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Weakly Connected Hop Domination in Graphs Resulting from Some Binary Operations

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Abstract. Let $G = (V(G), E(G))$ be a simple connected graph. A set $S \subseteq V(G)$ is a weakly connected hop dominating set of G if for every $q \in V \setminus S$, there exists $r \in S$ such that $d_G(q,r) = 2$, the subgraph weakly induced by S, denoted by $\langle S\rangle_w = \langle N_G[S], E_w\rangle$ where $E_w = \{qr \in E(G) : q \in S \text{ or } r \in S\}$ is connected and S is a dominating set of G. The minimum cardinality of a weakly connected hop dominating set of G is called weakly connected hop domination number and is denoted by $\gamma_{wh}(G)$. In this paper, the authors show and explore the concept of weakly connected hop dominating set. The weakly connected hop dominating set of some special graphs, shadow of graphs, join, corona and Lexicographic product of two graphs are characterized. Also, the weakly connected domination number of the aforementioned graphs are determined.

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Key Words and Phrases: Weakly connected set, hop dominating set, hop domination number, weakly connected hop dominating set, and weakly connected hop domination number

1. Introduction

The weakly connected domination in graph was studied by Dunbar, et.al. in [3]. They considered the weakly connected domination number $\gamma_w(G)$ of a graph G and some related domination parameters. They have shown that the problem of computing $\gamma_w(G)$ is NP-hard in general but linear for trees. In addition, several sharp upper and lower bounds for $\gamma_w(G)$ are obtained, and it was further investigated by Domke, et. al. in [2]. They provided a constructive characterization for trees T for which $\gamma(T) = \gamma_{wc}(T)$. Moreover, some related variations and parameters are studied in many classes of graphs including those results under some binary operations (see [4], [9], [10] and [11]).

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The concept of hop dominating set of a graph G was investigated and introduced by Ayyaswamy, et. al. in [1]. Later on, they presented the strong equality of hop domination and hop independent domination numbers for trees. Hop domination numbers of shadow graph and mycielskian graph are also discussed in [8]. Canoy, et.al. in [7], revisited the concept of hop domination, related it with other domination concepts, and investigated it in graphs resulting from some binary operations.

In this paper, we introduce the concept of weakly connected hop dominating set of a graph G. We show and investigate a parameter that is, defined in a manner that a well-known weakly connected set and hop dominating set are put into one. Indeed, notwithstanding hop dominating set of a graph G requires the subgraph weakly induced by S, denoted by $\langle S\rangle_w = \langle N_G[S], E_w\rangle$ where $E_w = \{qr \in E(G) : q \in S \text{ or } r \in S\}$ is connected.

The motivation of introducing the concept is to give further investigation on weakly connected, hop domination and some of its variations. In fact, it can be shown that every weakly connected hop dominating set is a hop dominating set of a graph G. Therefore, the hop domination number a graph G is at most equal to the weakly connected hop domination number of a graph G.

2. Terminology and Notation

A simple connected graph $G = (V(G), E(G))$, where $V(G)$ is a *vertex-set* of G and $E(G)$ is an *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. Two vertices u, v of a graph G are *adjacent*, or *neighbors*, if uv is an edge of G. The set of neighbors of a vertex u in G is called the *open neighborhood* of u in G and is denoted by $N_G(u)$. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$, defined by Harary in [5].

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among the dominating sets of G. A dominating set S with $|S| = \gamma(G)$ is said to be γ-*set* of G, defined by J. Tarr, et.al in [12].

Let $S \subseteq V(G)$. The *subgraph weakly induced* by S is the graph $\langle S\rangle_w = \langle N_G[S], E_w\rangle$, where $E_w = \{qr \in E(G) : q \in S \text{ or } r \in S\}$, Patangan, et.al [9]. A dominating set $S \subseteq V(G)$ is a *weakly connected dominating set* in G if the subgraph $\langle S\rangle_w$ weakly induced by S is connected. The *weakly connected domination number* $\gamma_w(G)$ of G is the minimum cardinality among all weakly connected dominating sets of G. A weakly connected dominating set S with $|S| = \gamma_w(G)$ is said to be γ_w-*set* of G, Sandueta, et.al in [10].

