u = I_{u'}$, $I_v = I_{v'}$, and $I_w = I_{w'}$. Hence, $u\theta u'$, $v\theta v'$ and $w\theta w'$.

Since θ is a congruence relation on H, by Lemma 3.7, $(u \odot v)\theta(u' \odot v')$. From $w' \in u \odot v$, there exists $a \in u' \odot v'$ such that $w'\theta a$ and so, $I_{w'} = I_a$. Thus, $I_w = I_{w'} = I_a$.

By $PHGR_1$ on H, $a \in u' \odot v' \subseteq (x \odot z) \odot (y \odot z) \ll x \odot y$. Hence, there exists $b \in x \odot y$ such that $a \ll b$, which means that $0 \in a \odot b$ and $0 \in a \circledast b$. Furthermore, $I_b \in I_x \circledcirc I_y$, $I_0 \in I_a \circledcirc I_b$, and$I_0 \in I_a \otimes I_b$. Since $I_w = I_{w'} = I_a$, we have $I_0 \in I_w \circledcirc I_b$ and $I_0 \in I_w \otimes I_b$ which means that $I_w \ll I_b$. This implies that $(I_x \circledcirc I_z) \circledcirc (I_y \circledcirc I_z) \ll I_x \circledcirc I_y$.

Similarly, we can show also that $(I_x \otimes I_z) \otimes (I_y \otimes I_z) \ll I_x \otimes I_y$.

Therefore, $[PHGR_1]$ holds.

$[PHGR_2]$ Let $I_w \in (I_x \circledcirc I_y) \otimes I_z$. Then there exists $I_u \in I_x \circledcirc I_y$ such that $I_w \in I_u \otimes I_z$. Since $I_u \in I_x \circledcirc I_y$, there exists $u' \in x \odot y$ such that $u\theta u'$, that is, $I_u = I_{u'}$. Hence, $I_{u'} \in I_x \circledcirc I_y$. Since $I_w \in I_u \otimes I_z = I_{u'} \otimes I_z$, there exists $w' \in u' \circledast z$ such that $w'\theta w$. Now, $w' \in u' \circledast z \subseteq (x \odot y) \circledast z = (x \circledast z) \odot y$, by $PHGR_2$ on H. Hence, $w' \in (x \circledast z) \odot y$ and $u' \circledast z \subseteq (x \odot y) \circledast z$. This means that there exists $b \in x \circledast z$ such that $w' \in b \odot y$. Since $b \in x \circledast z$, $I_b \in I_x \otimes I_z$. Also, $I_{w'} \in I_b \circledcirc I_y$. Thus, $I_w = I_{w'} \in I_b \circledcirc I_y \subseteq (I_x \otimes I_z) \circledcirc I_y$. Hence, $(I_x \circledcirc I_y) \otimes I_z \subseteq (I_x \otimes I_z) \circledcirc I_y$.

For the other set inclusion, let $I_w \in (I_x \otimes I_z) \circledcirc I_y$. Then there exists $I_u \in I_x \otimes I_z$ such that $I_w \in I_u \circledcirc I_y$. Since $I_u \in I_x \otimes I_z$, there exists $u' \in x \circledast z$ such that $u\theta u'$, that is, $I_u = I_{u'}$. Hence, $I_{u'} \in I_x \otimes I_z$. Since $I_w \in I_u \circledcirc I_y = I_{u'} \circledcirc I_y$, there exists $w' \in u' \odot y$ such that $w'\theta w$. Now, $w' \in u' \odot y \subseteq (x \circledast z) \odot y = (x \odot y) \circledast z$, by $PHGR_2$ on H. Hence, $w' \in (x \odot y) \circledast z$ and $u' \odot y \subseteq (x \circledast z) \odot y$. This means that there exists $b \in x \odot y$ such that $w' \in b \circledast z$. Now, $b \in x \odot y$ implies $I_b \in I_x \circledcirc I_y$. Also, $I_{w'} \in I_b \otimes I_z$. Thus, $I_w = I_{w'} \in I_b \otimes I_z \subseteq (I_x \circledcirc I_y) \otimes I_z$. Hence, $(I_x \otimes I_z) \circledcirc I_y \subseteq (I_x \circledcirc I_y) \otimes I_z$.

