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Classification of four dimensional train algebras of degree 2 and exponent 4

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Abstract. This paper is devoted to the classification of train algebras of degree 2 and exponent 4. This classification is made in dimension at most four and according to the type of the algebra. We first show that in four dimension, the type of algebra can only be of $(2, 0, 1, 1)$, $(2, 1, 0, 1)$ or $(2, 1, 1, 0)$.

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1. Introduction

Several nonassociative algebras verifying polynomial identities are used in algebra modeling of population genetics. We can cite among others, Bernstein algebras (cf [2],[6],[8]), Jordan algebras (see [10],[7]), power-associative train algebras ([5]) train algebras of degree 2 and exponent n ([9],[4]). In [1] the authors defined an algebra satisfying a train identity of degree 2 and exponent 4 as an algebra A such that for any x in A, we have: $(x^4)^2 = \omega(x)^4 x^4$, where K is an infinite and algebrically closed commutative field of characteristic different from 2 and 3. If such an algebra does not verify a polynomial identity of degree less than or equal to 7, we say that it is a train algebras of degree 2 and exponent 4. This class of algebras models populations whose genetic potential becomes stable in the fourth generation. In particular, this class contains Bernstein algebras([3]). In this paper we are interested in the classification of these algebras in dimension 4; it is made according to the type of the algebra A; this one cannot be of lower dimension.

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2. Preliminaries

Let K be a commutative field and A a commutative K-algebra, not necessarily associative. For any element x of A we define the principal powers of x by: $x^1 = x$, $x^{k+1} = xx^k$ for any integer $k \geq 1$.

We will say that the algebra A is a baric algebra if there exists a non-zero homomorphism of algebras $\omega : A \to K$ called the weight function of the algebra A. The weight of an element x of A is the scalar $\omega(x)$.

A baric K-algebra (A, ω) is a Bernstein algebra if $(x^2)^2 = \omega(x)^2 x^2$ for any x in A.

In the following K is a commutative algebraically closed field of characteristic distinct from 2 and 3, and A is a commutative nonassociative algebra. A baric algebra (A, ω) satisfies a train identity of degree 2 and exponent 4 if for any x in A, we have:

$$(x^4)^2 = \omega(x)^4 x^4. \tag{1}$$

An algebra A is called a train algebra of degree 2 and exponent 4 if it satisfies a train identity of degree 2 and exponent 4 and does not verify a polynomial identity of degree less than 8.

For an element x of weight 1 in A (i.e. $\omega(x) = 1$), the identity (1) gives $(x^4)^2 = x^4$ and thus x^4 is an non zero idempotent of A.

In [1], the authors have established the following two theorems:

Theorem 1. *Let A be an algebra satisfying a train identity of degree 2 and exponent 4. Then A has a Peirce decomposition $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$, where $e^2 = e \neq 0$ and $A_\alpha = \{y \in ker(\omega) \mid ey = \alpha y\}$ for $\alpha \in \{0; 1/2; \lambda = -\dfrac{1 + i\sqrt{7}}{4}; \overline{\lambda} = \dfrac{-1 + i\sqrt{7}}{4}\}$.*

Theorem 2. *Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be a Peirce decomposition relative to a non-zero idempotent e of an algebra satisfying a train identity of degree 2 and exponent 4. Then:*

(*i*) $A_0^2 \subset A_0 \oplus A_{1/2}$;

(*ii*) $A_{1/2}^2 \subset A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$;

(*iii*) $A_\lambda^2 \subset A_{1/2}$;

(*i***v**) $A_{\overline{\lambda}}^2 \subset A_{1/2}$;

(**v**) $A_{1/2}A_0 \subset A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$;

(**v***i*) $A_0 A_\lambda \subset A_{1/2}$;

(**v***ii*) $A_0 A_{\overline{\lambda}} \subset A_{1/2}$;

(**v***iii*) $A_{1/2}A_\lambda \subset A_{1/2} \oplus A_0 \oplus A_{\overline{\lambda}}$;

(*i***x**) $A_{1/2}A_{\overline{\lambda}} \subset A_{1/2} \oplus A_0 \oplus A_\lambda$;

(x) $A_\lambda A_{\overline{\lambda}} \subset A_{1/2}$.

Lemma 1. *Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be a Peirce decomposition relative to a non-zero idempotent e of an algebra satisfying a train identity of degree 2 and exponent 4, then for all $x_{1/2}$, x_0, x_λ and $x_{\overline{\lambda}}$ respectively in $A_{1/2}$, A_0, A_λ, $A_{\overline{\lambda}}$, the following assertions are verified:*

(i) $2x_{1/2}(e(ex_{1/2}^2)) + 2e(x_{1/2}(ex_{1/2}^2)) + x_{1/2}(ex_{1/2}^2) + 2e(ex_{1/2}^3) + ex_{1/2}^3 + x_{1/2}^3 = 0$;

(ii) $2x_{1/2}(x_{1/2}(ex_{1/2}^2)) + 2x_{1/2}(ex_{1/2}^3) + 2ex_{1/2}^4 + x_{1/2}^4 + (e(ex_{1/2}^2) + ex_{1/2}^2 + x_{1/2}^2)^2 = 0$;

(iii) $2e(x_0(ex_0^2)) + 2e(ex_0^3) - x_0(ex_0^2) - ex_0^3 = 0$;

(iv) $8ex_0^4 + (ex_0^2)^2 - 4x_0^4 = 0$;

(v) $4e(ex_\lambda^3) + 8\lambda ex_\lambda^3 - (4\lambda + 1)x_\lambda^3 = 0$;

(vi) $32ex_\lambda^4 + (1 - 32\lambda)(x_\lambda^2)^2 - 16x_\lambda^4 = 0$;

(vii) $4e(ex_{\overline{\lambda}}^3) + 8\overline{\lambda} ex_{\overline{\lambda}}^3 - (4\overline{\lambda} + 1)x_{\overline{\lambda}}^3 = 0$;

(viii) $32ex_{\overline{\lambda}}^4 + (1 - 32\overline{\lambda})(x_{\overline{\lambda}}^2)^2 - 16x_{\overline{\lambda}}^4 = 0$.

Proof. It suffices to set $x = e + \alpha x_{1/2} + \beta x_0 + \gamma x_\lambda + \mu x_{\overline{\lambda}}$ and identify the coefficients of $\alpha^i\beta^j\gamma^k\mu^\ell$ $(1 \leq i + j + k + \ell \leq 8)$, equality $x^4 = (x^4)^2$ allowing to obtain respectively (*i*), (*ii*), (*iii*), (*iv*), (*v*), (*vi*), (*vii*) and (*viii*).

Lemma 2. *Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be a Peirce decomposition relative to a non-zero idempotent e of an algebra satisfying a train identity of degree 2 and exponent 4. If $A_\lambda = A_{\overline{\lambda}} = 0$ then, for all $x_0 \in A_0$ and $x_{1/2} \in A_{1/2}$, we have the following identities:*

(i) $2e(x_0x_{1/2}^2) + 2x_{1/2}(x_0x_{1/2}) - x_0x_{1/2}^2 = 0$;

(ii) $x_{1/2}^3 = 0$;

(iii) $(x_{1/2}^2)^2 = 0$;

(iv) $(ex_0^2)^2 + 8ex_0^4 - 4x_0^4 = 0$.

(v) $x_{1/2}(ex_0^2) - x_{1/2}x_0^2 = 0$;

(vi) $2e(x_0(x_{1/2}(x_0x_{1/2}))) + 2e(x_{1/2}(x_{1/2}x_0^2)) + x_{1/2}^2(ex_0^2) - x_0(x_{1/2}(x_0x_{1/2})) - x_{1/2}(x_{1/2}x_0^2) + 4(x_0x_{1/2})^2 = 0$;

(vii) $4x_{1/2}(x_{1/2}(x_{1/2}x_0)) + 2e(x_{1/2}(x_0x_{1/2}^2)) + 2x_{1/2}(e(x_0x_{1/2}^2)) + x_{1/2}(x_0x_{1/2}^2) + 4x_{1/2}^2(x_0x_{1/2}) = 0$;

(viii) $(ex_0^2)(x_0x_{1/2})+4x_{1/2}(ex_0^3)+4x_{1/2}(x_0(ex_0^2))+4e(x_0(x_0^2x_{1/2}))+4e(x_{1/2}x_0^3)-2x_0(x_{1/2}x_0^2)-2x_{1/2}x_0^3=0$.

Proof. The proof is similar to that of the Lemma 1.

According to Propositions 3.1 and 3.2 of [1], the possible types of A in dimension 4 are: $(2,2,0,0)$, $(2,1,0,1)$, $(2,1,1,0)$ and $(2,0,1,1)$.