A set $S \subseteq V(G)$ is a *hop dominating* set of G if for every $q \in V(G) \setminus S$, there exists $r \in S$ such that $d_G(q, r) = 2$. The hop domination number of G, denoted by $\gamma_h(G)$, is the minimum cardinality among the hop dominating sets of G. Any hop dominating set S of G with $|S| = \gamma_h(G)$ is refered to as $\gamma_h - set$ of G, Ayyaswamy, et.al. in [1].

3. Results

Definition 1. A set $S \subseteq V(G)$ is a *weakly connecetd hop dominating* set of G if S is both a weakly connected set and hop dominating set of G. The minimum cardinality of a weakly connected hop dominating set of G is called weakly connected hop domination number and is denoted by $\gamma_{wh}(G)$. A weakly connected hop dominating set S with $|S| = \gamma_{wh}(G)$ is said to be γ_{wh}-set of G.

Example 1. Let G be the graph in Figure 1 and $S = \{b, d, h\} \subseteq V(G)$. Observe that for all vertices $a, c, e, f, g \in V(G) \setminus S$, there are $b, d, h \in S$ such that $d_G(b, g) = 2$, $d_G(d, f) = 2$, $d_G(h, a) = 2$, $d_G(h, e) = 2$, and $d_G(h, c) = 2$, and $N_G[S] = V(G)$. Thus, S is a hop dominating set. Moreover, $\langle S\rangle_w = \langle N_G(S), \{ba, bc, bh, dc, de, dh, hg, hf\}\rangle$ is connected. Therefore, S is a weakly connected hop dominating set of G.

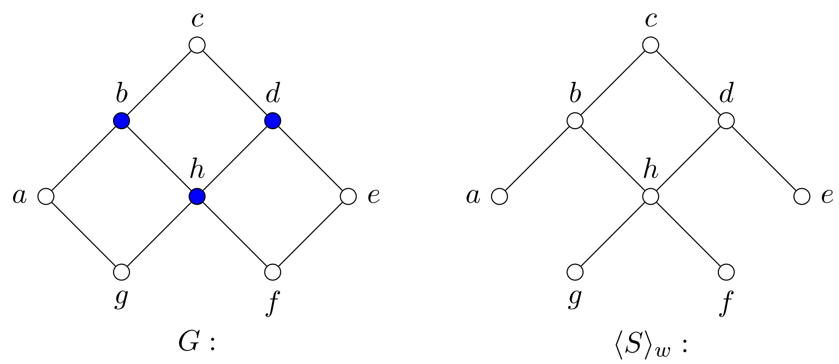

Figure 1: Graph G with $\gamma_{wh}(G) = 3$ and its $\langle S\rangle_w$

Remark 1. *Every weakly connected hop dominating set of a connected graph G is a weakly connected dominating set of G.*

Proposition 1. *Every weakly connected hop dominating set of a connected graph G is a hop dominating set of G. Thus, $\gamma_h(G) \leq \gamma_{wh}(G)$.*

Proof. Let $S \subseteq V(G)$ be a weakly connected hop dominating set of G. Then S is a weakly connected and a hop dominating set of G. Thus, the inequality holds. □

Proposition 2. *Let G be any connected graph. Then $S \subseteq V(G)$ is weakly connected hop dominating set of G if and only if S is hop dominating set of G.*

Proof. By Proposition 1, we only need to show that a hop dominating set S of G is a weakly connected hop dominating set of G. Suppose there exists a set $S \subseteq V(G)$ such that S is not a weakly connected hop dominating set. Since by assumption, S is hop dominating set of G. It follows that for all $q \in V(G) \setminus S$, there exists $r \in S$ such that $d_G(q, r) = 2$. Thus, $rq \in E(\langle S\rangle_w)$ where $q \in V(G) \setminus S$ and $r \in S$. Hence, $\langle S\rangle_w = \langle N_G[S], E_w\rangle$ with

$E_w = \{rq \in E(G) : r \in S \text{ or } q \in S\}$. Finally, since for all $q \in V(G) \setminus S$, there exists $r \in S$ such that $d_G(q, r) = 2$ which implies that there exists $w \in V(G)$ with qw, $wr \in E(\langle S\rangle_w)$. Thus, $\langle S\rangle_w$ is connected. Consequently, S is a weakly connected hop dominating set of G. □

