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On g-Regularity and g-Normality in Fuzzy Soft Topological Spaces

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Abstract. The main aim of this work is to introduce and study the notions of generalized regularity, normality, and symmetric in fuzzy soft topological spaces via fuzzy soft generalized closed sets. Some of their basic properties are investigated. Many related theorems and relations of these notions are presented. Moreover, the hereditary property and some preservation theorems are discussed.

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1. Introduction and Preliminaries

Levine [18] introduced the notion of generalized closed set, briefly g-closed in general topology. A subset B of a topological space (X, τ) is called g-closed, if $cl(B) \subseteq U$ whenever $B \subseteq U$ and U is open in (X, τ). This notion has been studied extensively in topology and fuzzy topology by many authors as in ([3, 5, 7, 8, 13, 17, 25–27, 32]). The investigation of g-closed sets has led to several new and interesting concepts, e.g. g-regular, g-normal spaces, their generalizations which are studied in ([12, 15, 21–24]), and new separation axioms weaker than T_1 are presented. In recent time, the topological structures play an important role in many applications of complex real-life problems in various field, specially the fields that concerned with handling all cases that contain uncertainties such as medical diagnosis and decision making,...etc see e. g. ([10, 11]).

After the discovery of fuzzy set theory by Zadeh [33], many authors generalized and applied this idea in different aspects see e. g. ([1, 2, 14, 19]. The concept of fuzzy topological space was introduced by Chang in [6]. Balasubramanian et. al [5] introduced the

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concept of fuzzy generalized closed sets. Then M. El-Shafei [12] introduced and studied some applications of fuzzy generalized closed sets. Tanay et. al. [30] defined and studied the notion of topological structure for fuzzy soft sets and studied many related concepts. Recently, Tarrannum et. al. [31] introduced the concept of fuzzy soft generalized closed sets, fuzzy soft generalized continuous maps, and studied some properties for them.

In this paper, we define and study the notions of fuzzy soft generalized regular spaces, generalized normal spaces, and symmetric spaces by utilizing fuzzy soft generalized closed sets. We obtain some characterizations. Several related theorems and relationships of them are discussed. In addition, the hereditary property and some preservation theorems are presented.

Throughout this work, U refers to an universe set, E is the set of all parameters, $P(U)$ is the power set of U, I^U is the set of all fuzzy sets on U, where $I = [0, 1], FS$- refers to fuzzy soft, and (U, δ, E) means fuzzy soft topological space. In the next, we recall some basic definitions and notations which are used in this sequel.

A fuzzy set(or F-set) A in U is a mapping $A : U \longrightarrow I$ assigns the value $A(x) \in I$ for all $x \in U$. An F-point x_α is an F-set such that $x_\alpha(y) = \alpha > 0$ if $x = y$ and $x_\alpha(y) = 0$ otherwise for all $y \in U$. We write $x_\alpha \in A$ if $\alpha \leq A(x)$. The class of all F-points of U is denoted by $FP(U)$ [33].

A fuzzy soft set(or FS-set) $f_E = (f, E)$ on U is a mapping $f : E \longrightarrow I^U$ where $f(e) = f_e$ is an F-set on U. Thus f_E can be written as the set of ordered pairs $f_E = \{(e, f(e)) : e \in E, f(e) \in I^U\}$. The class of all FS-sets on U is denoted by $FSS(U)$ [19].

For two FS-sets f_E and g_E on U, we have[19]:
1) f_E is called a null (resp. universal) FS-set, symbolized by $\tilde{0}_E(resp.\tilde{1}_E)$ if $f(e) = \underline{0}(resp.f(e) = \underline{1})$ for all $e \in E$.
2) f_E is a subset of g_E if $f(e) \leq g(e)\ \forall\ e \in E$, symbolized by $f \sqsubseteq g$.
3) f_E and g_E are equal if $f_E \sqsubseteq g_E$ and $g_E \sqsubseteq f_E$. It is symbolized by $f_E = g_E$.
4) The union of f_E and g_E is an FS-set h_E defined by $h(e) = f(e) \vee g(e)$ for all $e \in E$. h_E is symbolized by $f_E \sqcup g_E$.
5) The intersection of f_E and g_E is an FS-set l_E defined by $l(e) = f(e) \wedge g(e)$ for all $e \in E$. l_E is symbolized by $f_E \sqcap g_E$.