Theorem 3. *Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be a Peirce decomposition relative to a non-zero idempotent e of an algebra satisfying a train identity of degree 2 and exponent 4. If $A_\lambda = A_{\overline{\lambda}} = 0$, then A verifies a polynomial equation of degree less than eight and therefore, A is not train of degree 2 and exponent 4.*

Proof. The type of A being $(2,2,0,0)$, then $A_\lambda = A_{\overline{\lambda}} = 0$, $A_{1/2}^2 \subset A_0$, $A_0^2 \subset A_{1/2} \oplus A_0$ and $A_{1/2}A_0 \subset A_{1/2}$; we can then set $A_{1/2} =< e_0 >$, $A_0 =< e_1, e_2 >$, so $A =< e, e_0, e_1, e_2 >$ such that:

$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = ee_2 = 0$, $e_0^2 = \alpha_0e_1 + \alpha_1e_2$, $e_1e_2 = \beta_0e_0 + \beta_1e_1 + \beta_2e_2$, $e_1^2 = \gamma_0e_0 + \gamma_1e_1 + \gamma_2e_2$, $e_2^2 = \mu_0e_0 + \mu_1e_1 + \mu_2e_2$, $e_0e_1 = \mu e_0$, $e_0e_2 = \gamma e_0$; According to the Lemma (2) we have $x_{1/2}^3 = (x_{1/2}^2)^2 = 0$ which allows us to obtain the following relations:

$x_{1/2}^3 = 0 \Rightarrow e_0^3 = (\alpha_0\mu + \alpha_1\gamma)e_0 = 0$ so : $\alpha_0\mu + \alpha_1\gamma = 0$. Moreover, $(x_{1/2}^2)^2 = 0 \Rightarrow (\alpha_0^2\gamma_0+\alpha_1^2\mu_0+2\alpha_0\alpha_1\beta_0)e_0+(\alpha_0^2\gamma_1+\alpha_1^2\mu_1+2\alpha_0\alpha_1\beta_1)e_1+(\alpha_0^2\gamma_2+\alpha_1^2\mu_2+2\alpha_0\alpha_1\beta_2)e_2 = 0$ so : $\alpha_0^2\gamma_0+\alpha_1^2\mu_0+2\alpha_0\alpha_1\beta_0 = 0$; $\alpha_0^2\gamma_1+\alpha_1^2\mu_1+2\alpha_0\alpha_1\beta_1 = 0$; and $\alpha_0^2\gamma_2+\alpha_1^2\mu_2+2\alpha_0\alpha_1\beta_2 = 0$.

Thus we have the following cases, the products not mentioned in the multiplication table of A are zero.

<u>1^{st} **Case**</u>: $A_{1/2}^2 \neq 0$.

We can set $e_1 = e_0^2$ and therefore $\alpha_0 = 1$, $\alpha_1 = 0$. Thus, we have $\mu = \gamma_0 = \gamma_1 = \gamma_2 = 0$ and using the others identities of the Lemma 2; $\mu_0 = \beta_2 = \beta_1 = \gamma = \mu_2 = 0$. Therefore, the multiplication table of A becomes $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_0^2 = e_1$, $e_1e_2 = \beta_0e_0$, $e_2^2 = \mu_1e_1$. Let $x = e + ae_0 + be_1 + ce_2$ an element in A of weight 1. We have $(x^2)^3 - (x^2)^2 = 0$, so for any x in A, $(x^2)^3 - \omega(x)^2(x^2)^2 = 0$.

<u>2^{nd} **Case**</u>: $A_{1/2}^2 = 0$.

We have $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = ee_2 = 0$, $e_1e_2 = \beta_0e_0+\beta_1e_1+\beta_2e_2$, $e_1^2 = \gamma_0e_0+\gamma_1e_1+\gamma_2e_2$, $e_2^2 = \mu_0e_0 + \mu_1e_1 + \mu_2e_2$, $e_0e_1 = \mu e_0$, $e_0e_2 = \gamma e_0$.

Determine the identity verified by A according to its multiplication table. Using the identities of the Lemma 2 we have $\gamma_1\mu+\gamma_2\gamma = 0$, $\mu_1\mu+\mu_2\gamma = 0$, $\gamma_1(\gamma_1^2+\gamma_2\beta_1)+\beta_1\gamma_2(\gamma_1+\beta_2) = 0$, $\gamma_2(\gamma_1^2 + \gamma_2\beta_1) + \beta_2\gamma_2(\gamma_1 + \beta_2) = 0$, $\mu_1\beta_1(\beta_1 + \mu_2^2) + \mu_1(\mu_1\beta_2 + \mu_2^3) = 0$ and $\mu_1\beta_2(\beta_1 + \mu_2^2) + \mu_2(\mu_1\beta_2 + \mu_2^3) = 0$.

i) $\gamma = 0$ and $\mu \neq 0$.

We can set $\mu = 1$ and then $\gamma_1 = \mu_1 = \mu_2 = \gamma_2\beta_1 = \gamma_2\beta_2 = 0$. The multiplication table of A becomes $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0e_0 + \beta_1e_1 + \beta_2e_2$, $e_1^2 = \gamma_0e_0 + \gamma_2e_2$, $e_2^2 = \mu_0e_0$, $e_0e_1 = e_0$.

$i.1)$ Suppose $\gamma_2 \neq 0$. Then $\beta_1 = \beta_2 = 0$, so $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0$, $e_1^2 = \gamma_0 e_0 + e_2$, $e_2^2 = \mu_0 e_0$, $e_0e_1 = e_0$ and for any x in A, $(x^3)^2 = \omega(x)^3 x^3$.

$i.2)$ $\gamma_2 = 0$. Then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$, $e_1^2 = \gamma_0 e_0$, $e_2^2 = \mu_0 e_0$, $e_0e_1 = e_0$. Using the identities of the Lemma 2 we have $\beta_1 = \beta_2 = 0$. Therefore, $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0$, $e_1^2 = \gamma_0 e_0$, $e_2^2 = \mu_0 e_0$, $e_0e_1 = e_0$. For any x in A, $(x^2)^2 = \omega(x)^2 x^2$.

$ii)$ $\mu = 0$ and $\gamma \neq 0$.
Then $\gamma_1 = \gamma_2 = \mu_2 = \mu_1\beta_1 = \mu_1\beta_2 = 0$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$, $e_1^2 = \gamma_0 e_0$, $e_2^2 = \mu_0 e_0 + \mu_1 e_1$, $e_0e_1 = \mu e_0$, $e_0e_2 = \gamma e_0$.
$ii.1)$ $\mu_1 \neq 0$, then $\beta_1 = \beta_2 = 0$, so $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0$, $e_1^2 = \gamma_0 e_0$, $e_2^2 = \mu_0 e_0 + e_1$ (we can set $\mu_1 = 1$), $e_0e_1 = \mu e_0$, $e_0e_2 = \gamma e_0$ and for any x in A, $(x^2)^2 = \omega(x)^2 x^2$.
$ii.2)$ $\mu_1 = 0$, then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$, $e_1^2 = \gamma_0 e_0$, $e_2^2 = \mu_0 e_0$, $e_0e_1 = \mu e_0$, $e_0e_2 = \gamma e_0$. Using the identities of the Lemma 2 we have $\beta_1 = \beta_2 = 0$. Therefore, $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0$, $e_1^2 = \gamma_0 e_0$, $e_2^2 = \mu_0 e_0$, $e_0e_1 = \mu e_0$, $e_0e_2 = \gamma e_0$. For any x in A, $(x^2)^2 = \omega(x)^2 x^2$.

$iii)$ $\gamma = \mu = 0$
$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$, $e_1^2 = \gamma_0 e_0 + \gamma_1 e_1 + \gamma_2 e_2$, $e_2^2 = \mu_0 e_0 + \mu_1 e_1 + \mu_2 e_2$.
$iii.1)$ $\gamma_1 = 0$ and $\gamma_2 \neq 0$, then $\beta_1 = \beta_2 = \mu_2 = 0$, so $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0$, $e_1^2 = \gamma_0 e_0 + e_2$, $e_2^2 = \mu_0 e_0 + \mu_1 e_1$. The identity $(x^4)^2 = \omega(x)^4 x^4$ implies that $\mu_1 = 0$ and for any x in A, we have $(x^3)^2 = \omega(x)^3 x^3$.

$iii.2)$ $\gamma_2 = 0$ and $\gamma_1 \neq 0$, this case is impossible.

$iii.3)$ $\gamma_2 = \gamma_1 = 0$, then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$, $e_1^2 = \gamma_0 e_0$, $e_2^2 = \mu_0 e_0 + \mu_1 e_1 + \mu_2 e_2$.

$iii.3.1)$ $\mu_1 = 0$, then $\mu_2 = 0$ and $\beta_1 = \beta_2 = 0$. Therefore for any x in A, we have $(x^2)^2 = \omega(x)^2 x^2$.

$iii.3.2)$ $\mu_1 \neq 0$, then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$, $e_1^2 = \gamma_0 e_0$, $e_2^2 = \mu_0 e_0 + \mu_1 e_1 + \mu_2 e_2$. If $\mu_2 = 0$, then $\beta_1 = \beta_2 = 0$ and for any x in A, we have $(x^2)^2 = \omega(x)^2 x^2$. If $\mu_2 \neq 0$, then $\mu_1\beta_2 = \mu_2\beta_1$. For $x = e_0$, the identity $(x^4)^2 = 0$ implies that $\beta_1 = -\mu_2$ or $\beta_1 = -\frac{\mu_2}{2}$. We show that this is impossible.

$iii.4)$ $\gamma_2\gamma_1 \neq 0$, then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $e_1e_2 = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$, $e_1^2 = \gamma_0 e_0 + \gamma_1 e_1 + \gamma_2 e_2$, $e_2^2 = \mu_0 e_0 + \mu_1 e_1 + \mu_2 e_2$.
$iii.4.1)$ $\mu_2 = 0$. Then $\mu_1 = 0$ and $\beta_1\beta_2 \neq 0$. without loss of generality. It suffices to make some basis transformations to prove that we can set $\gamma_2 = 1$ and therefore $\beta_1 = \gamma_1\beta_2$. We show this is impossible.