Therefore, $(I_x \otimes I_z) \circledcirc I_y = (I_x \circledcirc I_y) \otimes I_z$ and $[PHGR_2]$ holds.

$[PHGR_3]$ By $PHGR_3$ of H, $0 \in x \circledast x$ and $0 \in x \odot x$ which means that $I_0 \in I_x \otimes I_x$ and $I_0 \in I_x \circledcirc I_x$. This means that $I_x \ll I_x$. Therefore, $[PHGR_3]$ holds.

$[PGHR_4]$ Let $I_w \in I_0 \circledcirc (I_0 \otimes I_x)$. Then there exists $I_u \in I_0 \otimes I_x$ for which $I_w \in I_0 \circledcirc I_u$. Since $I_u \in I_0 \otimes I_x$, there exists $u' \in 0 \circledast x$ such that $u\theta u'$ and $I_u = I_{u'}$. Hence, $I_{u'} \in I_0 \otimes I_x$. Since $I_w \in I_0 \circledcirc I_u = I_0 \circledcirc I_{u'}$, there exists $w' \in 0 \odot u'$ such that $w\theta w'$ and $I_w = I_{w'}$. Now, $w' \in 0 \odot u' \subseteq 0 \odot (0 \circledast x) \ll x$, by $PHGR_4$ of H. This means that $w' \ll x$. Thus, $I_{w'} \ll I_x$. Now, $I_w = I_{w'}$ and so, $I_w \ll I_x$. Since $I_w \in I_0 \circledcirc (I_0 \otimes I_x)$, we have $I_0 \circledcirc (I_0 \otimes I_x) \ll I_x$. Therefore, $[PHGR_4]$ holds.

$[PHRG_5]$ Let $I_w \in (I_x \otimes I_y) \otimes I_z$. Then there is $I_u \in I_x \otimes I_y$ such that $I_w \in I_u \otimes I_z$. Since $I_u \in I_x \otimes I_y$, there exists $u' \in x \circledast y$ such that $u\theta u'$ and $I_u = I_{u'}$. Also, since $I_w \in I_u \circledast I_z$, there exists $w' \in u \circledast z$ such that $I_w = I_{w'}$. Since θ is a congruence on H and $u\theta u'$, by Lemma 3.7, $(u \circledast z)\theta(u' \circledast z)$. Then there exists $a \in u' \circledast z$ such that $w'\theta a$. Thus, $I_w = I_{w'} = I_a$. Now, $a \in u' \circledast z \subseteq (x \circledast y) \circledast z \ll y \odot z$, $PHGR_5$ on H. Hence, there exists $b \in y \odot z$ such that $a \ll b$. This means that $0 \in a \circledast b$ and $0 \in a \odot b$. Furthermore, $I_b \in I_y \circledcirc I_z$, $I_0 \in I_a \circledcirc I_b$, and $I_0 \in I_a \otimes I_b$. Since $I_w = I_{w'} = I_a$, we have $I_0 \in I_w \circledcirc I_b$ and $I_0 \in I_w \otimes I_b$ and hence, $I_w \ll I_b$. Note that $I_b \in I_y \circledcirc I_z$. Thus, $(I_x \otimes I_y) \circledast I_z \ll I_y \circledcirc I_z$. Hence, $[PHGR_5]$ holds.

Therefore, $(H/I; \otimes, \circledcirc, I)$ is a pseudo hyper GR-algebra. □

Lemma 3.9. Let θ and θ' be two regular congruences on H such that

$[0]_\theta = [0]_{\theta'}$. Then $\theta = \theta'$.

Proof. It is enough to show that $x\theta y \iff x\theta' y$.

If $x\theta y$, then $(x \circledast x)\bar{\theta}(x \circledast y)$. Since $0 \in x \circledast x$ and θ is a congruence on H, there exists $z \in x \circledast y$ such that $0\theta z$. Then, $z \in [0]_\theta = [0]_{\theta'}$ and $z \in [0]'_\theta$, that is, $0\theta' z$. Hence, $\{0\}\theta'(x \circledast y)$. Similarly, we can also show that $\{0\}\theta'(y \circledast x)$. Thus, $x\theta' y$ since θ' is regular.