Remark 2. *For any connected graph G of order $n \geq 3$, then*

$$2 \leq \gamma_h(G) \leq \gamma_{wh}(G) \leq n.$$

Theorem 1. *Let G be any connected graph of order $n \geq 3$. Then $\gamma_{wh}(G) = 2$ if and only if $\gamma_h(G) = 2$.*

Proof. If $\gamma_h(G) = 2$. By Remark 2, $\gamma_{wh}(G) = 2$. Let $S = \{v_1, v_2\}$ be a weakly connected hop dominating set of G. Then by Remark 1, S is a hop dominating set of G. Therefore, $\gamma_h(G) = 2$. □

Theorem 2. *Let G be any connected graph of order $n \geq 3$. Then, $\gamma_{wh}(G) = n$ if and only if $G = K_n$.*

Proof. Let G be any connected graph of order $n \geq 3$. If $G = K_n$, then $\gamma_{wh}(G) = n$. Suppose $\gamma_{wh}(G) = n$ and $G \neq K_n$. Let $S \subseteq V(G)$ be a weakly connected hop dominating set of G. Then for all $q \in V(G) \setminus S$ there exists $r \in S$ such that $d_G(q, r) = 2$. Thus, $S = V(G) \setminus \{q\}$. Consequently, $\gamma_{wh}(G) \leq |S| = n - 1$, this contradicts to the assumption. □

Theorem 3. *For a complete bipartite $K_{m,n}$ of order $m, n \geq 2$, $\gamma_{wh}(K_{m,n}) = 2$.*

Proof. Write $K_{m,n} = \overline{K_m} + \overline{K_n}$. Let X and Y be the partite sets of $V(K_{m,n})$ where $V(\overline{K_m}) = X$ and $V(\overline{K_n}) = Y$ and let $S = \{x_1, y_1\}$ where $x_1 \in X$ and $y_1 \in Y$ be a weakly connected and dominating set of $K_{m,n}$. Since for every vertex $x \in X \setminus S$ there exists $y_i \in S$, $1 \leq i \leq m$, such that $d_G(x, y_i) = 2$, and for every vertex $y \in Y \setminus S$ there exists $x_j \in S$, $1 < j < n$ such that $d_G(x_j, y) = 2$. It follows that, S is a hop dominating set of $K_{m,n}$. Thus, S is a weakly connected hop dominating set of $K_{m,n}$. Consquently, $\gamma_{wh}(K_{m,n}) =|S| = 2$. □

Corollary 1. *Let G beany connected grap with $\gamma(G) = 1$. Then $\gamma_{wh}(G) = 2$ if and only if $G = K_{m,n}$ for $m = 1$ and $n \geq 2$.*

Corollary 2. *For a star S_n of order $n \geq 2$, $\gamma_{wh}(S_n) = 2$.*

Theorem 4. *Let G be a connected graph of order $n = 4$. Then $\gamma_{wh}(G) = 2$ if and only if G is either P_4, C_4, or S_4.*

Proof. If $G = P_4$, $G = C_4$ or $G = S_4$, then $\gamma_{wh}(G) = 2$. Let $S = \{v_1, v_2\}$ be a weakly connected hop dominating set of G. Since S is a hop dominating set there exist $q, r \in V(G) \setminus S$ such that $d_G(q, v_1) = 2$ and $d_G(r, v_2) = 2$. Now, observe that

$N_G[v_1] = \{v_1, q, v_2\}$ and $N_G(v_2) = \{v_2, v_1, r\}$. Thus, $N_G[S] = \{q, r, v_1, v_2\} = V(G)$ and $E(\langle S\rangle_w) = \{qv_1, v_1v_2, v_1r\} = E_w$ which implies that $\langle S_w\rangle$ is connected for $E(G) = \{qv_1, v_1v_2, v_2r\}$ or $E(G) = \{qv_1, v_1v_2, v_2r, rq\}$. Therefore, G is either path P_4, cycle C_4 or star S_4. □

Proposition 3. *Let n be a positive integer, then*

(i) For a path P_n, $\gamma_{wh}(P_n) = \begin{cases} \lfloor \frac{n+1}{2} \rfloor, \text{if } n = 3,4,5 \text{ ;} \\ 3r+1, \text{if } n = 6r \text{ ;} \\ 3r + \lfloor \frac{s+1}{2} \rfloor, \text{if } n = 6r+s \text{ ; } 1 \leq s \leq 5. \end{cases}$