$iii.4.2)$ $\mu_2 \neq 0$. Then $\mu_1 \neq 0$ and $\beta_1\beta_2 \neq 0$, so $\mu_1\beta_2 = \mu_2\beta_1$. We show that $\beta_1 = -\mu_2$

or $\beta_1 = -\mu_2^2$ this is impossible.

We conclude that the type of A cannot be $(2, 2, 0, 0)$.

The above result allows us to discard the type $(2, 2, 0, 0)$ in the classification.

3. Classification of algebras of type $(2, 1, 1, 0)$

In this section we will set $x = e + \alpha e_0 + \beta e_1 + \theta e_2$ an element of weight 1 in A. Also we will use essentially the assertion of the Lemma 1.

If Type $A = (2, 1, 1, 0)$ then $A_{\overline{\lambda}} = 0$ and $A_{1/2}^2 \subset A_0 \oplus A_\lambda$, $A_0^2 \subset A_{1/2} \oplus A_0$, $A_\lambda^2 \subset A_{1/2}$, $A_{1/2}A_0 \subset A_{1/2} \oplus A_\lambda$, $A_{1/2}A_\lambda \subset A_{1/2} \oplus A_0$, $A_0 A_\lambda \subset A_{1/2}$; we can thus set $A_{1/2} =< e_0 >$, $A_0 =< e_1 >$, $A_\lambda =< e_2 >$. The multiplication table of A is given by: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_0 e_1 + \alpha_1 e_2$, $e_1^2 = \beta_0 e_0 + \beta_1 e_1$, $e_2^2 = \gamma e_0$, $e_0 e_1 = \mu_0 e_0 + \mu_1 e_2$, $e_0 e_2 = \gamma_0 e_0 + \gamma_1 e_1$, $e_1 e_2 = \mu e_0$.

The assertion (i) of the Lemma 1 allows to obtain the equality:

$2(\alpha_0\mu_0 - \lambda^2\alpha_1\gamma_0)e_0 = 0$ so:

$$\alpha_0\mu_0 = \lambda^2\alpha_1\gamma_0 \tag{2}$$

As for (ii), it leads to:

$[2(1 + \lambda)(\alpha_1\gamma_1\mu_0 + \alpha_0\gamma_0\mu_1) + \alpha_0^2\beta_0 + \alpha_0\alpha_1\mu(1 + \lambda) + \frac{1}{8}(1 + 3\lambda)\alpha_1^2\gamma]e_0 + [\alpha_0^2\beta_1 + (1 + 2\lambda)\alpha_0\mu_1\gamma_1 - 2\alpha_0^2\mu_0]e_1 + [(2 + 4\lambda)\alpha_1^2\gamma + (1 + 4\lambda)\alpha_1\gamma_1\mu_1 - (1 + \lambda)^2\alpha_1^2\gamma_0]e_2 = 0$ then:

$$2(1 + \lambda)(\alpha_1\gamma_1\mu_0 + \alpha_0\gamma_0\mu_1) + \alpha_0^2\beta_0 + \alpha_0\alpha_1\mu(1 + \lambda) + \frac{1}{8}(1 + 3\lambda)\alpha_1^2\gamma = 0. \tag{3}$$

$$\alpha_0^2\beta_1 + (1 + 2\lambda)\alpha_0\mu_1\gamma_1 - 2\alpha_0^2\mu_0 = 0. \tag{4}$$

$$(2 + 4\lambda)\alpha_1^2\gamma + (1 + 4\lambda)\alpha_1\gamma_1\mu_1 - (1 + \lambda)^2\alpha_1^2\gamma_0 = 0. \tag{5}$$

The assertion (iii) of the Lemma 1 leads to the equality:

$-\frac{1}{2}(3 + \lambda)\beta_0\mu_1 e_2 = 0$ or:

$$\beta_0\mu_1 = 0. \tag{6}$$

The assertion (iv) of the Lemma 1 allows to obtain:

$(\frac{1}{16}\alpha_0\beta_0^2 - \beta_1^3)e_1 + \frac{1}{16}\alpha_1\beta_0^2 e_2 = 0$ which gives us:

$$\alpha_0\beta_0^2 = 16\beta_1^3. \tag{7}$$

$$\alpha_1\beta_0 = 0. \tag{8}$$

Using the assertion (v) of the lemma 1 we have:

$(2\lambda + \frac{1}{2})\gamma\gamma_1 e_1 = 0$ thus:

$$\gamma\gamma_1 = 0. \tag{9}$$

The assertion (vi) of the lemma 1 makes it possible to obtain the equality:

$(1 - 32\lambda)\alpha_0\gamma^2 e_1 + (1 - 32\lambda)\alpha_1\gamma e_2 = 0$, so:

$$\alpha_0\gamma = 0. \tag{10}$$

$$\alpha_1\gamma = 0. \tag{11}$$

By multiplying (7) by γ we obtain $\gamma\beta_1 = 0$ since $\alpha_0\gamma = 0$. And by reasoning in the same way but with μ_1 we obtain $\mu_1\beta_1 = 0$; and we get $\alpha_1\beta_1 = 0$ with α_1. Finally the product of (7) with μ_0 gives $\mu_0\beta_1 = 0$. These equalities clearly show that some scalars cannot be non-zero simultaneously; this is the case of γ and γ_1, β_0 and μ_1 for examples and among many others.

Suppose that $\beta_1 \neq 0$. Then $\alpha_1 = \mu_1 = 0$ and the equalities (3) and (7) implies that $0 = \alpha_0^2\beta_0 = 16\beta_1^3$,which is absurd. Thus $\beta_1 = 0$. Thus we have $\beta_0\alpha_0 = \beta_0\alpha_1 = \beta_0\mu_1 = 0$, $\gamma\gamma_1 = \gamma\alpha_0 = \gamma\alpha_1 = 0$, $\alpha_0\mu_0 = \lambda^2\alpha_1\gamma_0$, $(1+4\lambda)\alpha_1\gamma_1\mu_1 - (1+\lambda)^2\alpha_1^2\gamma_0 = 0$, $(1+2\lambda)\alpha_0\gamma_1\mu_1 - 2\alpha_0^2\mu_0 = 0$, $2\alpha_1\gamma_1\mu_0 + 2\alpha_0\gamma_0\mu_1 + \alpha_0\alpha_1\mu = 0$.

Let us then distinguish the following cases which satisfy these equalities:

3.1. <u>1^{st} Case</u> : $\alpha_0 \neq 0$ and $\alpha_1 = 0$ then $\beta_0 = \gamma = \mu_0 = \gamma_0\mu_1 = \gamma_1\mu_1 = 0$.

i) $\mu_1 \neq 0$, then $\gamma_0 = \gamma_1 = 0$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = e_1$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = e_2$, $e_0e_2 = 0$, $e_1e_2 = \mu e_0$. This case is impossible because A not satisfy identity $(x^4)^2 - \omega(x)^4x^4 = 0$.

ii) $\mu_1 = \mu = \gamma_1 = 0$, then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = e_1$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_0e_0$, $e_1e_2 = 0$. A verifies the identity $x(x^2)^2 - 4\lambda\omega(x)x^4 + (4\lambda^2 + 2\lambda - 1)\omega(x)^2x^3 + (2\lambda - 4\lambda^2)\omega(x)^3x^2 = 0$.

iii) $\mu_1 = \mu = 0$, $\gamma_1 \neq 0$ then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = e_1$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_0e_0 + e_1$, $e_1e_2 = 0$. A verifies the identity $x(x^2)^2 - 4\lambda\omega(x)x^4 + (4\lambda^2 + 2\lambda - 1)\omega(x)^2x^3 + (2\lambda - 4\lambda^2)\omega(x)^3x^2 = 0$.

iv) $\mu_1 = 0$, $\gamma_1\gamma_0\mu \neq 0$ then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = e_1$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_0e_0 + \gamma_1e_1$, $e_1e_2 = e_0$. A verifies the identity $x(x^2)^2 - 4\lambda\omega(x)x^4 + (4\lambda^2 + 2\lambda - 1)\omega(x)^2x^3 + (2\lambda - 4\lambda^2)\omega(x)^3x^2 = 0$.

3.2. <u>2^{nd} Case</u> : $\alpha_0\alpha_1 \neq 0$, then $\beta_0 = \beta_1 = \gamma = \mu_0 = \gamma_0 = \mu_1\gamma_1 = 0$.

The multiplication table of A becomes: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_0e_1 + \alpha_1e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_1e_2$, $e_0e_2 = \gamma_1e_1$,$e_1e_2 = 0$.

i) $\mu_1 = 0$. The multiplication table for A is given as follows : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = e_1 + e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_1 e_1$,$e_1e_2 = 0$. Where e_1 is replaced by $\alpha_0^{-1}e_1$ and e_2 by $\alpha_1^{-1}e_2$. A verifies polynomial identity $(x^2)^3 - \omega(x)x(x^2)^2 - \omega(x)^2(x^2)^2 + \omega(x)^3x^3 = 0$.

ii) $\mu_1 \neq 0$, then $\gamma_1 = 0$. In this case, the multiplication table for A is given as follows : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = e_1 + e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_1 e_2$, $e_0e_2 = 0$,$e_1e_2 = 0$, where e_1 is replaced by $\alpha_0^{-1}e_1$ and e_2 by $\alpha_1^{-1}e_2$. We prove that A checks the polynomial identity $x(x^2)^2 - \lambda\omega(x)(x^2)^2 - \omega(x)^2x^3 + \lambda\omega(x)^3x^2 = 0$.