Following the same argument, $x\theta' y$ implies $x\theta y$. Therefore, $\theta = \theta'$. □

4. Isomorphism Theorems of Pseudo Hyper GR-algebras

This section discusses some hyper isomorphism theorems of pseudo hyper GR-algebras, namely, the first and the third hyper isomorphism theorems. All throughout, H and H' are pseudo hyper GR-algebras, unless otherwise stated.

Lemma 4.1. Let θ be a regular congruence on a pseudo hyper GR-algebra H and $I = [0]_\theta$. Then the map $\pi : H \longrightarrow H/I$ defined by $\pi(x) = I_x$, for all $x \in H$, is an epimorphism, called the canonical epimorphism.

Proof. Let $x, y \in H$ such that $x = y$. Then

$$\pi(x) = I_x = [x]_\theta = [y]_\theta = I_y = \pi(y).$$

Hence, π is a well-defined map.

Now, observe that $\pi(0) = I_0 = I$. Next, we will show that π is a homomorphism. Pick $x, y \in H$. Let $J \in \pi(x) \circledast \pi(y)$. Then $J \in I_x \circledast I_y$ and so there exists element $u \in x \circledast y$ such that $J = I_u = \pi(u) \in \pi(x \circledast y)$. Thus, $\pi(x) \circledast \pi(y) \subseteq \pi(x \circledast y)$. Now, let $L \in \pi(x \circledast y)$. Then there exists an element $v \in x \circledast y$ such that $L = \pi(v) = I_v$. Note that $I_v \in I_x \circledast I_y = \pi(x) \circledast \pi(y)$. Hence, we have $L \in \pi(x) \circledast \pi(y)$ and $\pi(x \circledast y) \subseteq \pi(x) \circledast \pi(y)$. Thus, $\pi(x \circledast y) = \pi(x) \circledast \pi(y)$. Similarly, for the hyperoperation $\odot$, we can show that $\pi(x \odot y) = \pi(x) \odot \pi(y)$. Thus, π is a hyper homomorphism.

Let $I_x \in H/I$ with $x \in H$. Then $\pi(x) = I_x \in H/I$. Therefore, π is a surjective map and so, an epimorphism. □

Theorem 4.2. (Homomorphism Theorem) Let θ be a regular congruence relation on H and $I = [0]_\theta$. If $f : H \longrightarrow H'$ is a homomorphism of pseudo hyper GR-algebras such that $f(x) \ll f(y)$ and $f(y) \ll f(x)$ imply that $f(x) = f(y)$ for all $x, y \in H$, then $\bar{f} : H/I \longrightarrow H'$, which is defined by $\bar{f}(I_x) = f(x)$, for all $x \in H$, is a unique homomorphism such that $\bar{f} \circ \pi = f$, where π denotes the canonical epimorphism and $\circ$ is the composition map. Moreover, if $I = ker\, f$, then $\bar{f}$ is a monomorphism.

Proof. Let θ be a regular congruence relation on H and $I = [0]_\theta$. Define $\bar{f} : H/I \longrightarrow H'$ by $\bar{f}(I_x) = f(x)$ for all $x \in H$. Let $x, y \in H$ such that $I_x = I_y$ and let $t \in I_x = I_y$. Since H/I is a pseudo hyper GR-algebra, by $[PHGR_3]$, $t \ll t$ and so, $I_x \ll I_y$. Hence $I \in I_x \circledast I_y$ and $I \in I_x \odot I_y$. It follows that there exist $z \in x \circledast y$ and $z \in x \odot y$ such that $I = I_z$. Thus, $z \in I \subseteq ker\, f$ and so $f(z) = 0'$. Since f is a hyper homomorphism, we have

$0' = f(z) = f(x \circledast y) = f(x) \circledast f(y)$ and $0' = f(z) = f(x \odot y) = f(x) \odot f(y)$. it follows that $f(x) \ll f(y)$. Using the same argument, picking $t' \in I_y = I_x$ will imply that $f(y) \ll f(x)$. Thus, by the hypothesis $f(x) \ll f(y)$ and $f(y) \ll f(x)$ imply that $f(x) = f(y)$. Hence, $\bar{f}(I_x) = \bar{f}(I_y)$ and $\bar{f}$ is a well-defined map.