(ii) For a cycle C_n, $\gamma_{wh}(C_n) = \begin{cases} \lfloor \frac{n+1}{2} \rfloor, \text{if } n = 4,5 \text{ ;} \\ 3r+1, \text{if } n = 6r \text{ ;} \\ 3r + \lfloor \frac{s+1}{2} \rfloor, \text{if } n = 6r+s \text{ ; } 1 \leq s \leq 5. \end{cases}$

(iii) For a wheel W_n of order n with $n-1$ spokes, $\gamma_{wh}(W_n) = 3$.

(iv) For a fan F_n of order n, $\gamma_{wh}(F_n) = 3$.

(v) For a Petersen graph P, $\gamma_{wh}(P) = 4$.

The *shadow graph* $Sh(G)$ of a graph G is constructed by taking two copies of G, say G_1 and G_2. Join each vertex $u \in V(G_1)$ to the neighbors of the corresponding vertex $u' \in V(G_2)$, defined by Jayagopal, et.al. [6].

Theorem 5. (Natarajan, et. al., in [8]) For any connected graph G,

$$\gamma_h(Sh(G)) = \gamma_h(G).$$

Theorem 6. *Let G be any graph of order $n \geq 3$, then $\gamma_{wh}(Sh(G)) = \gamma_{wh}(G)$.*

Proof. Let S be a weakly connected hop dominating set of G. By Remark 1, S is a hop dominating set of G and by Theorem 5, $\gamma_h(Sh(G)) = \gamma_h(G)$. So we are left to show that $\langle S\rangle_w = \langle N_G[S], E_w\rangle$ is connected, where $E_w = \{uv' \in E(Sh(G)) : u \in S \text{ or } v' \in S\}$. Let $u' \in V(Sh(G))$ be a twin vertex of $u \in V(G)$. For every vertex $u' \in V(Sh(G))$, there exists $v \in S \setminus \{u\}$ such that $v \in N_{Sh(G)}[v] \cup N_{Sh(G)}[u]$. Now, $u' \in N_{Sh(G)}(v)$ implies that $u'v \in E(Sh(G))$. Hence, $\langle S\rangle_w$ is connected. Since $d_{Sh(G)}(u', v) = 1$. This implies that $d_{Sh(G)}(u', u) = 2$. Thus, S is a weakly connected hop dominating set of $Sh(G)$. This concludes that $\gamma_{wh}(Sh(G)) = \gamma_{wh}(G)$. □

The next results show the equality of $\gamma_{wh}(Sh(G)) = \gamma_{wh}(G)$.

Corollary 3. *For a shadow graph of P_n, C_n, K_n, and $K_{m,n}$ with m, n vertices.*

(i) For a path P_n of order n,

$$\gamma_{wh}(Sh(P_n)) = \begin{cases} \lfloor \frac{n+1}{2} \rfloor, \text{if } n = 3,4,5 \text{ ;} \\ 3r+1, \text{if } n = 6r \text{ ;} \\ 3r + \lfloor \frac{s+1}{2} \rfloor, \text{if } n = 6r+s \text{ ; } 1 \leq s \leq 5. \end{cases}$$
$$= \gamma_{wh}(P_n)$$

(ii) For a cycle C_n of order n,

$$\gamma_{wh}(Sh(C_n)) = \begin{cases} \lfloor\frac{n+1}{2}\rfloor, \text{if } n = 4, 5 \ ; \\ 3r + 1, \text{if } n = 6r \ ; \\ 3r + \lfloor\frac{s+1}{2}\rfloor, \text{if } n = 6r + s \ ; \ 1 \leq s \leq 5. \end{cases}$$
$$= \gamma_{wh}(C_n)$$

(iii) For a complete graph K_n of order n, $\gamma_{wh}(Sh(K_n)) = n = \gamma_{wh}(K_n)$.

(iv) For a complete bipartite $K_{m,n}$ of order $m, n \geq 2$, $\gamma_{wh}(Sh(K_{m,n})) = 2 = \gamma_{wh}(K_{m,n})$.

The join of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G + H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G), v \in V(H)\}$, Harary [5].

