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Weak $\alpha(\Lambda, sp)$-continuity for multifunctions

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Abstract. This paper deals with the concepts of upper and lower weakly $\alpha(\Lambda, sp)$-continuous multifunctions. Moreover, several characterizations of upper and lower weakly $\alpha(\Lambda, sp)$-continuous multifunctions are established.

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Key Words and Phrases: $\alpha(\Lambda, sp)$-open set, upper weakly $\alpha(\Lambda, sp)$-continuous multifunction, lower weakly $\alpha(\Lambda, sp)$-continuous multifunction

1. Introduction

Weaker and stronger forms of open sets play an important role in the generalization of different forms of continuity. Using different forms of open sets, several authors have introduced and investigated various types of continuity for functions and multifunctions. In 1987, Noiri [8] introduced the class of functions called weakly α-continuous functions. Popa and Noiri [11], Rose [14] and Sen and Bhattacharyya [15] studied some properties of weakly α-continuous functions. Several different forms of continuous multifunctions have been introduced and studied over the years. Many authors have researched and studied several stronger and weaker forms of continuous functions and multifunctions. Neubrunn [7] introduced and investigated the notions of upper and lower α-continuous multifunctions. Popa and Noiri [12] studied some characterizations of upper and lower α-continuous multifunctions. Moreover, Popa and Noiri [13] defined a class of multifunctions called weakly α-continuous multifunctions. Cao and Dontchev [4] and Popa and Noiri [13] investigated several characterizations of weakly α-continuous multifunctions. In [3], the present authors introduced and studied the notions of upper and lower (τ_1, τ_2)-precontinuous multifunctions. Viriyapong and Boonpok [17] introduced and investigated the concepts of upper and lower weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Noiri and Hatir [9] introduced the notions of Λ_{sp}-closed and spg-closed sets and investigated properties of these sets. In [2] by considering the notion of Λ_{sp}-sets, introduced and investigated

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(Λ, sp)-closed sets, (Λ, sp)-open sets and the (Λ, sp)-closure operator. The purpose of the present paper is to introduce the concepts of upper and lower weakly $\alpha(\Lambda, sp)$-continuous multifunctions. Moreover, several characterizations of upper and lower weakly $\alpha(\Lambda, sp)$-continuous multifunctions are discussed.

2. Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [5] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. Let A be a subset of a topological space (X, τ). A subset $\Lambda_{sp}(A)$ [9] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$.

Lemma 1. [9] *For subsets A, B and $A_\alpha (\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_{sp}(A)$.

(2) If $A \subseteq B$, *then* $\Lambda_{sp}(A) \subseteq \Lambda_{sp}(B)$.

(3) $\Lambda_{sp}(\Lambda_{sp}(A)) = \Lambda_{sp}(A)$.

(4) If $U \in \beta(X, \tau)$, *then* $\Lambda_{sp}(U) = U$

(5) $\Lambda_{sp}(\cap\{A_\alpha | \alpha \in \nabla\}) \subseteq \cap\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

(6) $\Lambda_{sp}(\cup\{A_\alpha | \alpha \in \nabla\}) = \cup\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

A subset A of a topological space (X, τ) is called a *Λ_{sp}-set* [9] if $A = \Lambda_{sp}(A)$.

Lemma 2. [9] *For subsets A and $A_\alpha (\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $\Lambda_{sp}(A)$ *is a* Λ_{sp}*-set.*

(2) If A is β-open, then A is a Λ_{sp}-set.

(3) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cap_{\alpha \in \nabla} A_\alpha$ is a Λ_{sp}-set.

(4) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cup_{\alpha \in \nabla} A_\alpha$ is a Λ_{sp}-set.

A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [2] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*. Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [2] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [2] of A and is denoted by $A^{(\Lambda, sp)}$.

Lemma 3. [2] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ *and* $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)} = \cap\{F | A \subseteq F$ *and* F *is* (Λ, sp)*-closed*$\}$.

(4) $A^{(\Lambda,sp)}$ *is* (Λ, sp)*-closed.*

(5) A *is* (Λ, sp)*-closed if and only if* $A = A^{(\Lambda,sp)}$.

The union of all (Λ, sp)-open sets contained in A is called the (Λ, sp)*-interior* [2] of A and is denoted by $A_{(\Lambda,sp)}$.