3.3. 3^{rd} Case : $\alpha_0 = \alpha_1 = 0$ then $\beta_0\mu_1 = \gamma\gamma_1 = 0$.

i) $\beta_0\gamma \neq 0$ and $\mu_1 \neq 0$, then $\gamma_1 = 0$.
$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = \beta_0 e_0$, $e_2^2 = \gamma e_0$, $e_0e_1 = \mu_0 e_0$, $e_0e_2 = \gamma_0 e_0$, $e_1e_2 = \mu e_0$. Then algebra A verifies polynomial identity $(x^2)^2 - \omega(x)^2x^2 = 0$.

ii) $\beta_0 \neq 0$ and $\gamma = 0$, then $\mu_1 = 0$. $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = \beta_0 e_0$, $e_2^2 = 0$, $e_0e_1 = \mu_0 e_0$, $e_0e_2 = \gamma_0 e_0 + \gamma_1 e_1$, $e_1e_2 = \mu e_0$. This case is impossible because A not satisfy identity $(x^4)^2 - \omega(x)^4x^4 = 0$.

iii) $\beta_0 = \gamma = 0$, then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_0 e_0 + \mu_1 e_2$, $e_0e_2 = \gamma_0 e_0 + \gamma_1 e_1$, $e_1e_2 = \mu e_0$. If $\mu_1 = 0$, the algebra A verifies the identity $(x^2 - 2\lambda\omega(x))^3(x^2 - 2\lambda\omega(x))^2 - (1-2\lambda)\omega(x)^2((x^2 - 2\lambda\omega(x))^2)^2 - \frac{(1-2\lambda)^2}{2}\omega(x)^4(x^2 - 2\lambda\omega(x))^3 + \frac{(1-2\lambda)^3}{2}\omega(x)^6(x^2 - 2\lambda\omega(x))^2 = 0$.

Suppose now, $\mu_1 \neq 0$. Then $\gamma_1 = 0$ and we can set $\mu_1 = 1$ and the multiplication table of A is one of:

*iii.*1) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$, $e_1e_2 = 0$.

*iii.*2) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_0 e_0 + e_2$, $e_0e_2 = e_0$, $e_1e_2 = 0$.

*iii.*3) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$, $e_1e_2 = \mu e_0$.

*iii.*4) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_0 e_0 + e_2$, $e_0e_2 = e_0$, $e_1e_2 = \mu e_0$.

*iii.*5) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_0e_0 + e_2$, $e_0e_2 = 0$, $e_1e_2 = \mu e_0$. Then the algebra A verifies polynomial identity $(x^2)^3 - (1+\lambda)\omega(x)^2(x^2)^2 + \lambda\omega(x)^4x^2 = 0$.

iv) $\beta_0 = 0$, $\gamma \neq 0$, then $\gamma_1 = 0$. $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = \gamma e_0$, $e_0e_1 = \mu_0e_0 + \mu_1e_2$, $e_0e_2 = \gamma_0e_0$, $e_1e_2 = \mu e_0$.

If $\mu_1 = 0$, then the algebra A verifies polynomial identity $4\lambda^2(x^3)^2 + 4\lambda\omega(x)^2x^3x^2 + \omega(x)^2(x^2)^2 + 2\omega(x)^3x^3 - (2\lambda+1)\omega(x)^4x^2 = 0$. If $\mu_1 \neq 0$, then this is impossible because A not satisfy identity $(x^4)^2 - \omega(x)^4x^4 = 0$.

3.4. 4^{th} case : $\alpha_0 = 0$ and $\alpha_1 \neq 0$ then $\beta_0 = \gamma = \gamma_0 = \gamma_1\mu_0 = \gamma_1\mu_1 = 0$.

$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_1e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_0e_0 + \mu_1e_2$, $e_0e_2 = \gamma_1e_1$, $e_1e_2 = \mu e_0$.

i) $\alpha_0 = \gamma = \gamma_0 = \mu_1 = \mu_0 = 0$ and $\gamma_1 \neq 0$
$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_1e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_1e_1$, $e_1e_2 = \mu e_0$. We can suppose $\alpha_1 = \gamma_1 = 1$. If $\mu = 0$, the algebra A verifies polynomial identity $(x^2)^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$. If $\mu \neq 0$, it is impossible because A not satisfy identity $(x^4)^2 - \omega(x)^4x^4 = 0$.

ii) $\alpha_0 = \gamma = \gamma_0 = \gamma_1 = \mu_0 = \mu_1 = 0$.
$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_1e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = 0$, $e_0e_2 = 0$, $e_1e_2 = \mu e_0$. If $\mu = 0$, the algebra A verifies polynomial identity $(x^2)^3 - \omega(x)^2(x^2)^2 = 0$. If $\mu \neq 0$, we can set $\alpha_1 = \mu = 1$ and the algebra A verifies polynomial identity $(x^2)^3 - (1+\lambda)\omega(x)^2(x^2)^2 + \lambda\omega(x)^4x^2 = 0$.

iii) $\alpha_0 = \gamma = \gamma_0 = \gamma_1 = \mu_1 = 0$, $\mu_0 \neq 0$.
$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_1e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_0e_0$, $e_0e_2 = 0$, $e_1e_2 = \mu e_0$. If $\mu = 0$, the algebra A verifies polynomial identity $(x^2)^3 - (1+\lambda)\omega(x)^2(x^2)^2 + \lambda\omega(x)^4x^2 = 0$.If $\mu \neq 0$, we can set $\alpha_1 = \mu = 1$ and A verifies the same identity.

iv) $\alpha_0 = \gamma = \gamma_0 = \gamma_1 = \mu_0 = 0$, $\mu_1 \neq 0$.
$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_1e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_1e_2$, $e_0e_2 = 0$, $e_1e_2 = \mu e_0$. If $\mu = 0$, the algebra A verifies polynomial identity $(x^2)^3 - \omega(x)^2(x^2)^2 = 0$. If $\mu \neq 0$, we can set $\alpha_1 = \mu = 1$ and the algebra A verifies polynomial identity $(x^2)^3 - (1+\lambda)\omega(x)^2(x^2)^2 + \lambda\omega(x)^4x^2 = 0$.

v) $\alpha_0 = \gamma = \gamma_0 = \gamma_1 = 0$, $\mu_0\mu_1 \neq 0$.
$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_1e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_0e_1 = \mu_0e_0 + \mu_1e_2$, $e_0e_2 = 0$, $e_1e_2 = \mu e_0$. We can set $\alpha_1 = \mu_1 = 1$. If $\mu = 0$, the algebra A verifies polynomial identity $(x^2)^3 - \omega(x)^2(x^2)^2 = 0$. If $\mu \neq 0$, the algebra A verifies polynomial identity $(x^2)^3 - (1+\lambda)\omega(x)^2(x^2)^2 + \lambda\omega(x)^4x^2 = 0$.

Thus we have the following theorem:

Theorem 4. *Let A be a train algebra of degree 2 and exponent 4. If the type of A is $(2,1,1,0)$ then we have the algebras whose multiplication tables are given as follows, the products not mentioned are zero. If one or both parameters α and β appear in the multiplication table, the algebra will be denoted $A(\alpha)$ or $A(\alpha,\beta)$.*

1°) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_2 = \lambda e_2$, $e_0e_1 = e_2$, $e_0e_2 = e_0$.

2°) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_2 = \lambda e_2$, $e_0e_1 = \alpha e_0 + e_2$, $e_0e_2 = e_0$.

3°) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_2 = \lambda e_2$, $e_0e_1 = e_2$, $e_0e_2 = e_0$, $e_1e_2 = \alpha e_0$. *For α and α' in $K^\star$, $A(\alpha)$ and $A(\alpha')$ are isomorph if and only if it exist $k \in K^\star$ such that $\alpha' = k^2\alpha$, so $\alpha'\alpha^{-1} \in (K^\star)^2$.*

4°) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_2 = \lambda e_2$ $e_0e_1 = \alpha e_0 + e_2$, $e_0e_2 = e_0$, $e_1e_2 = \beta e_0$. *For α, α', β, β' in $K^\star$, $A(\alpha,\beta)$ and $A(\alpha',\beta')$ are isomorph if and only if it exist $k \in K^\star$ such that $\beta' = k^2\beta$, so $\beta'\beta^{-1} \in (K^\star)^2$.*

4. Classification of algebras of type $(2,0,1,1)$

The type of A being $(2,0,1,1)$ then $A_0 = 0$ and we have : $A_{1/2}^2 \subset A_\lambda \oplus A_{\overline{\lambda}}$, $A_\lambda^2 \subset A_{1/2}$, $A_{\overline{\lambda}}^2 \subset A_{1/2}$, $A_{1/2}A_\lambda \subset A_{1/2} \oplus A_{\overline{\lambda}}$, $A_{1/2}A_{\overline{\lambda}} \subset A_{1/2} \oplus A_\lambda$, $A_\lambda A_{\overline{\lambda}} \subset A_{1/2}$. We can set:

$A_{1/2} =< e_0 >$, $A_\lambda =< e_1 >$, $A_{\overline{\lambda}} =< e_2 >$, such that $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = \alpha_0e_1 + \alpha_1e_2$, $e_1^2 = \gamma e_0$, $e_2^2 = \mu e_0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0e_0 + \beta_1e_2$, $e_0e_2 = \gamma_0e_0 + \gamma_1e_1$.