Lemma 1. *Let $G + H$ be the join of G and H. Then $S \subseteq V(G + H)$ is a weakly connected hop dominating set of $G + H$ if and only if either S is a hop dominating set of G or S is a hop dominating set of H.*

Proof. Suppose S is a weakly connected hop dominating set of $G + H$. Suppose $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$. Let $u, v \in S$ and let $u, v \in V(G)$ or $u, v \in V(H)$. Pick $u \in S \cap V(G)$ and $v \in S \cap V(H)$. Then for every $x \in V(G) \setminus S$ there exists $u \in S \cap V(G)$ such that $d_G(x, u) = 2$ or for every $x \in V(H) \setminus S$, there exists $v \in S \cap V(H)$ such that $d_G(x, v) = 2$. Thus, S is a hop dominating set of G or S is a hop dominating set of H.

Conversely, suppose without loss of generality, S is a hop dominating set of G or S is a hop dominating set of H. It follows that S is a hop dominating set of $G + H$. Consider the following cases:

Case i. Let $u, v \in S$. Pick $x \in V(G)$. Then there exists $xu, xv \in E(G + H)$ such that $u, v \in S$. Hence, $x \in N_{G+H}(S)$. It follows that, $N_{G+H}[S] = V(G)$. Hence, $\langle S \rangle_w$ is connected.

Case ii. Let $u, v \in S$. Pick $x \in V(H)$. Similar process to Case i, $\langle S \rangle_w$ is connected.

Therefore, we conclude the fact that S is a weakly connected hop dominating set of $G + H$. □

Proposition 4. *Let $G + H$ be the join of G and H, A set $S \subseteq V(G + H)$ is a weakly connected hop dominating set if and only if one of the following is satisfied:*

(i) $S \subseteq V(G)$ and S is a weakly connected hop dominating set in G.

(ii) $S \subseteq V(H)$ and S is a weakly connected hop dominating set in H.

(iii) $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$.

Theorem 7. *Let $G + H$ be the join of G of order m and H of order n with $\gamma_h(G) = 2$ or $\gamma_h(H) = 2$. Then $\gamma_{wh}(G + H) = 4$.*

Proof. Let $S = \{u_1, u_2, v_1, v_2\}$ where $u_1, u_2 \in V(G)$ and $v_1, v_2 \in V(H)$. Observe that $u_1, u_2, v_1, v_2 \in V(G+H)$ will compromise a set that will induce a complete graph of order 4 in $G+H$. By Theorem 2, $\gamma_{wh}(K_4) = 4$. Also, for every vertex $q \in V(G+H) \setminus S$, there exists an element in S say u_2 or v_2 such that $d_G(q, u_2) = 2$ or $d_G(q, v_2) = 2$. Moreover, $\langle S \rangle_w$ is connected since for every vertex q in $V(G+H) \setminus S$ is adjacent to $u_1, u_2 \in V(G)$ or $v_1, v_2 \in V(H)$ and S is dominating. Hence, S is a weakly connected hop dominating set in $G+H$. Thus $\gamma_{wh}(G+H) = |S| = 4$. □

Corollary 4. *The weakly connected hop domination number of the join of path and cycle graphs of order $m, n \geq 2$ are given as follows:*

(i) $\gamma_{wh}(P_m + P_n) = 4$.

(ii) $\gamma_{wh}(C_m + C_n) = 4$.

(iii) $\gamma_{wh}(P_m + C_n) = 4$.

Proposition 5. *Let $G+H$ be the join of G of order $m \geq 2$ and H of order $n \geq 2$, then $\gamma_{wh}(G+H) = 4$.*

The corona $G \circ H$ of graphs G and H, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex of the i^{th} copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$, $v \in V(G)$, Harary [5].

Theorem 8. *Let G and H be connected graph of orders m and n, respectively. Let $S \subseteq V(G \circ H)$ be a weakly connected hop dominating set of $G \circ H$ and $H^1, H^2, ..., H^n$ be the copies of H. Then*

(i) $S \cap V(H^i) = \varnothing$ *for all* $1 \leq i \leq n$.

(ii) $S \cap V(G) \neq \varnothing$.

Proof. Let $S \subseteq V(G \circ H)$ be γ_{wh}-set of $G \circ H$.