An FS-point x_α^e on U is an FS-set (x_α^e, E) given by $x_\alpha^e(e') = x_\alpha$ if $e' = e$ and $x_\alpha^e(e') = \underline{0}$ otherwise, where x_α is an F-point in U with the support x and the value α, $\alpha \in (0, 1]$. An FS-point $x_\alpha^e \tilde{\in} f_E$ if $\alpha \leq f(e)(x)$. The set of all FS-points in U is denoted by $FSP(U)$. We can write $x_\alpha^e \neq y_\beta^e$ if $x \neq y$ [4, 9].

The triple (U, δ, E) is called a fuzzy soft topological space (or $FSTS$) where E is a fixed set of parameters and δ is the class of FS-sets on U which is closed under a finite intersection, an arbitrary union, and 0_E , 1_E belong to δ. The family $FSOS(U)$ (resp. $FSCS(U)$) refers to the set of all FS-open(resp. FS-closed) sets on U [4, 30].

Notation. [29] For $x^e_\alpha \in FSP(U)$, $O_{x^e_\alpha}$ refers to an FSO-set contains x^e_α and is called FSO-nbd of x^e_α , $N_E(x^e_\alpha)$ refers to the set of all FSO-nbds of x^e_α. In general O_{f_E} refers to an FSO-set contains f_E.

An FS-closure of an FS-set h_E in (U, δ, E) denoted by $cl(h_E)$ is the smallest FSC-set on U which contains h_E, and an FS-interior of h_E denoted by $int(h_E)$ is the largest FSO-set contained in h_E. It is clear that $x^e_\alpha \widetilde{\in} int(h_E)$ if and only if there exists $O_{x^e_\alpha} \in \delta$ such that $O_{x^e_\alpha} \sqsubseteq h_E$ [4].

Definition 1. *[16] Let $FSS(U)$ and $FSS(V)$ be two classes of all FS-sets on U, V respectively, and let $p : U \longrightarrow V$ and $u : E \longrightarrow K$ be two maps, then the map f_{up} : $FSS(U) \longrightarrow FSS(V)$ is called an FS-map for which:*
i) If $h_E \in FSS(U_E)$, then the image of h_E denoted by $f_{up}(h_E)$ is an FS-set on V given by $f_{up}(h_E)(k) = sup\{p(h(e)) : e \in u^{-1}(k)\}$ if *$u^{-1}(k) \neq \emptyset$ and $f_{up}(h_E)(k) = 0_K$, otherwise $\forall k \in K$.*
ii) If $g_K \in FSS(V)$, then the preimage of g_K denoted by $f^{-1}_{up}(g_K)$ is an FS-set on U defined by $f^{-1}_{up}(g_K)(e) = p^{-1}(g(u(e)))$ for all $e \in E$.

An FS-map f_{up} is called one-one(onto) if u and p are one-one(onto).
For more details about the properties of image and preimage of the FS-sets see [16].

Definition 2. *[4] The FS-sets h_E and g_E on U are called FS-quasi coincident, denoted by $h_E q g_E$ if there is $e \in E$ and $x \in U$ such that $h(e)(x) + g(e)(x) > 1$. If h_E is not quasi coincident with g_E, we write $h_E \tilde{q} g_E$. In particular, $x^e_\alpha q g_E$ if $\alpha + g(e)(x) > 1$.*

Proposition 1. *[4, 29]*

(i) $f_E \tilde{q} g_E \Leftrightarrow f_E \sqsubseteq g^c_E$.

(ii) $f_E \sqcap g_E = 0_E \Rightarrow f_E \tilde{q} g_E$.

(iii) $f_E \tilde{q} g_E$, $h_E \sqsubseteq g_E \Rightarrow f_E \tilde{q} h_E$.

(iv) $x^e_\alpha \tilde{q} f_E \Leftrightarrow x^e_\alpha \widetilde{\in} f^c_E$.

(v) $f_E \sqsubseteq g_E \Leftrightarrow (x^e_\alpha q f_E \Rightarrow x^e_\alpha q g_E)$.

(vi) $f_E \tilde{q} f^c_E$.

Lemma 1. *[29] For an $FSTS$ (U, δ, E) and $x^e_\alpha \in FSP(U)$, we have:*

(i) $g_E \tilde{q} f_E$ if and only if $g_E \tilde{q} cl(f_E)$ $\forall$ $g_E \in \delta$,

(ii) $x^e_\alpha \tilde{q} cl(f_E)$ if and only if $O_{x^e_\alpha} \tilde{q} f_E$ $\forall$ $O_{x^e_\alpha} \in \delta$.