Lemma 4. [2] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-interior, the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ *and* $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ *is* (Λ, sp)*-open.*

(4) A *is* (Λ, sp)*-open if and only if* $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

A subset A of a topological space (X, τ) is called $\alpha(\Lambda, sp)$*-open* (resp. $s(\Lambda, sp)$*-open*) if $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ (resp. $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$) [2]. The complement of an $\alpha(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open) set is called $\alpha(\Lambda, sp)$*-closed* (resp. $s(\Lambda, sp)$*-closed*). The family of all $\alpha(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $\alpha\Lambda_{sp}O(X, \tau)$ (resp. $s\Lambda_{sp}O(X, \tau)$). Let A be a subset of a topological space (X, τ). The intersection of all $\alpha(\Lambda, sp)$-closed (resp. $s(\Lambda, sp)$-closed) sets of X containing A is called the $\alpha(\Lambda, sp)$*-closure* (resp. $s(\Lambda, sp)$*-closure*) of A and is denoted by $A^{\alpha(\Lambda,sp)}$ (resp. $A^{s(\Lambda,sp)}$). The union of all $\alpha(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open) sets of X contained in A is called the $\alpha(\Lambda, sp)$*-interior* (resp. $s(\Lambda, sp)$*-interior*) of A and is denoted by $A_{\alpha(\Lambda,sp)}$ (resp. $A_{s(\Lambda,sp)}$).

Lemma 5. *Let A be a subset of a topological space (X, τ). Then, $x \in A^{s(\Lambda,sp)}$ if and only if $U \cap A \neq \emptyset$ for every $U \in s\Lambda_{sp}O(X, \tau)$ containing x.*

Lemma 6. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A *is* (Λ, sp)*-open in* X, *then* $A^{s(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$.

(2) A is $\alpha(\Lambda, sp)$-open in X if and only if $U \subseteq A \subseteq U^{s(\Lambda,sp)}$ for some (Λ, sp)-open set U of X.

(3) $A^{\alpha(\Lambda,sp)} = A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

Proof. (1) Let $x \in [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and G be any $s(\Lambda, sp)$-open set of X containing x. Then, there exists a (Λ, sp)-open set U of X such that $U \subseteq G \subseteq U^{(\Lambda,sp)}$. Since $x \in G \subseteq U^{(\Lambda,sp)}$ and $x \in [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $\emptyset \neq U \cap [A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq U \cap A^{(\Lambda,sp)} \subseteq [U \cap A]^{(\Lambda,sp)}$. Thus, $A \cap U \neq \emptyset$ and hence $A \cap G \neq \emptyset$. This shows that $x \in A^{s(\Lambda,sp)}$. On the other hand, assume that $x \notin [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Then, $x \in [[X - A]_{(\Lambda,sp)}]^{(\Lambda,sp)} \in s\Lambda_{sp}O(X, \tau)$. Since A is (Λ, sp)-open, we have $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $A \cap [[X - A]_{(\Lambda,sp)}]^{(\Lambda,sp)} = \emptyset$. This shows that $x \notin A^{s(\Lambda,sp)}$. Therefore, $A^{s(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$.

(2) Suppose that A is $\alpha(\Lambda, sp)$-open. Then, $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Let $U = A_{(\Lambda,sp)}$. Thus, $U \subseteq A \subseteq [U^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and by (1), $U \subseteq A \subseteq U^{s(\Lambda,sp)}$.

Conversely, assume that there exists a (Λ, sp)-open set U such that $U \subseteq A \subseteq U^{s(\Lambda,sp)}$. By (1), we have $U \subseteq A \subseteq [U^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, $U \subseteq A_{(\Lambda,sp)}$ and hence

$$[U^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

Since $A \subseteq [U^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$. This shows that A is $\alpha(\Lambda, sp)$-open.

(3) We observe that

$$\begin{aligned}
[[[A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} &\subseteq [[[A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\
&\subseteq [[A^{(\Lambda,sp)} \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\
&= [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\
&\subseteq A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}.
\end{aligned}$$

Thus, $A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is $\alpha(\Lambda, sp)$-closed and hence $A_{\alpha(\Lambda,sp)} \subseteq A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. On the other hand, since $A_{\alpha(\Lambda,sp)}$ is $\alpha(\Lambda, sp)$-closed, we have

$$[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [[[A_{\alpha(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A_{\alpha(\Lambda,sp)}.$$

Therefore, $A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A_{\alpha(\Lambda,sp)}$. Thus, $A^{\alpha(\Lambda,sp)} = A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then, F is said to be a *surjection* if $F(X) = Y$, or equivalently, if for each $y \in Y$, there exists an $x \in X$ such that $y \in F(x)$. Moreover, $F : X \to Y$ is called *upper semi-continuous* (resp. *lower semi-continuous*) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [10].