(i) Suppose that $S \cap V(H^i) = \varnothing$ for some i, $1 \leq i \leq n$. Since every vertex of H^i for all i, $1 \leq i \leq n$ is adjacent to exactly one vertex of G. It follows that S is a γ_{wh}-set of $G \circ H$.

(ii) Suppose that $S \cap V(G) \neq \varnothing$. Let u be a vertex of $S \cap V(G)$. Hence $u \in S$ and $u \in V(G)$. Since $|S| = |V(G)|$, $d_G(u, v) = 2$ for all $v \in V(H^i)$ for all i, $1 \leq i \leq n$. Therefore, S is a γ_{wh}-set of $G \circ H$.

□

Theorem 9. (Sandueta, et. al., in [10]) Let G be a connected graph of order $m \geq 2$ and H a graph of order n. Then $S \subseteq V(G \circ H)$ is a weakly connected dominating set of $G \circ H$ if and only if $V(v + H^v) \cap C$ is a dominating set of $v + H^v$ for all $v \in V(G)$ and $V(G) \cap C$ is a weakly connected dominating set of G.

Theorem 10. *Let G be a connected graph of order $m \geq 2$ and H a graph of order n. Then $S \subseteq V(G \circ H)$ is a weakly connected hop dominating set of $G \circ H$ if and only if $S \cap V(v + H^v)$ is a dominating set of $v + H^v$ for all $v \in V(G)$ and $S \cap V(G)$ is a weakly connected hop dominating set of G.*

Proof. Suppose S is a weakly conncted hop dominating set of $G \circ H$ and $u \in V(G)$. By Remark 1, S is a hop dominating set of $G \circ H$. By Theorem 9, if S is a weakly connected dominating set of $G \circ H$ and $u \in V(G)$, then $S \cap V(v + H^v)$ is a dominating set of $(v + H^v)$.

Since S is a weakly connected hop dominating set of $G \circ H$, $S \cap V(G) \neq \varnothing$ (unless, $\langle S \rangle_w$ is disconneted). By Theorem 9, $S \cap V(G)$ is a weakly connected dominating set of G. So we are left to show that $S \cap V(G)$ is hop dominating set of G. Let $u \in S \cap V(G)$ for all $u \in S$ and $u \in V(G)$. Then $|S| = |V(G)|$. Since every vertex of H^v is adjacent to exactly one vertex of $V(G)$, say $x \in V(v + H^v)$, this implies that for each $x \in V(v + H^v)$, there exists $u \in S \cap V(G)$ such that $d_G(u, x) = 2$. Therefore, the conclusion follows.

Conversely, assume that $S \cap V(v + H^v)$ is a dominating set of $v + H^v$ for all $v \in V(G)$ and $S \cap V(G)$ is a weakly connected hop dominating set of G. Suppose $S \subseteq V(G \circ H)$ is not a dominating set of $G \circ H$. Let $x \in V(G) \setminus S$ or $x \in V(v + H^v) \setminus S$. Then there exists $u \in V(G \circ H) \setminus S$ such that $ux \in E(G \circ H)$, which is a contradiction in both cases.

Suppose $\langle S \rangle_w$ is disconnected. Then $\langle S \cap V(G) \rangle_w$ must be disconnected. This implies that $S \cap V(G)$ is not weakly connected hop dominating set of G, which is a contradiction to our assumption. Therefore, $S \subseteq V(G \circ H)$ is a weakly connected hop dominating set of $G \circ H$. □

Theorem 11. *Let G be any connected graph of order $m \geq 2$ and H a graph of order n. A set $S \subseteq V(G \circ H)$ is a weakly connected hop dominating set if $|S| = |V(G)|$.*

Proof. Let $S_1 = V(G)$. Then by Theorem 10, S_1 is a weakly connected hop dominating set of $G \circ H$. Thus, $\gamma_{wh}(G \circ H) \leq |S_1| = |V(G)|$. Next, suppose S_2 is a γ_{wh}-set of $G \circ H$. Then $|S_2 \cap V(v + H^v)| = 1$ for all $v \in V(G)$ since every vertex of H^v is adjacent to exactly one vertex of $V(G)$. Thus, $\gamma_{wh}(G \circ H) = |S_2| = |V(G)|$. Therefore, $\gamma_{wh}(G \circ H) = |V(G)|$. □

Proposition 6. *Let G be a complete graph of order $m \geq 2$ and H be any noncomplete graph of order n. Then $\gamma_{wh}(G \circ H) = |V(G)|$.*

Proposition 7. *Let G be a path of order $m \geq 2$ and H be any noncomplete graph of order n. Then $\gamma_{wh}(G \circ H) = |V(G)|$.*

Proposition 8. *For any positive integers $m, n \geq 2$, we have*

(i) $\gamma_{wh}(P_m \circ K_n) = |V(P_m)|$.