Let $I_x, I_y \in H/I$. We will show that $\bar{f}(I_x \odot I_y) = \bar{f}(I_x) \odot \bar{f}(I_y)$. Let $w \in \bar{f}(I_x \odot I_y)$. Then there exists $I_t \in I_x \odot I_y$ such that $w = \bar{f}(I_t) = f(t)$. Now, $I_t \in I_x \odot I_y$ implies that $t \in x \odot y$ and

$$w = f(t) \in f(x \odot y) = f(x) \odot f(y) = \bar{f}(I_x) \odot \bar{f}(I_y),$$

Thus, $\bar{f}(I_x \odot I_y) \subseteq \bar{f}(I_x) \odot \bar{f}(I_y)$.

Now, let $u \in \bar{f}(I_x) \odot \bar{f}(I_y) = f(x) \odot f(y) = f(x \odot y)$. Then there exists an element $v \in x \odot y$ such that $u = f(v)$. Hence, $I_v \in I_x \odot I_y$, and we have $u = f(v) = \bar{f}(I_v) \in \bar{f}(I_x \odot I_y)$. Thus, $\bar{f}(I_x) \odot \bar{f}(I_y) \subseteq \bar{f}(I_x \odot I_y)$. Hence, $\bar{f}(I_x \odot I_y) = \bar{f}(I_x) \odot \bar{f}(I_y)$. In a similar manner, we can show that $\bar{f}(I_x \circledast I_y) = \bar{f}(I_x) \circledast \bar{f}(I_y)$. Hence, $\bar{f}$ is a homomorphism.

Now, $dom(\bar{f} \circ \pi) = H = dom\, f$ and for all $x \in H$, $(\bar{f} \circ \pi)(x) = \bar{f}(\pi(x)) = \bar{f}(I_x) = f(x)$. Hence, $\bar{f} \circ \pi = f$.

To show the uniqueness of $\bar{f}$, we suppose that there is another homomorphism g such that $g \circ \pi = f$. Let $x \in H$. Then

$$g(I_x) = g(\pi(x)) = f(x) = \bar{f}(\pi(x)) = \bar{f}(I_x).$$

Now, we will show that if $I = ker\, f$, then $\bar{f}$ is a monomorphism. Suppose that $\bar{f}(I_x) = \bar{f}(I_y)$ with $x, y \in H$. Then $f(x) = f(y)$. Since f is a homomorphism and by $[PHGR_3]$,

$$0_{H'} = f(0_H) \in f(x \circledast x) = f(x) \circledast f(x) = f(x) \circledast f(y) = f(x \circledast y).$$

So, there exists $u \in x \circledast y$ such that $f(u) = 0_{H'}$. Hence, $u \in ker\, f = I$ and so, $u\theta 0$. It follows that $(x \circledast y)\theta\{0\}$. Also,

$$0_{H'} = f(0_H) \in f(x \circledast x) = f(x) \circledast f(x) = f(y) \circledast f(x) = f(y \circledast x).$$

So, there exists $v \in y \circledast x$ such that $f(v) = 0_{H'}$. Then $v \in ker\, f = I$ and so, $v\theta 0$. Hence, $(y \circledast x)\theta\{0\}$.

Similarly,

$$0_{H'} = f(0_H) \in f(x \odot x) = f(x) \odot f(x) = f(x) \odot f(y) = f(x \odot y).$$

So, there exists $u \in x \odot y$ such that $f(u) = 0_{H'}$. This means that $u \in ker\, f = I$ and so, $u\theta 0$ which implies that $(x \odot y)\theta\{0\}$. Lastly,

$$0_{H'} = f(0_H) \in f(x \odot x) = f(x) \odot f(x) = f(y) \odot f(x) = f(y \odot x).$$

Thus, there exists $v \in y \odot x$ such that $f(v) = 0_{H'}$. Moreover, $v \in ker\, f = I$ and $v\theta 0$. Thus, $(y \odot x)\theta\{0\}$. Since θ is a regular congruence relation, it follows that $x\theta y$. Thus, $I_x = I_y$. Therefore, $\bar{f}$ is a one-to-one map. This proves the theorem. □

Before we prove the First Isomorphism Theorem, let us consider the following example which is a specific case of the next theorem.