3. Characterizations of upper and lower weakly $\alpha(\Lambda, sp)$-continuous multifunctions

We begin this section by introducing the notions of upper and lower weakly $\alpha(\Lambda, sp)$-continuous multifunctions.

Definition 1. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

(1) upper weakly $\alpha(\Lambda, sp)$-continuous at $x \in X$ if, for each $s(\Lambda, sp)$-open set U of X containing x and each (Λ, sp)-open set V of Y containing $F(x)$, there exists a nonempty (Λ, sp)-open set G of X such that $G \subseteq U$ and $F(G) \subseteq V^{s(\Lambda,sp)}$;

(2) lower weakly $\alpha(\Lambda, sp)$-continuous at $x \in X$ if, for each $s(\Lambda, sp)$-open set U of X containing x and each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a nonempty (Λ, sp)-open set G of X such that $G \subseteq U$ and $F(z) \cap V^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in G$;

(3) upper (lower) weakly $\alpha(\Lambda, sp)$-continuous if F has this property at each point of X.

Theorem 1. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper weakly $\alpha(\Lambda, sp)$-continuous at a point $x \in X$;

(2) for any (Λ, sp)-open set V of Y containing $F(x)$, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V^{(\Lambda,sp)}$;

(3) $x \in [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y containing $F(x)$;

(4) $x \in [[[F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y containing $F(x)$.

Proof. (1) ⇒ (2): Let V be any (Λ, sp)-open set of Y containing $F(x)$. By $s\Lambda_{sp}O(X, x)$, we denote the family of all $s(\Lambda, sp)$-open set of X containing x. For each $s(\Lambda, sp)$-open set U of X containing x, there exists a nonempty (Λ, sp)-open set G_U of X such that $G_U \subseteq U$ and $F(G_U) \subseteq V^{(\Lambda,sp)}$. Let $W = \cup\{G_U \mid U \in s\Lambda_{sp}O(X, x)\}$. Put $S = W \cup \{x\}$, then W is (Λ, sp)-open in X, $x \in W^{s(\Lambda,sp)}$ and $F(W) \subseteq V^{(\Lambda,sp)}$. Therefore, we have S is an $\alpha(\Lambda, sp)$-open set of X containing x by Lemma 6 and $F(S) \subseteq V^{(\Lambda,sp)}$.

(2) ⇒ (3): Let V be any (Λ, sp)-open set of Y containing $F(x)$. Then, there exists an $\alpha(\Lambda, sp)$-open set S of X containing x such that $F(S) \subseteq V^{(\Lambda,sp)}$. Thus,

$$x \in S \subseteq F^+(V^{(\Lambda,sp)})$$

and hence $x \in [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$.

(3) ⇒ (4): Let V be any (Λ, sp)-open set of Y containing $F(x)$. Now put

$$U = [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}.$$

Then, $U \in \alpha\Lambda_{sp}O(X,\tau)$ and $x \in U \subseteq F^+(V^{(\Lambda,sp)})$. This shows that

$$x \in [[[F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

(4) $\Rightarrow$ (1): Let U be any $s(\Lambda,sp)$-open set of X containing x and let V be any (Λ,sp)-open set of Y containing $F(x)$. Then, we have

$$x \in [[[F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [[F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{s(\Lambda,sp)}.$$

It follows from Lemma 5 that $\emptyset \neq U \cap [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)} \in s\Lambda_{sp}O(X,\tau)$. Put

$$G = [U \cap [F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

Then, G is a nonempty (Λ,sp)-open set of Y, $G \subseteq U$ and $F(G) \subseteq V^{(\Lambda,sp)}$.

Theorem 2. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) F is lower weakly $\alpha(\Lambda,sp)$-continuous at a point $x \in X$;

(2) for any (Λ,sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an $\alpha(\Lambda,sp)$-open set U of X containing x such that $F(z) \subseteq V^{(\Lambda,sp)} \neq \emptyset$ for every $z \in U$;

(3) $x \in [F^-(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ for every (Λ,sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$;

(4) $x \in [[[F^-(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ for every (Λ,sp)-open set V of Y such that

$$F(x) \cap V \neq \emptyset.$$

Proof. The proof is similar to that of Theorem 1.