Definition 3. *[20] An FS-set h_E in (U, δ, E) is said to be regular open (resp. regular closed) if $h_E = int(cl(h_E))$ (resp. $h_E = cl(int(h_E))$). The family of all FS-regular open (resp. all FS-regular closed) on U is denoted by $FSRO(U)$ (resp. $FSRC(U)$).*

Definition 4. *[31] An FS-set f_E in (U,δ,E) is said to be fuzzy soft generalized closed (or FSg-closed) if $cl(f_E) \sqsubseteq h_E$ for all $f_E \sqsubseteq h_E$ and $h_E \in FSOS(U)$. The collection of all FSg-closed sets in (U,δ,E) is denoted by $FSgCS(U)$. The complement of an FSg-closed set is called an FSg-open set.*

Note. Clearly, every FSC-set is an FSg-closed set.

Definition 5. *[28] An $FST(U,\delta,E)$ is said to be:*

(i) FST_0 iff for any $x^e_\alpha, y^e_\beta \in FSP(U)$ with $x^e_\alpha \tilde{q} y^e_\beta$ implies *$x^e_\alpha \tilde{q} cl(y^e_\beta)$ or $cl(x^e_\alpha)\tilde{q} y^e_\beta$.*

(ii) FST_1 iff for any $x^e_\alpha, y^e_\beta \in FSP(U)$ with $x^e_\alpha \tilde{q} y^e_\beta$ implies *$x^e_\alpha \tilde{q} cl(y^e_\beta)$ and $cl(x^e_\alpha)\tilde{q} y^e_\beta$.*

(iii) FST_2 iff for any x^e_α , $y^e_\beta \in FSP(U)$ with $x^e_\alpha \tilde{q} y^e_\beta$, there are $O_{x^e_\alpha}, O_{y^e_\beta} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{y^e_\beta}$.

Definition 6. *[28] An $FSTS$ (U,δ,E) is said to be:*

(i) FSR_2(or FS-regular) iff for any $x^e_\alpha \in FSP(U)$ with $x^e_\alpha \tilde{q} f_E$, f_E is an FSC-set, there are $O_{x^e_\alpha}, O_{f_E} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{f_E}$.

*(ii) FSR_3 (or FS-*normal*) iff for any FSC-sets f_E, g_E with $f_E \tilde{q} g_E$, there are $O_{f_E}, O_{g_E} \in \delta$ such that $O_{f_E} \tilde{q} O_{g_E}$.*

(iii) FST_3 (resp. FST_4) iff it is FSR_2 (resp. FSR_3) and FST_1.

Theorem 1. *[28] $FST_4 \Rightarrow FST_3 \Rightarrow FST_2 \Rightarrow FST_1 \Rightarrow FST_0$.*

Definition 7. *[29] Let (U,τ) be a topological space. The family $\delta = \{\widetilde{\chi}_A : A \in \tau\}$ defines an FST on U induced by τ.*

Definition 8. *[31] An FS-map $f_{up} : (U,\delta,E) \longrightarrow (V,\vartheta,K)$ is said to be:*

(i) FSg-continuous if $f^{-1}_{up}(h_E) \in FSgCS(U)$ for any $h_E \in FSCS(V)$.

(ii) $FSgc$-irresolute if $f^{-1}_{up}(g_E) \in FSgCS(U)$ for any $g_E \in FSgCS(V)$.

Note. Clearly, every $FSgc$-irresolute map is FSg-continuous.

2. Fuzzy soft g-regular spaces

Definition 9. *An $FSTS$ (U,δ,E) is said to be:*

(i) $FST_{\frac{1}{2}}$ iff any FSg-closed set on U is an FSC-set.

(ii) $FST_{2\frac{1}{2}}$ iff for any x^e_α , $y^e_\beta \in FSP(U)$ with $x^e_\alpha \tilde{q} y^e_\beta$, there are $O_{x^e_\alpha}, O_{y^e_\beta} \in \delta$ such that $clO_{x^e_\alpha} \tilde{q} clO_{y^e_\beta}$.