Example 4.3. Consider the pseudo hyper GR-algebra $H = \{0, 1, 2\}$ in Example 3.6. We can verify that the given congruence relation θ on H is regular.

In our case, $I = [0]_\theta = \{0, 1\}$ and $I_2 = \{2\}$. Then $H/I = \{I, I_2\}$ whose Cayley table is shown below

$\circledast$	I	I_2
I	$\{I\}$	$\{I\}$
I_2	$\{I, I_2\}$	$\{I, I_2\}$

$\odot$	I	I_2
I	$\{I\}$	$\{I, I_2\}$
I_2	$\{I, I_2\}$	$\{I, I_2\}$

By routine calculations, H/I is a pseudo hyper GR-algebra.

Now, consider the set $H' = \{0, 1\}$ together with the Cayley tables below

$\circledast$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0, 1\}$	$\{0, 1\}$

$\odot$	0	1
0	$\{0\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{0, 1\}$

By routine calculations, H' is a pseudo hyper GR-algebra.

Define the map $f : H \longrightarrow H'$ by $f(0) = 0 = f(1)$ and $f(2) = 1$. Then f is a homomorphism as shown in table below

x	y	$x \circledast y$	$f(x \circledast y)$	$f(x)$	$f(y)$	$f(x) \circledast f(y)$
0	0	$\{0\}$	$\{0\}$	0	0	$\{0\}$
0	1	$\{0\}$	$\{0\}$	0	0	$\{0\}$
0	2	$\{0\}$	$\{0\}$	0	1	$\{0\}$
1	0	$\{0, 1\}$	$\{0\}$	0	0	$\{0\}$
1	1	$\{0, 1\}$	$\{0\}$	0	0	$\{0\}$
1	2	$\{0, 2\}$	$\{0\}$	0	1	$\{0\}$
2	0	$\{0, 2\}$	$\{0, 1\}$	1	0	$\{0, 1\}$
2	1	$\{0, 2\}$	$\{0, 1\}$	1	0	$\{0, 1\}$
2	2	$\{0, 2\}$	$\{0, 1\}$	1	1	$\{0, 1\}$

x	y	$x \odot y$	$f(x \odot y)$	$f(x)$	$f(y)$	$f(x) \odot f(y)$
0	0	$\{0\}$	$\{0\}$	0	0	$\{0\}$
0	1	$\{0,1\}$	$\{0\}$	0	0	$\{0\}$
0	2	$\{0,2\}$	$\{0,1\}$	0	1	$\{0,1\}$
1	0	$\{0,1\}$	$\{0\}$	0	0	$\{0\}$
1	1	$\{0,1\}$	$\{0\}$	0	0	$\{0\}$
1	2	$\{0,1,2\}$	$\{0,1\}$	0	1	$\{0,1\}$
2	0	$\{0,2\}$	$\{0,1\}$	1	0	$\{0,1\}$
2	1	$\{0,1,2\}$	$\{0,1\}$	1	0	$\{0,1\}$
2	2	$\{0,2\}$	$\{0,1\}$	1	1	$\{0,1\}$

According how f is being defined, we have $ker\, f = \{0,1\} = [0]_\theta = I$ and $Im\, f = \{0,1\} = H'$.

Let us define the map $\varphi : H/I \longrightarrow H'$ by

$$\varphi(I_x) = \begin{cases} 0 & \text{if } I_x = I \\ 1 & \text{otherwise.} \end{cases}$$

We can verify that φ is an isomorphism. Moreover, $H/ker\, f \cong Im\, f$.

Corollary 4.4. (First Isomorphism Theorem) Let θ be a regular congruence relation on H and $I = [0]_\theta$. If $f : H \longrightarrow H'$ is a hyper homomorphism of pseudo hyper GR-algebras such that $f(x) \ll f(y)$ and $f(y) \ll f(x)$ imply that $f(x) = f(y)$ and $ker\, f = I$, then $H/ker\, f \cong Im\, f$.