Definition 2. [2] *Let A be a subset of a topological space (X,τ). The $\theta(\Lambda,sp)$-closure of A, $A^{\theta(\Lambda,sp)}$, is defined as follows:*

$$A^{\theta(\Lambda,sp)} = \{x \in X \mid A \cap\ U^{(\Lambda,sp)} \neq \emptyset \text{ for each } U \in \Lambda_{sp}O(X,\tau) \text{ containing } x\}.$$

Lemma 7. [2] *Let A be a subset of a topological space (X,τ). Then, $x \in A^{(\Lambda,sp)}$ if and only if $U \cap A \neq \emptyset$ for every $U \in \Lambda_{sp}O(X,\tau)$ containing x.*

Lemma 8. [2] *For a subset A of a topological space (X,τ), the following properties hold:*

(1) If A is (Λ,sp)-open in (X,τ), then $A^{(\Lambda,sp)} = A^{\theta(\Lambda,sp)}$.

(2) $A^{\theta(\Lambda,sp)}$ is (Λ,sp)-closed for every subset A of X.

Theorem 3. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) F *is upper weakly* $\alpha(\Lambda, sp)$*-continuous;*

(2) for each $x \in X$ *and each* (Λ, sp)*-open set* V *of* Y *containing* $F(x)$*, there exists an* $\alpha(\Lambda, sp)$*-open set* U *of* X *containing* x *such that* $F(U) \subseteq V^{(\Lambda,sp)}$*;*

(3) $F^+(V) \subseteq [[[F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(4) $[[[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq F^-(K)$ *for every* (Λ, sp)*-closed set* K *of* Y*;*

(5) $[F^-(K_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(K)$ *for every* (Λ, sp)*-closed set* K *of* Y*;*

(6) $[F^-([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$ *for every subset* B *of* Y*;*

(7) $F^+(B_{(\Lambda,sp)}) \subseteq [F^+([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ *for every subset* B *of* Y*;*

(8) $F^+(V) \subseteq [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(9) $[F^-(K_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(K)$ *for every* $r(\Lambda, sp)$*-closed set* K *of* Y*;*

(10) $[F^-(V)]^{\alpha(\Lambda,sp)} \subseteq F^-(V^{(\Lambda,sp)})$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(11) $[F^-(B^{\theta(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(B^{\theta(\Lambda,sp)})$ *for every subset* B *of* Y*.*

Proof. (1) $\Rightarrow$ (2): The proof follows immediately from Theorem 1.

(2) $\Rightarrow$ (3): Let V be any (Λ, sp)-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V^{(\Lambda,sp)}$. Therefore, $x \in U \subseteq F^+(V^{(\Lambda,sp)})$. Since U is an $\alpha(\Lambda, sp)$-open set containing x, we have $x \in U \subseteq [[[F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$.

(3) $\Rightarrow$ (4): Let K be any (Λ, sp)-closed set of Y. Then, $Y - K$ is (Λ, sp)-open in Y and by (3), $F^+(Y - K) \subseteq [[[F^+([Y - K]^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$. By the straightforward calculations, we obtain $[[[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq F^-(K)$.

(4) $\Rightarrow$ (5): Let K be any (Λ, sp)-closed set of Y. Then, we have

$$[[[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq F^-(K)$$

and hence $[F^-(K_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(K)$ by Lemma 6.

(5) $\Rightarrow$ (6): Let B be any subset of Y. Then, $B^{(\Lambda,sp)}$ is (Λ, sp)-closed in Y. Thus, by (5), we have $[F^-([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$.

(6) $\Rightarrow$ (7): Let B be any subset of Y. Then,

$$\begin{aligned} X - F^+(B_{(\Lambda,sp)}) &= F^-([Y - B]^{(\Lambda,sp)}) \\ &\supseteq [F^-([[Y - B]^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \\ &= [F^-([Y - [B_{(\Lambda,sp)}]^{(\Lambda,sp)}])]^{\alpha(\Lambda,sp)} \\ &= [X - F^+([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \\ &= X - [F^+([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}. \end{aligned}$$

Thus, $F^+(B_{(\Lambda,sp)}) \subseteq [F^+([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$.

(7) ⇒ (8): The proof is obvious.

(8) ⇒ (1): Let x be any point of X and let V be any (Λ, sp)-open set of Y containing $F(x)$, by Lemma 6, $x \in F^+(V) \subseteq [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)} \subseteq [[[F^+(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence F is upper weakly $\alpha(\Lambda, sp)$-continuous at x by Theorem 1.

(5) ⇒ (9): The proof is obvious.