According the type, we use here identities of Lemma (1)

The assertion (i) leads to equality: $(\alpha_0\beta_0(1+\lambda) + \alpha_1\gamma_0(1+\overline{\lambda}))e_0 = 0$ so:

$$\alpha_0\beta_0(1+\lambda) + \alpha_1\gamma_0(1+\overline{\lambda}) = 0. \tag{12}$$

The assertion (ii) leads to equality: $[\alpha_0\beta_1\gamma_0 + \alpha_1\beta_0\gamma_1 + \frac{1}{8}(\gamma\alpha_0^2(1+3\lambda) + 4\alpha_0\alpha_1\rho + \mu\alpha_1(1+3\lambda))]e_0 + (4\alpha_0\beta_0\lambda + 2\alpha_0^2\beta_0 + 2\alpha_0\beta_1\gamma_1\lambda + \alpha_1\gamma_0\alpha_0)e_1 + (4\alpha_1^2\gamma_0\overline{\lambda} + 2\alpha_1^2\gamma_0 + 2\alpha_1\beta_1\gamma_1\overline{\lambda} + \alpha_0\alpha_1\beta_0)e_2 = 0$

or:

$$\alpha_0\beta_1\gamma_0 + \alpha_1\beta_0\gamma_1 + \frac{1}{8}(\gamma\alpha_0^2(1+3\lambda) + 4\alpha_0\alpha_1\rho + \mu\alpha_1^2(1+3\lambda)) = 0. \tag{13}$$

$$4\alpha_0^2\beta_0\lambda + 2\alpha_0^2\beta_0 + 2\alpha_0\beta_1\gamma_1\lambda + \alpha_1\gamma_0\alpha_0 = 0. \tag{14}$$

$$4\alpha_1^2\gamma_0\overline{\lambda} + 2\alpha_1^2\gamma_0 + 2\alpha_1\beta_1\gamma_1\overline{\lambda} + \alpha_0\alpha_1\beta_0 = 0. \tag{15}$$

The assertion (v) leads to: $2(1-\lambda)\gamma\beta_1e_2 = 0$ so:

$$\gamma\beta_1 = 0. \tag{16}$$

The assertion (vi) leads to: $(1-32\lambda)\alpha_0\gamma^2 e_1 + (1-32\lambda)\alpha_1\alpha_1\gamma^2 e_2 = 0$, so:

$$\alpha_0\gamma = 0. \tag{17}$$

$$\alpha_1\gamma = 0. \tag{18}$$

The assertion (vii) leads to: $2(1-\overline{\lambda})\mu\gamma_1 e_1 = 0$ thus:

$$\mu\gamma_1 = 0. \tag{19}$$

Finally the assertion (vii) allows to obtain $(1-32\lambda)\alpha_0\mu^2 e_1 + (1-32\lambda)\alpha_1\mu^2 e_2 = 0$ so:

$$\alpha_0\mu = 0. \tag{20}$$

$$\alpha_1\mu = 0. \tag{21}$$

Let us distinguish the following cases satisfying the preceding equalities:

4.1. 1^{st} Case: $\alpha_0 \neq 0$ and $\alpha_1 = 0$

Then $\gamma = \mu = \beta_0 = \beta_1\gamma_1 = \beta_1\gamma_0 = 0$ and the multiplication table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = \alpha_0 e_1$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_1 e_2$, $e_0e_2 = \gamma_0 e_0 + \gamma_1 e_1$.

i) $\beta_1 \neq 0, \gamma_1 = \gamma_0 = 0$, $\rho \neq 0$.

$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = e_1$, $e_1e_2 = \rho e_0$ where e_1 is replaced by $\alpha_0^{-1}e_1$ and e_2 is replaced by $\alpha_0^{-1}\beta_1^{-1}e_2$. A verifies the polynomial identity : $(x^2 - 2\overline{\lambda}\omega(x)x)^3 - (\lambda - 2\overline{\lambda})\omega(x)^2(x^2 - 2\overline{\lambda}\omega(x)x)^2 + (\lambda + 2\overline{\lambda} - 2)\omega(x)^4(x^2 - 2\overline{\lambda}\omega(x)) = 0$.

ii) $\beta_1 \neq 0, \gamma_1 = \gamma_0 = \rho = 0$

$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = e_1$, $e_1e_2 = 0$ and $e_0e_1 = e_2$, where e_1 is replaced by $\alpha_0^{-1}e_1$ and e_2 by $\alpha_0^{-1}\beta_1^{-1}e_2$. A verifies the identity $(x^2)^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$.

iii) $\beta_1 = 0$

$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = e_1$ and $e_1e_2 = \rho e_0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_0 e_0 + \gamma_1 e_1$, where e_1 is replaced by $\alpha_0^{-1}e_1$. A verifies the identity $(x^2)^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$.

4.2. 2^{nd} Case: $\alpha_0 \neq 0$ and $\alpha_1 \neq 0$, then $0 = \gamma = \rho = \mu = \beta_1\gamma_1$.

i) $\beta_0 = \beta_1 = \gamma_0 = \gamma_1 = 0$. The multiplication table of A is : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = e_1 + e_2$. Where e_1 is replaced by $\alpha_0^{-1}e_1$ et e_2 par $\alpha_1^{-1}e_2$. A verifies the polynomial identity: $x^4 - \frac{1}{2}\omega(x)x^3 - \frac{1}{2}\omega(x)^3 x = 0$.

ii) $\beta_0 = \gamma_0 = \gamma_1 = 0$ and $\beta_1 \neq 0$. The multiplication table of A is : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = e_1 + e_2$, $e_0e_1 = e_2$. Where e_1 is replaced by $\alpha_0^{-1}e_1$ and e_2 by $\alpha_1^{-1}e_2$. A verifies the polynomial identity: $x^4 - \frac{1}{2}\omega(x)x^3 - \frac{1}{2}\omega(x)^3 x = 0$

iii) $\beta_0 = \gamma_0 = \beta_1 = 0$ and $\gamma_1 \neq 0$. The multiplication table of A is : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = e_1 + e_2$, $e_0e_2 = e_1$. Where e_1 is replaced by $\alpha_0^{-1}e_1$ and e_2 by $\alpha_1^{-1}e_2$, so A verifies the polynomial identity : $(x^2 - 2\overline{\lambda}\omega(x)x)^3 - (1+\lambda)\omega(x)^2(x^2 - 2\overline{\lambda}\omega(x)x)^2 + \lambda\omega(x)^4(x^2 - 2\overline{\lambda}\omega(x)) = 0$.

4.3. 3^{rd} Case: $\alpha_0 = 0$ and $\alpha_1 \neq 0$.

Then $\mu = \gamma = \gamma_0 = \beta_1\gamma_1 = \beta_0\gamma_1 = 0$, so the multiplication table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = \alpha_1 e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0 e_0 + \beta_1 e_2$, $e_0e_2 = \gamma_1 e_1$. We distinguish the algebras whose multiplication tables are:

i) $\gamma_1 \neq 0$, then $\beta_0 = \beta_1 = 0$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = \alpha_1 e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_1 e_1$. We can set $\alpha_1 = \gamma_1 = 1$. If $\rho = 0$, A verifies the polynomial identity $(x^2)^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$

ii) $\gamma_1 = \beta_1 = \beta_0 = 0$, then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = \alpha_1 e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = 0$, $e_0e_2 = 0$. We can set $\alpha_1 = 1$ and A verifies the polynomial identity : $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

iii) $\gamma_1 = 0$, $\beta_1\beta_0 \neq 0$, $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = \alpha_1 e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0 e_0 + \beta_1 e_2$, $e_0e_2 = 0$. We can set $\alpha_1 = 1$ and A verifies the polynomial identity: $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

iv) $\gamma_1 = \beta_1 = 0$, $\beta_0 \neq 0$,$e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = \alpha_1 e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0 e_0$, $e_0e_2 = 0$. We can set $\alpha_1 = \beta_0 = 1$ and A verifies the polynomial identity: $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

v) $\gamma_1 = \beta_0 = 0$, $\beta_1 \neq 0$, $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = \alpha_1 e_2$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_1 e_2$, $e_0e_2 = 0$.We can set $\alpha_1 = \beta_1 = 1$ and A verifies the polynomial identity: $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

4.4. 4^{th} case: $\alpha_0 = \alpha_1 = 0$

i) $\alpha_0 = \alpha_1 = \gamma_1 = \gamma_0 = \beta_1 = 0$. Then the multiplication table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = \gamma e_0$, $e_2^2 = \mu e_0$, $e_0e_1 = \beta_0 e_0$, $e_0e_2 = 0$, $e_1e_2 = \rho e_0$. The classification of algebras in this case amounts to the classification of the quadratic form q of polar forms φ defined from $A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$ to K by : $\varphi(e_0, e_0) = q(e_0) = 0$, $\varphi(e_1, e_1) = q(e_1) = \gamma$, $\varphi(e_2, e_2) = q(e_2) = \mu$, $\varphi(e_0, e_1) = \beta_0$, $\varphi(e_0, e_2) = 0$, $\varphi(e_1, e_2) = \rho$. The matrix of q in the basis (e_0, e_1, e_2) is given by:

$$M = \begin{pmatrix} 0 & \beta_0 & 0 \\ \beta_0 & \gamma & \rho \\ 0 & \rho & \mu \end{pmatrix}, \text{ so } det(M) = -\beta_0^2\mu.$$