Proof. By Theorem 4.2, the map $\bar{f} : H/ker\, f \longrightarrow H'$ is a monomorphism and by Remark **??** (ii), $\bar{f} : H/ker\, f \cong Im\, \bar{f}$ is an isomorphism. Thus, $H/ker\, f \cong Im\, \bar{f}$. Since $\bar{f}(I_x) = f(x)$ for all $x \in H$, $Im\, \bar{f} = Im\, f$. Hence, the result follows. □

Proposition 4.5. Let K be a pseudo hyper subGR-algebra of H and θ a regular congruence on H and $I = [0]_\theta$. Define $K/I = \{I_x \in H/I \mid x \in K\}$. Then K/I is a pseudo hyper subGR-algebra of H/I.

Proof. Since K is a pseudo hyper subGR-algebra of H, $0 \in K$ and so, $[0]_\theta = I \in K/I$ which means that K/I is nonempty. Let $I_x, I_y \in K/I$. Then $x, y \in K$. Since K is a pseudo hyper subGR algebra of H, $x \circledast y \subseteq K$. Suppose that $I_z \in I_x \circledast I_y$. Then $z \in x \circledast y \subseteq K$. Thus, $I_z \in K/I$ and $I_x \circledast I_y \subseteq K/I$. Similarly, suppose that $I_z \in I_x \odot I_y$. Then, $z \in x \odot y \subseteq K$, and so, $I_z \in K/I$. Thus, $I_x \odot I_y \subseteq K/I$. Therefore, K/I is a pseudo hyper subGR-algebra of H/I. □

Corollary 4.6. Let θ_1 and θ_2 be regular congruence relations on H, with $J = [0]_{\theta_1}$ and $I = [0]_{\theta_2}$. Define $J/I = \{I_x \in H/I \mid x \in J\}$. If J is a pseudo hyper subGR-algebra of H, then J/I is a pseudo hyper subGR-algebra of H/I.

Proof. By definition of J/I, it is easy to see that $J/I \subseteq H/I$. Since J is a pseudo hyper subGR-algebra of H, by Proposition 4.5, J/I is pseudo hyper subGR-algebra of H. □

Lemma 4.7. Let θ_1 and θ_2 be regular congruence relations on H, with $J = [0]_{\theta_1}$ and $I = [0]_{\theta_2}$. Define $J/I = \{I_x \in H/I \,|\, x \in J\}$ and a relation θ on H/I by $I_x\theta I_y$ if and only if $x\theta_1 y$, for all $I_x, I_y \in H/I$. Then θ is a regular congruence relation and $[I]_\theta = J/I$.

Proof. Define a relation θ on H/I by $I_x\theta I_y$ if and only if $x\theta_1 y$, for all $I_x, I_y \in H/I$. We will show that θ is a regular congruence relation on H/I.

We will show first that θ is an equivalence relation on H/I. Note that θ_1 is an equivalence relation on H. Let $I_x \in H/I$. Then $x \in H$ and $x\theta_1 x$ on H, that is, $I_x\theta I_x$, which means that reflexivity of θ on H/I holds. Now, let $I_x, I_y \in H/I$ such that $I_x\theta I_y$. Then $x, y \in H$ and $x\theta_1 y$ on H. Since θ_1 is a symmetric relation on H, $y\theta_1 x$ on H and so $I_y\theta I_x$ on H/I, that is, θ is a symmetric relation on H/I. Assume that $I_x\theta I_y$ and $I_y\theta I_z$, where $I_x, I_y, I_z \in H/I$. Then $x, y, z \in H$ and $x\theta_1 y$ and $y\theta_1 z$ on H. Since θ_1 is transitive relation on H, we have $x\theta_1 z$ which tells us that $I_x\theta I_z$. Hence, θ is a transitive relation on H/I. Therefore, θ is an equivalence relation.