(9) ⇒ (10): Let V be any (Λ, sp)-open set of Y. Then, $V^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed in Y and hence $[F^-(V)]^{\alpha(\Lambda,sp)} \subseteq [F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-(V^{(\Lambda,sp)})$.

(10) ⇒ (8): Let V be any (Λ, sp)-open set of Y. Thus,

$$\begin{aligned} X - [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)} &= [X - F^+(V^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \\ &= [F^-(Y - V^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \\ &\subseteq F^-([Y - V^{(\Lambda,sp)}]^{(\Lambda,sp)}) \\ &= X - F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)}) \end{aligned}$$

and hence $F^+(V) \subseteq F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)}) \subseteq [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$.

(10) ⇒ (11): Let B be any subset of Y. Put $V = [B^{\theta(\Lambda,sp)}]_{(\Lambda,sp)}$. By Lemma 8, we have $B^{\theta(\Lambda,sp)}$ is (Λ, sp)-closed in Y and by (10), $[F^-([B^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^-[B^{\theta(\Lambda,sp)}]$.

(11) ⇒ (9): Let K be any $r(\Lambda, sp)$-closed set of Y. By (11) and Lemma 8,

$$\begin{aligned} [F^-(K_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} &= [F^-([K^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \\ &= [F^-([[K_{(\Lambda,sp)}]^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \\ &\subseteq F^-([K_{(\Lambda,sp)}]^{\theta(\Lambda,sp)}) \\ &= F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)}) \\ &= F^-(K). \end{aligned}$$

Theorem 4. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower weakly $\alpha(\Lambda, sp)$-continuous;

(2) for each $x \in X$ and each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $U \subseteq F^-(V^{(\Lambda,sp)})$;

(3) $F^-(V) \subseteq [[[F^-(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(4) $[[[F^+(K_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq F^+(K)$ for every (Λ, sp)-closed set K of Y;

(5) $[F^+(K_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^+(K)$ for every (Λ, sp)-closed set K of Y;

(6) $[F^+([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^+(B^{(\Lambda,sp)})$ for every subset B of Y;

(7) $F^-(B_{(\Lambda,sp)}) \subseteq [F^-([B_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ *for every subset* B *of* Y*;*

(8) $F^-(V) \subseteq [F^-(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(9) $[F^+(K_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^+(K)$ *for every* $r(\Lambda, sp)$*-closed set* K *of* Y*;*

(10) $[F^+(V)]^{\alpha(\Lambda,sp)} \subseteq F^+(V^{(\Lambda,sp)})$ *for every* (Λ, sp)*-open set* V *of* Y*;*

(11) $[F^+([B^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq F^+(B^{\theta(\Lambda,sp)})$ *for every subset* B *of* Y*.*

Proof. The proof is similar to that of Theorem 3.

Lemma 9. *If* $F : (X, \tau) \to (Y, \sigma)$ *is lower weakly* $\alpha(\Lambda, sp)$*-continuous, then for each* $x \in X$ *and each subset* B *of* Y *with* $F(x) \cap B_{\theta(\Lambda,sp)} \neq \emptyset$*, there exists an* $\alpha(\Lambda, sp)$*-open set* U *of* X *containing* x *such that* $U \subseteq F^-(B)$*.*

Proof. Since $F(x) \cap B_{\theta(\Lambda,sp)} \neq \emptyset$, there exists a nonempty (Λ, sp)-open set U of Y such that $V \subseteq V^{(\Lambda,sp)} \subseteq B$ and $F(x) \cap V \neq \emptyset$. Since F is lower weakly $\alpha(\Lambda, sp)$-continuous, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $F(z) \cap V^{(\Lambda,sp)} \neq \emptyset$ for every $z \in U$. Thus, $U \subseteq F^-(B)$.

Theorem 5. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is lower weakly* $\alpha(\Lambda, sp)$*-continuous;*

(2) $[F^+(B)]^{\alpha(\Lambda,sp)} \subseteq F^+(B^{\theta(\Lambda,sp)})$ *for every subset* B *of* Y*;*

(3) $F(A^{\alpha(\Lambda,sp)}) \subseteq [F(A)]^{\theta(\Lambda,sp)}$ *for every subset* A *of* X*.*

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that

$$x \in F^-(Y - B^{\theta(\Lambda,sp)}) = F^-([Y - B]_{\theta(\Lambda,sp)}).$$

By Lemma 9, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that

$$U \subseteq F^-(Y - B) = X - F^+(B).$$

This shows that $U \cap F^+(B) = \emptyset$. Therefore, $x \in X - [F^+(B)]^{\alpha(\Lambda,sp)}$.