If q is degenerate, then $det(M) = 0$, so $\mu\beta_0 = 0$. Suppose $\beta_0 \neq 0$, then $\mu = 0$ and we have the cases:

i.1) $\gamma = \rho = 0$. The multiplication table of A is given by : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0e_1 = e_0$. Where e_1 is replaced by $\beta_0^{-1}e_1$.A verifies the polynomial identity : $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

i.2) $\gamma = 0$ and $\rho \neq 0$. The multiplication table of A is given by : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0e_1 = e_0$, $e_1e_2 = e_0$. Where e_1 is replaced by $\beta_0^{-1}e_1$ et e_2 par $\beta_0\rho^{-1}e_2$.A verifies the polynomial identity: $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

i.3) $\rho = 0$ and $\gamma \neq 0$. The multiplication table of A is given by : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0e_1 = e_0$ and $e_1^2 = e_0$. Where e_1 is replaced by$\beta_0^{-1}e_1$ et e_0 par $\beta_0^2\gamma^{-1}e_0$. A verifies the polynomial identity: $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

i.4) $\rho \neq 0$ and $\gamma \neq 0$. The multiplication table of A is given by: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0e_1 = e_0$, $e_1^2 = e_0$ et $e_1e_2 = \rho e_0$, where e_1 is replaced by $\beta_0^{-1}e_1$ and e_0 by $\beta_0^2\gamma^{-1}e_0$. A verifies the polynomial identity: $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

Suppose $\beta_0 = 0$, then $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = \gamma e_0$, $e_2^2 = \mu e_0$, $e_0e_1 = 0$, $e_0e_2 = 0$, $e_1e_2 = \rho e_0$.

i.5) $\mu = 0$. The multiplication table of A is given by : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = \gamma e_0$, $e_2^2 = 0$, $e_0e_1 = 0$, $e_0e_2 = 0$, $e_1e_2 = \rho e_0$. A verifies the polynomial identity : $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

i.6) $\mu\gamma\rho \neq 0$. We can set $\rho = \gamma = 1$ and the multiplication table of A is given by: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = e_0$, $e_2^2 = \mu e_0$, $e_0e_1 = 0$, $e_0e_2 = 0$, $e_1e_2 = e_0$.

$i.7)$ $\mu \neq 0$, $\rho = \gamma = 0$. We can set $\mu = 1$. The multiplication table of A is given by: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_0e_1 = 0$, $e_0e_2 = 0$, $e_1e_2 = 0$. A verifies the polynomial identity:
$(x^2-2\bar{\lambda}\omega(x)x)^3-(\lambda-2\bar{\lambda})\omega(x)^2(x^2-2\bar{\lambda}\omega(x)x)^2+(\lambda+2\bar{\lambda}-2)\omega(x)^4(x^2-2\bar{\lambda}\omega(x)) = 0$.

$i.8)$ $\mu\gamma \neq 0$, $\rho = 0$. We can set $\gamma = 1$. The multiplication table of A is given by: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = e_0$, $e_2^2 = \mu e_0$, $e_0e_1 = 0$, $e_0e_2 = 0$, $e_1e_2 = 0$. A verifies the polynomial identity: $(x^3)^2 + 2\omega(x)^2x^4 - 2\omega(x)^3x^3 + \omega(x)^4x^2 - 2\omega(x)^5x = 0$.

$i.9)$ $\mu\rho \neq 0$, $\gamma = 0$. We can set $\rho = \rho = 1$. The multiplication table of A is given by : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_0e_1 = 0$, $e_0e_2 = 0$, $e_1e_2 = e_0$. A verifies the polynomial identity: $(x^2 - 2\bar{\lambda}\omega(x)x)^3 - (\lambda - 2\bar{\lambda})\omega(x)^2(x^2 - 2\bar{\lambda}\omega(x)x)^2 + (\lambda + 2\bar{\lambda} - 2)\omega(x)^4(x^2 - 2\bar{\lambda}\omega(x)) = 0$.

If q is regular then $det(M) \neq 0$ therefore $\beta_0\mu \neq 0$ and we have:

$i.10)$ $\gamma = \rho = 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0e_1 = e_0$, $e_2^2 = e_0$. Where e_1 is replaced by $\beta_0^{-1}e_1$ and e_0 by $\mu^{-1}e_0$. A verifies the polynomial identity: $(x^3)^2 + 2\omega(x)^2x^4 - 2\omega(x)^3x^3 + \omega(x)^4x^2 - 2\omega(x)^5x = 0$.

$i.11)$ $\gamma = 0$ and $\rho \neq 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0e_1 = e_0$, $e_1e_2 = \rho e_0$, $e_2^2 = e_0$, where e_1 is replaced by $\beta_0^{-1}e_1$ and e_0 by $\mu^{-1}e_0$. A verifies the polynomial identity : $(x^3)^2 + 2\omega(x)^2x^4 - 2\omega(x)^3x^3 + \omega(x)^4x^2 - 2\omega(x)^5x = 0$.

$i.12)$ $\rho = 0$ and $\gamma \neq 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0e_1 = e_0$, $e_1^2 = \rho e_0$, $e_2^2 = e_0$, where e_1 is replaced by $\beta_0^{-1}e_1$ and e_0 by $\mu^{-1}e_0$. A verifies the polynomial identity: $(x^3)^2 + 2\omega(x)^2x^4 - 2\omega(x)^3x^3 + \omega(x)^4x^2 - 2\omega(x)^5x = 0$.

$i.13)$ $\rho \neq 0$ and $\gamma \neq 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0e_1 = e_0$, $e_1e_2 = e_0$, $e_1^2 = e_0$, $e_2^2 = \rho e_0$, where e_1 is replaced by $\beta_0^{-1}e_1$ and e_2 by $\beta_0\rho^{-1}e_2$ and e_0 by $\beta_0^2\gamma^{-1}e_0$. A verifies the polynomial identity: $(x^3)^2 + 2\omega(x)^2x^4 - 2\omega(x)^3x^3 + \omega(x)^4x^2 - 2\omega(x)^5x = 0$.

$ii)$ $\alpha_0 = \alpha_1 = \gamma_1 = \beta_1 = 0$, $\gamma_0 \neq 0$. Then the multiplication table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = \gamma e_0$, $e_2^2 = \mu e_0$, $e_0e_1 = \beta_0 e_0$, $e_0e_2 = \gamma_0 e_0$, $e_1e_2 = \rho e_0$. We can set $\gamma_0 = 1$.

$ii.1)$ Suppose $\beta_0 \neq 0$, then we can set $\beta_0 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda} e_2$, $e_0^2 = 0$, $e_1^2 = \gamma e_0$, $e_2^2 = \mu e_0$, $e_0e_1 = e_0$, $e_0e_2 = e_0$, $e_1e_2 = \rho e_0$. A verifies the polynomial identity: $(x^3)^2 + 2\omega(x)^2x^4 - 2\omega(x)^3x^3 + \omega(x)^4x^2 - 2\omega(x)^5x = 0$.

Suppose $\beta_0 = 0$, then $detM = -\gamma$.
Let q be the quadratic of polar forms φ defined from $A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$ to K by:

$\varphi(e_0,e_0) = q(e_0) = 0$, $\varphi(e_1,e_1) = q(e_1) = \gamma$, $\varphi(e_2,e_2) = q(e_2) = \mu$, $\varphi(e_0,e_1) = 0$, $\varphi(e_0,e_2) = 1$, $\varphi(e_1,e_2) = \rho$. The matrix of q in the basis (e_0,e_1,e_2) is given by:

$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \gamma & \rho \\ 1 & \rho & \mu \end{pmatrix}$, so $detM = -\gamma$.

ii.2) If $detM = 0$, $\gamma = 0$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = \mu e_0$, $e_0e_1 = 0$, $e_0e_2 = e_0$, $e_1e_2 = \rho e_0$. A verifies the polynomial identity: $(x^3)^2 + 2\omega(x)^2x^4 - 2\omega(x)^3x^3 + \omega(x)^4x^2 - 2\omega(x)^5x = 0$.

ii.3) If $detM \neq 0$, we can set $\gamma = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = e_0$, $e_2^2 = \mu e_0$, $e_0e_1 = 0$, $e_0e_2 = e_0$, $e_1e_2 = \rho e_0$.A verifies the polynomial identity: $(x^3)^2 + 2\omega(x)^2x^4 - 2\omega(x)^3x^3 + \omega(x)^4x^2 - 2\omega(x)^5x = 0$.

iii) $\alpha_1 = \alpha_0 = \beta_1 = \gamma_0 = \mu = 0$ and $\gamma_1 \neq 0$. We can set $\gamma_1 = 1$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = \gamma e_0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0e_0$, $e_0e_2 = e_1$.

iii.1) Suppose $\beta_0 \neq 0$, then we can set $\beta_0 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = \gamma e_0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = e_0$, $e_0e_2 = e_1$.