Now, we will show that θ is a congruence relation on H/I. Note that θ_1 is a congruence relation on H. Let $I_a, I_x, I_y \in H/I$ for some $a, x, y \in H$ such that $I_x\theta I_y$ on H/I. Note that θ_1 is a regular congruence. By definition of θ on H/I, $x\theta_1 y$ on H and Definition 3.1(viii), $(a \circledast x)\bar{\theta_1}(a \circledast y)$, $(x \circledast a)\bar{\theta_1}(y \circledast a)$, $(a \odot x)\bar{\theta_1}(a \odot y)$, and $(x \odot a)\bar{\theta_1}(y \odot a)$. Thus, for each $u \in a \circledast x$, there exists $v \in a \circledast y$ such that $u\theta_1 v$ and for each $v \in a \circledast y$, there exists $u \in a \circledast x$ such that $u\theta_1 v$. This means that for each $I_u \in I_a \circledast I_x$, there exists $I_v \in I_a \circledast I_y$ such that $I_u\theta I_v$ and for every $I_v \in I_a \circledast I_y$, there exists $I_u \in I_a \circledast I_x$ such that $I_u\theta I_v$. Hence, $(I_a \circledast I_x)\bar{\theta}(I_a \circledast I_y)$. In a similar manner, we can also show that $(I_x \circledast I_a)\bar{\theta}(I_y \circledast I_a)$. On the other hand, for each $r \in a \odot x$, there exists $s \in a \odot y$ such that $r\theta_1 s$ and for all $s \in a \odot y$, there exists $r \in a \odot x$ such that $r\theta_1 s$. This means that for each $I_r \in I_a \odot I_x$, there exists $I_s \in I_a \odot I_y$ such that $I_r\theta I_s$ and for every $I_s \in I_a \circledast I_y$, there exists $I_r \in I_a \circledast I_x$ such that $I_r\theta I_s$. Hence, $(I_a \odot I_x)\bar{\theta}(I_a \odot I_y)$. In a similar manner, we can also show that $(I_x \odot I_a)\bar{\theta}(I_y \circledast I_a)$. Therefore, θ is a congruence relation on H/I.

Finally, we will show that θ is a regular congruence relation on H/I. Suppose that $(I_x \circledast I_y)\theta\{I\}$, $(I_y \circledast I_x)\theta\{I\}$, $(I_x \odot I_y)\theta\{I\}$, and $(I_y \odot I_x)\theta\{I\}$. Then there exist $u \in x \circledast y$, $v \in y \circledast x$, $r \in x \odot y$, and $s \in y \odot x$ such that $I_u\theta\{I\}$, $I_v\theta\{I\}$, $I_r\theta\{I\}$, and $I_s\theta\{I\}$, that is, $u\theta_1 0$, $v\theta_1 0$, $r\theta_1 0$, and $s\theta_1 0$. So we have, $(x \circledast y)\theta_1\{0\}$, $(y \circledast x)\theta_1\{0\}$,$(x \odot y)\theta_1\{0\}$, and $(y \odot x)\theta_1\{0\}$. Since θ_1 is a regular congruence relation on H, $x\theta_1 y$, that is, $I_x\theta I_y$. Therefore, θ is a regular congruence relation on H/I. Now,

$$\begin{aligned}[I]_\theta = \{I_x \in H/I \,|\, I_x\theta I\} &= \{I_x \in H/I \,|\, x\theta_1 0\} \\ &= \{I_x \in H/I \,|\, x \in [0]_{\theta_1} = J\} \\ &= \{I_x \in H/I \,|\, x \in J\} \\ &= J/I\end{aligned}$$

Theorem 4.8. (Third Isomorphism Theorem) Let $f : H \longrightarrow H'$ be a homomorphism of pseudo hyper GR-algebras H and H' such that $f(x) \ll f(y)$ and $f(y) \ll f(x)$ imply that $f(x) = f(y)$. Suppose further that θ_1 and θ_2 are regular congruence relations on H

with $J = [0]_{\theta_1}$ and $I = [0]_{\theta_2}$. Then $(H/I)/(J/I) \cong H/J$, where $J/I = \{I_x \in H/I \,|\, x \in J\}$.

Proof. Let $f : H \longrightarrow H'$ be a homomorphism of pseudo hyper GR-algebras H and H' such that $f(x) \ll f(y)$ and $f(y) \ll f(x)$ imply that $f(x) = f(y)$. Suppose further that θ_1 and θ_2 are regular congruence relations on H with $J = [0]_{\theta_1}$ and $I = [0]_{\theta_2}$.

Let θ be a regular congruence relation on H/I defined in Lemma 4.7. We define the map $\varphi : H/I \to H/J$ by $\varphi(I_x) = J_x$.