(ii) $\gamma_{wh}(C_m \circ K_n) = |V(C_m)|$.

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex - set $V(G[H]) = V(G) \times V(H)$ such that $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$.

Observe that every non-empty set $F \subseteq V(G) \times V(H)$ can be expressed as $F = \bigcup_{x \in s} [\{x\} \times T_x]$, where $S \subseteq V(G)$, $T_x \subseteq V(H)$ for each $x \in S$ and $T_x = \{y \in V(H) | (x,y) \in C\}$, Harary [5].

Lemma 2. *For any non-trivial connected graphs G and H, a set $F = \bigcup_{x \in s} [\{x\} \times T_x] \subseteq V(G[H])$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$, is a weakly connected hop dominating set of $G[H]$ if and only if S is a weakly connected hop dominating set of G.*

Proof. Suppose F is a weakly connected hop dominating set of $G[H]$. Let $S \subseteq V(G)$ and $y \notin V(G) \setminus S$. Choose $k \in V(H)$. Then $(y,k) \notin V(G[F])$. Observe that for $(y,k) \notin V(G[H]) \setminus F$, there exists $(x,l) \in F$ such that $d_{G[H]}((y,k),(x,l)) = 2$. for all $x \in S$ and $l \in T_x$. It follows that $d_G(x,l) = 2$. Thus, S is a hop dominating set of G. Now, let $w, x \in S$, with $x \neq y$ and $xy \in E(G)$. If $m \in T_w$ and $l \in T_x$. Then $(m,w),(l,x) \in F$ and $d_{G[H]}((m,w),(x,l)) = 1$. This implies that $(m,w)(x,l) \in E(G[H])$. Since $d_{G[H]}((y,k),(x,l)) = 2$ for all $(y,k) \in V(G[H]) \setminus F$, $(y,k)(m,w),(y,k)(x,l) \in E(G[H])$. It follows that $xy, kl \in E(G[H])$ for all $x, y \in S$. Hence, $N_G(S) = V(G)$. So, $\langle S \rangle_w$ is connected. Therefore, S is a weakly connected hop dominating set of G.

Conversely, suppose that $\langle S \rangle_w$ is connected and S is a weakly connected hop dominating set of G. Observe that for every $w \in S$, there exists $y \in S$ such that $d_G(w,y) = 2$. Choose $m \in T_x$. Then $(m,x) \in F$ and $d_{G[H]}((y,k),(x,m)) = 2$. Hence, $(y,k)(x,l) \in E(G[H])$ for all $(y,k) \in V(G(H)) \setminus F$. Thus, $N_{G[H]}(F) = V(G[H])$. Therefore, $\langle F \rangle_w$ is connected. Consequently, F is a weakly connected hop dominating set of $G[H]$. □

Theorem 12. *For any non-trivial connected graphs G and H. a set $F = \bigcup_{x \in s} [\{x\} \times T_x] \subseteq V(G[H])$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$, is a weakly connected hop dominating set of $G[H]$ if and only if S is a weakly connected hop dominating set of G and T_x is a dominating set in H for all $x \in S$.*

Proof. Suppose F is a weakly connected hop dominating set of $G[H]$. Let $y \in V(G) \setminus S$ and choose $k \in V(H)$. Then $(y,k) \in V[G]$. Since F is a dominating set of $G[H]$, there exists $(x,l) \in F$ such that $(y,k)(x,l) \in E(G[H])$. It follows that $x \in S$ and $y \in N_G(x)$. This shows that S is a dominating set of G. Furthermore, by Lemma 2, $\langle S \rangle_w$ is connected. Now, let $v \in V(G) \setminus S$ and choose $k \in V(H)$ for all $k \in T_x$. Then $(x,m) \in V(G[H]) \setminus F$ which implies that $km \in E(H)$ for all $k \in T_x$. Therefore, $N_H[T_x] = V(H)$. That is T_x is a dominating set in H.