Definition 10. *An $FSTS(U,\delta,E)$ is said to be FSg-regular (or FS-GR_2) if for any FSg-closed set h_E and any FS-point x^e_α with $x^e_\alpha \tilde{q} h_E$, there are $O_{x^e_\alpha}$, $O_{h_E} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{h_E}$.*

Remark 1. *Clearly, every FS-GR_2 space is FSR_2. The next example shows that the converse is not necessarily true.*

Example 1. *Let $U = \{a, b\}$, $E = \{e, t\}$, and $\delta = \{0_E,\ 1_E,\ f_E = \{(e, \{a_{0.5}, b_{0.3}\}),$ $(t, \{a_{0.5}, b_{0.7}\})\},\ g_E = \{(e, \{a_{0.5}, b_{0.3}\}), (t, \{a_{0.5}, b_{0.7}\})\}\ \}$, then δ is FST on U. One can easy to verify that (U, δ, E) is FSR_2 but not FS-GR_2.*

Theorem 2. *An $FSTS$ (U, δ, E) is FS GR_2 if and only if it is FSR_2 and $FST_{\frac{1}{2}}$.*

Proof. If (U, δ, E) is FS-GR_2, then by Remark 1 it is FSR_2. For any FSg-closed set f_E and any FS-point x^e_α with $x^e_\alpha \tilde{q} f_E$ *i.e.* $x^e_\alpha \tilde{\in} f^C_E$, there are $O_{x^e_\alpha}$, $O_{f_E} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{f_E} \implies O_{x^e_\alpha} \tilde{q} f_E \implies x^e_\alpha \tilde{q} cl\,(f_E)$ implies that $x^e_\alpha \tilde{\in} [cl\,(f_E)]^C$. Thus $f^C_E \sqsubseteq [cl\,(f_E)]^C \implies cl\,(f_E) \sqsubseteq f_E$ and so, $f_E = cl\,(f_E)$ this means every FSg-closed set in (U, δ, E) is an FSC-set. The result holds.
Conversely, it is clear.

Theorem 3. *An $FSTS(U, \delta, E)$ is FS-GR_2 if and only if for any FS-point x^e_α and any FSg-open set $O_{x^e_\alpha}$, there is an FSO-set $O^*_{x^e_\alpha}$ such that $cl(O^*_{x^e_\alpha}) \sqsubseteq O_{x^e_\alpha}$.*

Proof. Let (U, δ, E) be FS-GR_2 and $O_{x^e_\alpha}$ be any FSg-open set containing FS-point x^e_α, then $O^c_{x^e_\alpha} = f_E$ which is an FSg-closed set. Since $O_{x^e_\alpha} \tilde{q} O^c_{x^e_\alpha}$ we have, $x^e_\alpha \tilde{q} O^c_{x^e_\alpha}$. Since (U, δ, E) is FS-GR_2, there are $O^*_{x^e_\alpha}$, $O_{O^c_{x^e_\alpha}} \in \delta$ such that $O^*_{x^e_\alpha} \tilde{q} O_{O^c_{x^e_\alpha}} = O_{f_E}$ implies $O^*_{x^e_\alpha} \sqsubseteq O^c_{f_E}$ and so, $cl(O^*_{x^e_\alpha}) \sqsubseteq O^c_{f_E}$. Since $O^c_{x^e_\alpha} \sqsubseteq O_{O^c_{x^e_\alpha}} = O_{f_E}$, we obtain $O^c_{f_E} \sqsubseteq O_{x^e_\alpha}$. Hence $cl(O^*_{x^e_\alpha}) \sqsubseteq O_{x^e_\alpha}$.
Conversely, let x^e_α be any FS-point and g_E be any FSg-closed set with $x^e_\alpha \tilde{q} g_E$, then $x^e_\alpha \in g^c_E = O_{x^e_\alpha}$ which is an FSg-open set containing x^e_α. So by hypothesis, there exists an FSO-set $O^*_{x^e_\alpha}$ such that $cl(O^*_{x^e_\alpha}) \sqsubseteq O_{x^e_\alpha} = g^c_E$ implies $g_E \sqsubseteq [cl(O^*_{x^e_\alpha})]^C = O_{g_E}$ and $cl(O^*_{x^e_\alpha}) \tilde{q} [cl(O^*_{x^e_\alpha})]^C = O_{g_E}$. Therefore $O^*_{x^e_\alpha} \tilde{q} O_{g_E}$. Hence the result holds.