(2) ⇒ (1): Let V be any (Λ, sp)-open set of Y. Since $V^{(\Lambda,sp)} = V^{\theta(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y, we have $[F^+(V)]^{\alpha(\Lambda,sp)} \subseteq F^+(V^{(\Lambda,sp)})$ and by Theorem 4, F is lower weakly $\alpha(\Lambda, sp)$-continuous.

(2) ⇒ (3): Let A be any subset of X. By (2), we have

$$A^{\alpha(\Lambda,sp)} \subseteq [F^+(F(A))]^{\alpha(\Lambda,sp)} \subseteq F^+([F(A)]^{\theta(\Lambda,sp)})$$

and hence $F(A^{\alpha(\Lambda,sp)}) \subseteq [F(A)]^{\theta(\Lambda,sp)}$.

(3) ⇒ (2): Let B be any subset of Y. By (3),

$$F([F^+(B)]^{\alpha(\Lambda,sp)}) \subseteq [F(F^+(B))]^{\theta(\Lambda,sp)} \subseteq B^{\theta(\Lambda,sp)}.$$

Thus, $[F^+(B)]^{\alpha(\Lambda,sp)} \subseteq F^+(B^{\theta(\Lambda,sp)})$.

Definition 3. *A function $f : (X, \tau \to (Y, \sigma))$ is said to be weakly $\alpha(\Lambda, sp)$-continuous if, for each $x \in X$ and each (Λ, sp)-open set V containing $f(x)$, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $f(U) \subseteq V^{(\Lambda,sp)}$.*

Theorem 6. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is weakly $\alpha(\Lambda, sp)$-continuous;

(2) $f^{-1}(V) \subseteq [f^{-1}(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(3) $[f^{-1}(K_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(K)$ for every $r(\Lambda, sp)$-closed set K of Y;

(4) $[f^{-1}(V)]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y;

(5) $[f^{-1}([B^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(B^{\theta(\Lambda,sp)})$ for every subset B of Y;

(6) $[[[f^{-1}(V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ for every (Λ, sp)-open set V of Y;

(7) $f^{-1}(V) \subseteq [[[f^{-1}(V^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(8) $f([[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \subseteq [f(A)]^{\theta(\Lambda,sp)}$ for every subset A of X;

(9) $[[[f^{-1}(B)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq f^{-1}(B^{\theta(\Lambda,sp)})$ for every subset B of Y.

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, by $F^{(\Lambda,sp)} : (X, \tau) \to (Y, \sigma)$ (resp. $F^{\alpha(\Lambda,sp)} : (X, \tau) \to (Y, \sigma)$ [6]) we denote a multifunction defined as follows: $F^{(\Lambda,sp)}(x) = [F(x)]^{(\Lambda,sp)}$ (resp. $F^{\alpha(\Lambda,sp)}(x) = [F(x)]^{\alpha(\Lambda,sp)}$) for each $x \in X$.

Definition 4. [6] *A subset A of a topological space (X, τ) is said to be:*

(1) (Λ, sp)-paracompact if every cover of A by (Λ, sp)-open sets of X is refined by a cover of A which consists of (Λ, sp)-open sets of X and is locally finite in X;

(2) (Λ, sp)-regular if for each $x \in A$ and each (Λ, sp)-open set U of X containing x, there exists a (Λ, sp)-open set V of X such that $x \in V \subseteq V^{(\Lambda,sp)} \subseteq U$.

Lemma 10. [6] *If A is a (Λ, sp)-regular (Λ, sp)-paracompact set of a topological space (X, τ) and U is a (Λ, sp)-open neighbourhood of A, then there exists a (Λ, sp)-open set V of X such that $A \subseteq V \subseteq V^{(\Lambda,sp)} \subseteq U$.*

Lemma 11. [6] *If $F : (X, \tau) \to (Y, \sigma)$ is (Λ, sp)-regular and (Λ, sp)-paracompact, then $[F^{\alpha(\Lambda,sp)}]^+(V) = F^+(V)$ for every (Λ, sp)-open set V of Y.*

Theorem 7. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is (Λ, sp)-paracompact and (Λ, sp)-regular for each $x \in X$. Then, the following properties are equivalent:*

(1) F is upper weakly $\alpha(\Lambda, sp)$-continuous;