Suppose that $I_x = I_y$. Since θ is a reflexive relation on H/I, $I_x \theta I_y$ on H/I which means that $x\theta_1 y$. Note that $J = [0]_{\theta_1}$. Now, if $x \in J = [0]_{\theta_2}$, then $x\theta_1 0$ and $0\theta_1 x$. Since, $x\theta_1 y$, by transitivity of θ_1 on H, $0\theta_1 y$ and so $y\theta_1 0$ which will imply that $y \in [0]_{\theta_1} = J$. Thus, $\varphi(I_x) = J_x = J = J_y = \varphi(I_y)$ and φ is a well-defined map.

We will now show that φ is a homomorphism. Note that $\varphi)(I) = J$. Let $I_x, I_y \in H/I$. Let $J_z \in \varphi(I_x \circledast I_y)$. Then, there exists $I_u \in I_x \circledast I_y$ such that $\varphi(I_u) = J_z$, that is, $J_u = \varphi(I_u) = J_z$. Since $I_u \in I_x \circledast I_y$, $u \in x \circledast y$. Hence, $J_u \in J_x \circledast J_y$. Thus, $J_z = J_u \in J_x \circledast J_y = \varphi(I_x) \circledast \varphi(I_y)$ and so, $\varphi(I_x \circledast I_y) \subseteq \varphi(I_x) \circledast \varphi(I_y)$. Next, let $J_{z'} \in \varphi(I_x) \circledast \varphi(I_y) = J_x \circledast J_y$. Then, there exists an element $u' \in x \circledast y$ such that $J_{z'} = J_{u'} = \varphi(I_{u'})$. Since $u' \in x \circledast y$, $I_{u'} \in I_x \circledast I_y$ and $J_{z'} = \varphi(I_{u'}) \in \varphi(I_x \circledast I_y)$. So, $\varphi(I_x) \circledast \varphi(I_y) \subseteq \varphi(I_x \circledast I_y)$. Therefore, $\varphi(I_x \circledast I_y) = \varphi(I_x) \circledast \varphi(I_y)$.

Now, let $J_z \in \varphi(I_x \odot I_y)$. Then, there exists an element $I_u \in I_x \odot I_y$ such that $\varphi(I_u) = J_z$, that is, $J_u = \varphi(I_u) = J_z$. Since $I_u \in I_x \odot I_y$, $u \in x \odot y$. Hence, $J_u \in J_x \odot J_y$. Thus, $J_z = J_u \in J_x \odot J_y = \varphi(I_x) \odot \varphi(I_y)$ and so, $\varphi(I_x \odot I_y) \subseteq \varphi(I_x) \odot \varphi(I_y)$. Next, let $J_{z'} \in \varphi(I_x) \odot \varphi(I_y) = J_x \odot J_y$. Then, there exists an element $u' \in x \odot y$ such that $J_{z'} = J_{u'} = \varphi(I_{u'})$. Since $u' \in x \odot y$, $I_{u'} \in I_x \odot I_y$ and so, $J_{z'} = \varphi(I_{u'}) \in \varphi(I_x \odot I_y)$. So, $\varphi(I_x) \odot \varphi(I_y) \subseteq (I_x \odot I_y)$. Hence, $\varphi(I_x \odot I_y) = \varphi(I_x) \odot \varphi(I_y)$ and φ is a homomorphism.

Now,

$$\begin{aligned} ker\,\varphi &= \{I_x \in H/I \,|\, \varphi(I_x) = [0]_{\theta_1}\} \\ &= \{I_x \in H/I \,|\, J_x = [0]_{\theta_1} = J\} \\ &= \{I_x \in H/I \,|\, x \in J\} \\ &= J/I. \end{aligned}$$

By Lemma 4.7, $ker\,\varphi = J/I = [I]_\theta$. Lastly, we will show that φ is onto. Let $z \in H$ and $I_z \in H/I$. Thus, there exists an element $I_z \in H/I$ such that $\varphi(I_z) = J_z$ and so φ is onto. Therefore, by the First Isomorphism Theorem, $(H/I)/(J/I) \cong Im\,\varphi = H/J$. □

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