Suppose $\beta_0 = 0$, then $detM = -\gamma$.
Let q be the quadratic of polar forms φ defined from $A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$ to K by : $\varphi(e_0,e_0) = q(e_0) = 0$, $\varphi(e_1,e_1) = q(e_1) = \gamma$, $\varphi(e_2,e_2) = q(e_2) = 0$, $\varphi(e_0,e_1) = 0$, $\varphi(e_0,e_2) = 1$, $\varphi(e_1,e_2) = \rho$. The matrix of q in the basis (e_0,e_1,e_2) is given by:

$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \gamma & \rho \\ 1 & \rho & 0 \end{pmatrix}$, so $detM = -\gamma$.

iii.2) If $detM = 0$, $\gamma = 0$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = 0$, $e_0e_2 = e_1$. A verifies the polynomial identity: $(x^2-2\bar{\lambda}\omega(x)x)^3-(\lambda-2\bar{\lambda})\omega(x)^2(x^2-2\bar{\lambda}\omega(x)x)^2+(\lambda+2\bar{\lambda}-2)\omega(x)^4(x^2-2\bar{\lambda}\omega(x)) = 0$.

iii.3) If $detM \neq 0$, $\gamma \neq 0$. We can set $\gamma = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = e_0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = 0$, $e_0e_2 = e_1$. A verifies the polynomial identity: $(x^2 - 2\bar{\lambda}\omega(x)x)^3 - (\lambda - 2\bar{\lambda})\omega(x)^2(x^2 - 2\bar{\lambda}\omega(x)x)^2 + (\lambda + 2\bar{\lambda} - 2)\omega(x)^4(x^2 - 2\bar{\lambda}\omega(x)) = 0$.

iv) $\alpha_1 = \alpha_0 = \beta_1 = \mu = 0$ and $\gamma_0\gamma_1 \neq 0$. The table of A is : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = \gamma e_0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0e_0$, $e_0e_2 = \gamma_0e_0 + \gamma_1e_1$.

*iv.*1) $\beta_0 = \gamma = 0$, then A verifies the polynomial identity: $(x^2 - 2\bar{\lambda}\omega(x)x)^3 - (\lambda - 2\bar{\lambda})\omega(x)^2(x^2 - 2\bar{\lambda}\omega(x)x)^2 + (\lambda + 2\bar{\lambda} - 2)\omega(x)^4(x^2 - 2\bar{\lambda}\omega(x)) = 0$.

*iv.*2) $\beta_0 = 0$, $\rho\gamma \neq 0$, then we can set $\gamma = \gamma_1 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = e_0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_0 e_0 + e_1$.

*iv.*3) $\beta_0 = \rho = 0$, $\gamma \neq 0$, then we can set $\gamma = \gamma_1 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = e_0$, $e_2^2 = 0$, $e_1e_2 = 0$, $e_0e_1 = 0$, $e_0e_2 = \gamma_0 e_0 + e_1$.

*iv.*4) $\beta_0\rho \neq 0$, $\gamma = 0$, then we can set $\beta_0 = \gamma_1 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = e_0$, $e_0e_2 = \gamma_0 e_0 + e_1$.

*iv.*5) $\beta_0 \neq 0$, $\rho = \gamma = 0$, then we can set $\beta_0 = \gamma_1 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = 0$, $e_0e_1 = e_0$, $e_0e_2 = \gamma_0 e_0 + e_1$.

*iv.*6) $\beta_0\gamma \neq 0$, then we can set $\beta_0 = \gamma = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = e_0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = e_0$, $e_0e_2 = \gamma_0 e_0 + e_1$.

v) $\alpha_1 = \alpha_0 = \gamma_1 = \gamma_0 = \gamma = 0$ and $\beta_1 \neq 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = \mu e_0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0 e_0 + \beta_1 e_2$, $e_0e_2 = 0$.

*v.*1) $\mu = 0$, A verifies the polynomial identity: $(x^2 - 2\lambda\omega(x)x)^3 - (\bar{\lambda} - 2\lambda)\omega(x)^2(x^2 - 2\lambda\omega(x)x)^2 + (\bar{\lambda} + 2\lambda - 2)\omega(x)^4(x^2 - 2\lambda\omega(x)) = 0$.

*v.*2) $\rho = \beta_0 = 0$, $\mu \neq 0$, then we can set $\mu = \beta_1 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_1e_2 = 0$, $e_0e_1 = e_2$, $e_0e_2 = 0$.

*v.*3) $\rho = 0$, $\beta_0\mu \neq 0$, then we can set $\mu = \beta_1 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_1e_2 = 0$, $e_0e_1 = \beta_0 e_0 + e_2$, $e_0e_2 = 0$.

*v.*4) $\beta_0 = 0$, $\rho\mu \neq 0$, then we can set $\mu = \beta_1 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_1e_2 = \rho e_0$, $e_0e_1 = e_2$, $e_0e_2 = 0$.

*v.*5) $\beta_0\rho\mu \neq 0$, then we can set $\mu = \beta_1 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0 e_0 + e_2$, $e_0e_2 = 0$.

vi) $\alpha_1 = \alpha_0 = \gamma_1 = \gamma = 0$ and $\gamma_0\beta_1 \neq 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = \mu e_0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0 e_0 + \beta_1 e_2$, $e_0e_2 = \gamma_0 e_0$.

$vi.1)$ $\beta_0 \neq 0$, $\rho = \mu = 0$. We can set $\gamma_0 = \beta_1 = 1$ and he table of A is : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = 0$, $e_0e_1 = \beta_0e_0 + e_2$, $e_0e_2 = e_0$.

$vi.2)$ $\beta_0 \neq 0$, $\rho = 0$, $\mu \neq 0$. We can set $\mu = \gamma_0 = \beta_1 = 1$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_1e_2 = 0$, $e_0e_1 = \beta_0e_0 + e_2$, $e_0e_2 = e_0$.

$vi.3)$ $\rho\beta_0 \neq 0$, $\mu = 0$. We can set $\beta_1 = \rho = \gamma_0 = 1$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = e_0$, $e_0e_1 = \beta_0e_0 + e_2$, $e_0e_2 = e_0$.

$vi.4)$ $\beta_0\rho\mu \neq 0$. We can set $\gamma_0 = \rho = \mu = 1$ and he table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_1e_2 = e_0$, $e_0e_1 = \beta_0e_0 + \beta_1e_2$, $e_0e_2 = e_0$.

$vi.5)$ $\beta_0 = 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = \mu e_0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_1e_2$, $e_0e_2 = \gamma_0e_0$.

Let q be the quadratic of polar forms φ defined from $A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$ to K by : $\varphi(e_0, e_0) = q(e_0) = 0$, $\varphi(e_1, e_1) = q(e_1) = 0$, $\varphi(e_2, e_2) = q(e_2) = \mu$, $\varphi(e_0, e_1) = \beta_1$, $\varphi(e_0, e_2) = \gamma_1$, $\varphi(e_1, e_2) = \rho$. The matrix of q in the basis (e_0, e_1, e_2) is given by:
$M = \begin{pmatrix} 0 & \beta_1 & \gamma_0 \\ \beta_1 & 0 & \rho \\ \gamma_0 & \rho & \mu \end{pmatrix}$, so $detM = \beta_1(2\gamma_0\rho - \beta_1\mu)$.

$vi.6)$ $\rho = \mu = 0$, then we can set $\beta_1 = \gamma_0 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = 0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$.

$vi.7)$ $\rho = 0$, $\mu \neq 0$, then we can set $\beta_1 = \gamma_0 = \mu = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_1e_2 = 0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$.

$vi.8)$ $\rho \neq 0$, $\mu = 0$, then we can set $\beta_1 = \gamma_0 = 1$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$.

$vi.9)$ $\rho\mu \neq 0$, then we can set $\beta_1 = \gamma_0 = \mu = 1$, $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = e_0$, $e_1e_2 = \rho e_0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$.

$vii)$ $\alpha_1 = \alpha_0 = \gamma_0 = \mu = \gamma = 0$ and $\gamma_1 \neq 0$, $\beta_1 \neq 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0e_0 + \beta_1e_2$,

$e_0e_2 = \gamma_1 e_1$.

vii.1) $\beta_0 = \rho = 0$, we can set $\beta_1 = \gamma_1 = 1$, the table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = 0$, $e_0e_1 = e_2$, $e_0e_2 = e_1$.

vii.2) $\beta_0 = 0$, $\rho \neq 0$, we can set $\beta_1 = \gamma_1$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = e_2$, $e_0e_2 = e_1$.

vii.3) $\beta_0 \neq 0$, $\rho = 0$. We can set γ_1 and the table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = 0$, $e_0e_1 = \beta_0e_0 + \beta_1e_2$, $e_0e_2 = e_1$.

vii.4) $\beta_0\rho \neq 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0e_0 + \beta_1e_2$, $e_0e_2 = \gamma_1e_1$. This case is impossible because A not verifies the identity $(x^4)^2 - \omega(x)^4x^4 = 0$.

viii) $\alpha_1 = \alpha_0 = \mu = \gamma = 0$ and $\gamma_0\gamma_1 \neq 0$, $\beta_1 \neq 0$. The table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = \beta_0e_0 + \beta_1e_2$, $e_0e_2 = \gamma_1e_1 + \gamma_0e_0$.

viii.1) $\beta_0 = \rho = 0$, without loss of generality. It suffices to make some basis transformations to prove that $\beta_1 = 1$ and the table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = 0$, $e_0e_1 = e_2$, $e_0e_2 = \gamma_1e_1 + \gamma_0e_0$.

viii.2) $\beta_0 = 0$, $\rho \neq 0$, without loss of generality. It suffices to make some basis transformations to prove that $\beta_1 = 1$ and the table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = \rho e_0$, $e_0e_1 = e_2$, $e_0e_2 = \gamma_1e_1+\gamma_0e_0$. This case is impossible because A not verifies the identity $(x^4)^2-\omega(x)^4x^4 = 0$. (for $x = e_0+e_2$)

viii.3) $\beta_0 \neq 0$, $\rho = 0$, without loss of generality. It suffices to make some basis transformations to prove that $\beta_1 = 1$ and the table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = 0$, $e_0e_1 = \beta_0e_0 + e_2$, $e_0e_2 = \gamma_1e_1 + \gamma_0e_0$. This case is impossible because A not verifies the identity $(x^4)^2 - \omega(x)^4x^4 = 0$.

viii.4) $\beta_0\rho \neq 0$, without loss of generality. It suffices to make some basis transformations to prove that $\rho = 1$ and the table of A is: $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0^2 = 0$, $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 = e_0$, $e_0e_1 = \beta_0e_0 + \beta_1e_2$, $e_0e_2 = \gamma_1e_1 + \gamma_0e_0$. This case is impossible because A not verifies the identity $(x^4)^2 - \omega(x)^4x^4 = 0$.