Theorem 4. *An $FSTS$ (U, δ, E) is FS-GR_2 if and only if for any FSg-closed set g_E and any FS-point x^e_α with $x^e_\alpha \tilde{q} g_E$, there are $O_{x^e_\alpha}, O_{g_E} \in \delta$ such that $cl(O_{x^e_\alpha}) \tilde{q} cl(O_{g_E})$.*

Proof. Let (U, δ, E) be an FS-GR_2 space and g_E be any FSg-closed set with $x^e_\alpha \tilde{q} g_E$, there are $O^*_{x^e_\alpha}, O_{f_E} \in \delta$ such that $O_{f_E} \tilde{q} O^*_{x^e_\alpha}$. From Lemma 1, we get $cl(O_{f_E}) \tilde{q} O^*_{x^e_\alpha}$ that is, $cl(O_{f_E}) \tilde{q} x^e_\alpha$. A gain, since (U, δ, E) is FS-GR_2, there are $O^{**}_{x^e_\alpha}$, $O_{cl(O_{f_E})} \in \delta$ such that $O^{**}_{x^e_\alpha} \tilde{q} O_{cl(O_{f_E})}$ implies that $cl(O^{**}_{x^e_\alpha}) \tilde{q} O_{cl(O_{f_E})}$ (by Lemma 1). Take $O_{x^e_\alpha} = O^*_{x^e_\alpha} \sqcap O^{**}_{x^e_\alpha}$ and by the above theorem, there exists $O_{x^e_\alpha} \in \delta$ such that $cl(O_{x^e_\alpha}) \sqsubseteq O^*_{x^e_\alpha}$. Since $cl(O_{f_E}) \tilde{q} O^*_{x^e_\alpha}$, we get $cl(O_{f_E}) \tilde{q} cl(O_{x^e_\alpha})$.
Conversely, It follows directly from hypothesis.

Definition 11. *An $FSTS$ (U, δ, E) is said to be FS-symmetric iff for any FS-points $x^e_\alpha, y^e_\beta \in FSP(U)$ with $x^e_\alpha \tilde{q} cl(y^e_\beta)$ implies $y^e_\beta \tilde{q} cl(x^e_\alpha)$.*

Theorem 5. *An $FSTS$ (U, δ, E) is FS-symmetric if and only if $cl(x^e_\alpha) \tilde{q} g_E$ for any FSC-set g_E with $x^e_\alpha \tilde{q} g_E$.*

Proof. Suppose that g_E is an FSC-set on U with $x_\alpha^e \tilde{q} g_E$. Clearly $cl\left(y_t^e\right) \sqsubseteq g_E$ for all $y_t^e \widetilde{\in} g_E$ and so, $x_\alpha^e \tilde{q} cl(y_t^e)$. Since (U, δ, E) is FS-symmetric, we have $y_\beta^e \tilde{q} cl(x_\alpha^e)$ for all $y_t^e \widetilde{\in} g_E$ and so, for all $y_t^e \widetilde{\in} g_E$ there is an FSO-set $O_{y_t^e}$ containing y_t^e such that $x_\alpha^e \tilde{q} O_{y_t^e}$. Put $h_E = \sqcup \left\{O_{y_t^e} : y_t^e \widetilde{\in} g_E \text{ and } x_\alpha^e \tilde{q} O_{y_t^e}\right\}$, then $h_E = O_{g_E}$ and $x_\alpha^e \tilde{q} h_E$. Thus $x_\alpha^e \widetilde{\in} h_E^C$ and so, $cl(x_\alpha^e) \sqsubseteq h_E^C$ implies $cl(x_\alpha^e) \tilde{q} h_E$. Therefore $cl(x_\alpha^e) \tilde{q} g_E$.
Conversely, it is obvious.