(2) $F^{\alpha(\Lambda,sp)}$ *is upper weakly* $\alpha(\Lambda, sp)$*-continuous;*

(3) $F^{(\Lambda,sp)}$ *is upper weakly* $\alpha(\Lambda, sp)$*-continuous.*

Proof. Similarly to Lemma 11, we put $G = F^{\alpha(\Lambda,sp)}$ or $F^{(\Lambda,sp)}$. First, suppose that F is upper weakly $\alpha(\Lambda, sp)$-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y containing $G(x)$. By Lemma 11, $x \in G^+(V) = F^+(V)$ and there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $F(z) \subseteq V^{(\Lambda,sp)}$ for each $z \in U$. Therefore, we have $F^{\alpha(\Lambda,sp)}(z) \subseteq F^{(\Lambda,sp)}(z) \subseteq V^{(\Lambda,sp)}$; hence $G(z) \subseteq V^{(\Lambda,sp)}$ for each $z \in U$. This shows that G is weakly $\alpha(\Lambda, sp)$-continuous.

Conversely, suppose that G is upper weakly $\alpha(\Lambda, sp)$-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y containing $F(x)$. By Lemma 11, $x \in F^+(V) = G^+(V)$ and hence $G(x) \subseteq V$. There exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $G(U) \subseteq V^{(\Lambda,sp)}$. Thus, $F(U) \subseteq V^{(\Lambda,sp)}$ and hence F is upper weakly $\beta(\Lambda, sp)$-continuous.

Lemma 12. [6] *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, it follows that for each* $\alpha(\Lambda, sp)$*-open set* V *of* Y $[F^{\alpha(\Lambda,sp)}]^-(V) = F^-(V)$*, where* G *denotes* $F^{\alpha(\Lambda,sp)}$ *or* $F^{(\Lambda,sp)}$*.*

Theorem 8. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is lower weakly* $\alpha(\Lambda, sp)$*-continuous;*

(2) $F^{\alpha(\Lambda,sp)}$ *is lower weakly* $\alpha(\Lambda, sp)$*-continuous;*

(3) $F^{(\Lambda,sp)}$ *is lower weakly* $\alpha(\Lambda, sp)$*-continuous.*

Proof. By utilizing Lemma 12, this can be proved in a similar way as Theorem 7.

Definition 5. *A subset* A *of a topological space* (X, τ) *is called* (Λ, sp)*-clopen if* A *is both* (Λ, sp)*-closed and* (Λ, sp)*-open.*

Lemma 13. *If* A *is* $\alpha(\Lambda, sp)$*-open and* $\alpha(\Lambda, sp)$*-closed in a topological space* (X, τ)*, then* A *is* (Λ, sp)*-clopen.*

Proof. Let A be $\alpha(\Lambda, sp)$-open and $\alpha(\Lambda, sp)$-closed. Then, $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A$. Thus, $A^{(\Lambda,sp)} = [[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence $A^{(\Lambda,sp)} \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A$. This shows that A is (Λ, sp)-closed. Therefore, $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$ and hence A is (Λ, sp)-open. Consequently, we obtain A is (Λ, sp)-clopen.

Lemma 14. *If a multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is upper weakly* $\alpha(\Lambda, sp)$*-continuous and lower weakly* $\alpha(\Lambda, sp)$*-continuous, then* $F^+(V)$ *is* (Λ, sp)*-clopen in* X *for every* (Λ, sp)*-clopen set* V *of* Y*.*

Proof. Let V be any (Λ, sp)-clopen set of Y. It follows from Theorem 3 that

$$F^+(V) \subseteq [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)} = [F^+(V)]_{\alpha(\Lambda,sp)}.$$

This shows that $F^+(V)$ is $\alpha(\Lambda, sp)$-open in X. Furthermore, since V is (Λ, sp)-open, it follows from Theorem 4 that $[F^+(V)]^{\alpha(\Lambda,sp)} \subseteq F^+(V^{(\Lambda,sp)}) = F^+(V)$. Thus, $F^+(V)$ is $\alpha(\Lambda, sp)$-closed and by Lemma 13, $F^+(V)$ is (Λ, sp)-clopen in X.