Theorem 5. *Let A be a train algebra of degree 2 and exponent 4. If the type of A is $(2, 0, 1, 1)$, then we have algebras whose multiplication tables are given as follows, the unmentioned products being zero, the algebra is denoted $A(\alpha)$ or $A(\alpha, \beta)$ if α and β are the parameters which appear in the multiplication table:*

1) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$,$ee_2 = \overline{\lambda}e_2$, $e_0^2 = e_2$, $e_1e_2 = \alpha e_0$, $e_0e_2 = e_1$. *for* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if it exist* $k \in K^\star$ *such that* $\alpha' = k^2\alpha$, *so* $\alpha'\alpha^{-1} \in (K^\star)^2$.

2) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_1^2 = e_0$, $e_2^2 = \alpha e_0$, $e_1e_2 = e_0$; *for* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if* $\alpha' = \alpha$.

3) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_1^2 = \alpha e_0$, $e_1e_2 = \beta e_0$, $e_0e_1 = e_0$, $e_0e_2 = e_1$; *for* α, α', β *and* β' *in* $K^\star$, $A(\beta)$ *and* $A(\beta')$ *are isomorph if and only if it exist* $k \in K^\star$ *such that* $\beta' = k^2\beta$, *so* $\beta'\beta^{-1} \in (K^\star)^2$.

4) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_1^2 = e_0$, $e_1e_2 = \alpha e_0$, $e_0e_2 = \beta e_0 + e_1$. *for* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if it exist* $k \in K^\star$ *such that* $\alpha' = k^2\alpha$, *so* $\alpha'\alpha^{-1} \in (K^\star)^2$.

5) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_2^2 = e_0$, $e_0e_1 = \alpha e_0 + e_2$.

6) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_2^2 = e_0$, $e_1e_2 = \alpha e_0$, $e_0e_1 = e_2$. *For* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if it exist* $k \in K^\star$ *such that* $\alpha' = k^2\alpha$, *so* $\alpha'\alpha^{-1} \in (K^\star)^2$.

7) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_2^2 = e_0$, $e_1e_2 = \alpha e_0$, $e_0e_1 = \beta e_0 + e_2$. *for* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if it exist* $k \in K^\star$ *such that* $\alpha' = k^2\alpha$, *so* $\alpha'\alpha^{-1} \in (K^\star)^2$.

8) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0e_1 = \alpha e_0 + e_2$, $e_0e_2 = e_0$.

9) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_2^2 = e_0$, $e_0e_1 = \alpha e_0 + e_2$, $e_0e_2 = e_0$. *For* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if* $\alpha' = \alpha$.

10) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_1e_2 = e_0$, $e_0e_1 = \alpha e_0 + e_2$, $e_0e_2 = e_0$. *For* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if* $\alpha' = \alpha$ *or* $\alpha' = -\alpha$.

11) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_2^2 = e_0$, $e_1e_2 = e_0$, $e_0e_1 = \alpha e_0 + \beta e_2$, $e_0e_2 = e_0$. *For* α, α', β *and* β' *in* $K^\star$, $A(\beta)$ *and* $A(\beta')$ *are isomorph if and only if it exist* $k \in K^\star$ *such that* $\beta' = k^2\beta$, *so* $\beta'\beta^{-1} \in (K^\star)^2$.

12) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0e_1 = e_2$, $e_0e_2 = e_0$.

13) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_2^2 = e_0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$.

14) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_1e_2 = \alpha e_0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$. *For* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if it exist* $k \in K^\star$ *such that* $\alpha' = k^2\alpha$, *so* $\alpha'\alpha^{-1} \in (K^\star)^2$.

15) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_2^2 = e_0$, $e_1e_2 = \alpha e_0$, $e_0e_1 = e_2$, $e_0e_2 = e_0$. *For* α *and* α' *in* $K^\star$, $A(\alpha)$ *and* $A(\alpha')$ *are isomorph if and only if* $\alpha' = \alpha$.

16) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0e_1 = e_2$, $e_0e_2 = e_1$.

17) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_1e_2 = \alpha e_0$, $e_0e_1 = e_2$, $e_0e_2 = e_1$. *For α and α' in $K^\star$, $A(\alpha)$ and $A(\alpha')$ are isomorph if and only if it exist $k \in K^\star$ such that $\alpha' = k^2\alpha$, so $\alpha'\alpha^{-1} \in (K^\star)^2$.*

18) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0e_1 = \alpha e_0 + \beta e_2$, $e_0e_2 = e_1$. *For α, α', β and β' in $K^\star$, $A(\beta)$ and $A(\beta')$ are isomorph if and only if it exist $k \in K^\star$ such that $\beta' = k^2\beta$, so $\beta'\beta^{-1} \in (K^\star)^2$.*

19) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \overline{\lambda}e_2$, $e_0e_1 = e_2$, $e_0e_2 = \alpha e_0 + \beta e_1$. *For α, α', β and β' in $K^\star$, $A(\beta)$ and $A(\beta')$ are isomorph if and only if it exist $k \in K^\star$ such that $\beta' = k^2\beta$, so $\beta'\beta^{-1} \in (K^\star)^2$.*

5. Classification of algebras of type $(2, 1, 0, 1)$

The type of A being $(2, 1, 0, 1)$ then $A_\lambda = 0$ and we have : $A_{1/2}^2 \subset A_0 \oplus A_{\overline{\lambda}}$, $A_0^2 \subset A_{1/2} \oplus A_0$, $A_{\overline{\lambda}}^2 \subset A_{1/2}$, $A_{1/2}A_0 \subset A_0 \oplus A_{\overline{\lambda}}$, $A_{1/2}A_{\overline{\lambda}} \subset A_{1/2} \oplus A_0$, $A_0A_{\overline{\lambda}} \subset A_{1/2}$. In this way, it is possible to set

$A_{1/2} =< e_0 >$, $A_0 =< e_1 >$, $A_{\overline{\lambda}} < e_2 >$ and the multiplication table of A is : $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $ee_2 = \overline{\lambda}e_2$; $e_0^2 = \alpha_0e_1 + \alpha_1e_2$, $e_1^2 = \beta_0e_0 + \beta_1e_1$, $e_2^2 = \gamma e_0$, $e_0e_1 = \mu_0e_0 + \mu_1e_2$, $e_0e_2 = \gamma_0e_0 + \gamma_1e_1$, $e_1e_2 = \mu e_0$.

The reasoning is similar to that of the type $(2, 1, 1, 0)$. This allows us to obtain the algebras given in the proposition below.

Theorem 6. *Let A be a train algebra of degree 2 and exponent 4. If the type of A is $(2, 1, 0, 1)$ then we have the algebras whose multiplication table are given as follows, the products not mentioned being zero. If one or both parameters α and β appear in the multiplication table, the algebra will be denoted $A(\alpha)$ or $A(\alpha, \beta)$.*

1°) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_2 = \bar{\lambda}e_2$, $e_0e_1 = e_2$, $e_0e_2 = e_0$.

2°) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_2 = \bar{\lambda}e_2$, $e_0e_1 = \alpha e_0 + e_2$, $e_0e_2 = e_0$.

3°) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_2 = \bar{\lambda}e_2$, $e_0e_1 = e_2$, $e_0e_2 = e_0$, $e_1e_2 = \alpha e_0$. *For α and α' in $K^\star$, $A(\alpha)$ and $A(\alpha')$ are isomorph if and only if it exist $k \in K^\star$ such that $\alpha' = k^2\alpha$, so $\alpha'\alpha^{-1} \in (K^\star)^2$.*

4°) $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_2 = \bar{\lambda}e_2$ $e_0e_1 = \alpha e_0 + e_2$, $e_0e_2 = e_0$, $e_1e_2 = \beta e_0$. *For α, α', β, β' in $K^\star$, $A(\alpha, \beta)$ and $A(\alpha', \beta')$ are isomorph if and only if it exist $k \in K^\star$ such that $\beta' = k^2\beta$, so $\beta'\beta^{-1} \in (K^\star)^2$.*

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