Conversely, suppose that S is weakly connected hop dominating set of G. Let $y \in V(G) \setminus S$ and choose $n \in V(H)$. Then $(y,n) \in V(G[H]) \setminus F$ and $ny \in E(H)$ for all $n \in T_x$. Hence,

$N_H[T_x] = V(H)$. This shows that T_x is a dominating set of G. Since $(y, n) \in V(G[H]) \setminus S$, there exists $(x, l), (x, m) \in F$ such that $d_{G[H]}((x, l), (y, n)) = 2$ for all $x \in S$ and $l, m \in V(H)$. This implies that $(x, m)(y, n) \in E(G[H])$. Hence, $N_{G[H]}[F] = V(G[H])$ and $\langle F \rangle_w$ is connected. Consequently, F is a weakly connected hop dominating set of $G[H]$. □

Proposition 9. *The weakly connected hop domination numbers of $P_m[P_n]$, $P_m[C_n]$, and $P_m[K_n]$ for positive integers m, n are given as follows:*

(i) $\gamma_{wh}(P_m[P_n]) = \gamma_{wh}(P_m) \cdot \gamma_{wh}(P_n)$, $m, n \geq 2$

(ii) $\gamma_{wh}(P_m[C_n]) = \gamma_{wh}(P_m) \cdot \gamma_{wh}(C_n)$, $m \geq 2$ *and* $n \geq 3$

(ii) $\gamma_{wh}(P_m[K_n]) = \gamma_{wh}(P_m) \cdot 2$, $m, n \geq 2$

4. Conclusion

This paper was able to introduced the concept of weakly connected hop dominating sets of some graphs and discussed its characterizations in the shadow graph, join, corona and lexicographic product of two graphs. The weakly connected hop domination number of these graphs are determined. For further study, the authors recommend to establish other variants related to the concept of the weakly connected hop dominating sets. Also, the characterization of the weakly connected hop dominating sets under some binary operations that are not discussed in the study and determine the exact values of the parameters of the said graphs are also encouraged.

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References

[1] S.K. Ayyaswamy and C. Natarajan. Hop domination in graphs. *Discussiones Mathematicae Graph Theory.*

[2] G. Domke, J. Hattingh, and L. Markus. On Weakly Connected Domination in Graphs II. *Discrete Mathematics*, 305:112 –122, 2005.

[3] J. Dunbar, J. Grossman, J. Hattingh, S. Hedetniemi, and A. McRae. On Weakly Connected domination in Graphs. *Discrete Mathematics*, pages 261–269, 1997.

[4] J. Hamja, I. Aniversario, and H. Rara. On Weakly Connected Closed Geodetic Domination in Graphs Under Some Binary Operations. *Europian Journal of Pure and Applied Mathematics*, 15(2):736–752, 2022.

[5] F. Harary. *Graph Theory.* Addison-Wesley Publishing Company, Inc., USA, 1969.

[6] R. Jayagopal and V. Raju. Domination Parameters in Shadow Graph and Path Connected Graph. *International Journal of Mathematics and its Applications*, 6(2-B):167–172, 2018.

[7] S. Canoy Jr., R. Mollejon, and J.G E. Canoy. Hop Dominating Sets in Graphs Under Binary Operations. *European Journal of Pure And Applied Mathematics*, 12(4):1455–1463, 2019.

[8] C. Natarajan and S.K. Ayyaswamy. Hop Domination in Graphs-II. *Versita.*, 23(2):187–199., 2015.

[9] R. Patangan, I. Aniversario, and Jr. A. Rosalio. Weakly Connected Closed Geodetic Numbers of Graphs. *International Journal of Mathematical Analysis .*, 10(6):257 – 270., 2016.

[10] E. Sandueta and Jr. S. Canoy. Weakly Connected Domination in Graphs Resulting from Some Graph Operations. *International Mathematical Forum.*, 6(21):1031 – 1035., 2011.

[11] E. Sandueta and Jr. S. Canoy. Weakly Connected Total Domination in Graphs. *International Mathematical Forum.*, 11(11):531 – 537., 2016.

[12] J. Tarr and S. Suen. *Domination in Graphs.* PhD thesis, University of South Florida, 2010.