Definition 6. [16] *A topological space (X, τ) is called Λ_{sp}-connected if X cannot be written as a disjoint union of two nonempty (Λ, sp)-open sets.*

Theorem 9. *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper weakly $\alpha(\Lambda, sp)$-continuous and lower weakly $\alpha(\Lambda, sp)$-continuous surjective multifunction. If (X, τ) is Λ_{sp}-connected and $F(x)$ is Λ_{sp}-connected for each $x \in X$, then (Y, σ) is Λ_{sp}-connected.*

Proof. Suppose that (Y, σ) is not Λ_{sp}-connected. There exist nonempty (Λ, sp)-open sets U and V of Y such that $U \cup V = Y$ and $U \cap V = \emptyset$. Since $F(x)$ is Λ_{sp}-connected for each $x \in X$, we have either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; $x \in F^+(U)$ and $y \in F^+(V)$. Thus,

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$;

(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$;

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

By Lemma 14, $F^+(U)$ and $F^+(V)$ are (Λ, sp)-clopen. This shows that (X, τ) is not Λ_{sp}-connected.

Definition 7. *Let A be a subset of a topological space (X, τ). The $\alpha(\Lambda, sp)$-frontier of A, $\alpha\Lambda_{sp}Fr(A)$, is defined as follows:*

$$\alpha\Lambda_{sp}Fr(A) = A^{\alpha(\Lambda,sp)} \cap [X - A]^{\alpha(\Lambda,sp)} = A^{\alpha(\Lambda,sp)} - A_{\alpha(\Lambda,sp)}.$$

Theorem 10. *The set of all points of X at which a multifunction $F : (X, \tau) \to (Y, \sigma)$ is not upper weakly $\alpha(\Lambda, sp)$-continuous is identical with the union of the $\alpha(\Lambda, sp)$-frontier of the upper inverse images of the (Λ, sp)-closures of (Λ, sp)-open sets of Y containing $F(x)$.*

Proof. Let x be a point of X at which F is not upper weakly $\beta(\Lambda, sp)$-continuous. Then, there exists a (Λ, sp)-open set V containing $F(x)$ such that $U \cap (X - F^+(V^{(\Lambda,sp)})) \neq \emptyset$ for every $\alpha(\Lambda, sp)$-open set U containing x. Then, we have $x \in [X - F^+(V^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)}$. Since $x \in F^+(V)$, $x \in [F^+(V^{(\Lambda,sp)})]^{\alpha(\Lambda,sp)}$ and hence $x \in \alpha\Lambda_{sp}Fr(F^+(V^{(\Lambda,sp)}))$. If F is upper weakly $\beta(\Lambda, sp)$-continuous at x, then there exists a $\alpha(\Lambda, sp)$-open set U containing x such that $F(U) \subseteq V^{(\Lambda,sp)}$; hence $U \subseteq F^+[V^{(\Lambda,sp)}]$. This shows that

$$x \in U \subseteq [F^+(V^{(\Lambda,sp)})]_{\alpha(\Lambda,sp)}.$$

This contradicts that $x \in \alpha\Lambda_{sp}Fr(F^+(V^{(\Lambda,sp)}))$. Thus, F is not upper weakly $\alpha(\Lambda, sp)$-continuous at x.

Theorem 11. *The set of all points of X at which a multifunction $F : (X, \tau) \to (Y, \sigma)$ is not lower weakly $\alpha(\Lambda, sp)$-continuous is identical with the union of the $\alpha(\Lambda, sp)$-frontier of the lower inverse images of the (Λ, sp)-closures of (Λ, sp)-open sets of Y meeting $F(x)$.*

Proof. The proof is similar to that of Theorem 10.

4. Conclusion

Topology is concerned with all questions directly or indirectly related to continuity. The concept of continuity is one of the most important tools in Mathematics and many different forms of generalizations of continuity have been introduced and studied. This paper deals with the concept of upper (resp. lower) weakly $\alpha(\Lambda, sp)$-continuous multifunctions. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called upper (resp. lower) weakly $\alpha(\Lambda, sp)$-continuous multifunctions at $x \in X$ if, for each $s(\Lambda, sp)$-open set U of X containing x and each (Λ, sp)-open set V of Y such that $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there exists a nonempty (Λ, sp)-open set G such that $G \subseteq U$ and $G \subseteq F^{+}(V^{s(\Lambda,sp)})$ (resp $G \subseteq F^{-}(V^{s(\Lambda,sp)})$). A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called upper (resp. lower) weakly $\alpha(\Lambda, sp)$-continuous multifunctions if F has this property at each point of X. Some characterizations and several properties concerning upper (resp. lower) weakly $\alpha(\Lambda, sp)$-continuous multifunctions are established. The ideas and results of this paper may motivate further research.

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