Corollary 1. *An $FSTS$ (U, δ, E) is said to be FS-symmetric if and only if every FS-point $x_\alpha^e \in FSP\left(U\right)$ is an FSg-closed set.*

Remark 2. *Clearly, every FST_1 space is FS-symmetric. The next example shows that the converse may not be true.*

Example 2. *Let $U = \{x\}$, $E = \{e\}$, and $\delta = \{0_E,\ 1_E,\ x_{0.5}^e\}$, then one can verify δ is FS-symmetric but not FST_1. Moreover, δ is not $FT_{\frac{1}{2}}$.*

Proposition 2. *An $FSTS$ (U, δ, E) is FST_1 if and only if it is FS-symmetric and FST_0.*

Proof. Clearly, if (U, δ, E) is FST_1, then it is FS-symmetric and FST_0.
Conversely, let (U, δ, E) be FS-symmetric and FST_0. Suppose $x_\alpha^e \tilde{q} y_t^e$. Then either $x_\alpha^e \tilde{q} cl(y_t^e)$ or $y_t^e \tilde{q} cl(x_\alpha^e)$. By FS-symmetry, we have $x_\alpha^e \tilde{q} cl(y_t^e)$ and $y_t^e \tilde{q} cl(x_\alpha^e)$ for any x_α^e , $y_t^e \in FSP(U)$. The result holds.

Theorem 6. *Every FS-GR_2 space is $FST_{2\frac{1}{2}}$.*

Proof. Let (U, δ, E) be FS-GR_2 and $x_\alpha^e, y_t^e \in FSP(U)$ with $x_\alpha^e \tilde{q} y_t^e$. Then (U, δ, E) is FS-symmetric and so x_α^e is an FSg-closed set. From Theorem 4 there are FSO-sets $O_{x_\alpha^e}$ and $O_{y_t^e}$ such that $cl(O_{x_\alpha^e}) \tilde{q} cl(O_{y_t^e})$. Hence the result holds.

Proposition 3. *For an FS-symmetric space (U, δ, E). The next properties are equivalent:*

(1) (U, δ, E) is FST_0,

(2) (U, δ, E) is $FST_{\frac{1}{2}}$,

(3) (U, δ, E) is FST_1.

Proof. It is obvious.

Definition 12. *An $FSTS$ (U, δ, E) is called FSG_3 iff it is FS-GR_2 and FS-symmetric.*

Proposition 4. *Every FSG_3 space is FST_2.*

Proof. It follows directly from the above Definition, Definition 10, and Corollary 1.

The next example shows that the converse of the above proposition may not be true.

Example 3. *Let U be an infinite set and $E = \{e\}$. For $x,\ y \in U$, $x \neq y$, let h_E be an FS-set on U given by $h(e)(z) = 1$ if $z = x$, $h(e)(z) = 0$ if $z = y$, and $h(e)(z) = 0.5$ if $z \neq x, z \neq y$. Now for any $z \in U$. Consider the FST δ on U generated by the class $\{(h_E)_{x,y} : x, y \in U,\ x \neq y\}$. Then one can check that δ is FST_2 but not FS-GR_2 and so not FSG_3.*

Theorem 7. *An $FSTS(U, \delta, E)$ is FSG_3 if and only if it is FST_3 .*

Proof. Let (U, δ, E) be an FSG_3-space, then it is FS-GR_2 and FS-symmetric. Now, every FS-GR_2 is FSR_2 and every FSG_3 is FST_2. Thus (U, δ, E) is FSR_2 and FST_2. So the result holds.
Conversely, let (U, δ, E) be FST_3, then it is FSR_2 and FT_1 and so, it is $FST_{\frac{1}{2}}$ and FS-symmetric. Thus (U, δ, E) is FSR_2 and $FST_{\frac{1}{2}}$ which implies that (U, δ, E) is FS-GR_2. Since (U, δ, E) is FS-symmetric. Hence (U, δ, E) is FSG_3.

3. Fuzzy soft g-normal spaces

Definition 13. *An $FSTS$ (U, δ, E) is said to be FSg-normal (or FS-GR_3) if for every FSg- closed sets f_E and h_E with $f_E \tilde{q} h_E$, there are FSO-sets O_{f_E} and O_{h_E} containing f_E and h_E respectively, such that $O_{f_E} \tilde{q}\ O_{h_E}$.*

Remark 3. *Clearly, every FS-GR_3 space is FSR_3.*

Theorem 8. *An $FSTS$ (U, δ, E) is FS-GR_3 if and only if for any FSg-closed set f_E and for any FSO-set O_{f_E} containing f_E, there is $O^*_{f_E} \in \delta$ such that $cl(O^*_{f_E}) \sqsubseteq O_{f_E}$.*

Proof. Let (U, δ, E) be an FS-GR_3 space, h_E be any FSg-closed set, and let O_{h_E} be any FSO-set containing h_E, then $O^c_{h_E}$ is an FSC-set. It is known that $O_{h_E} \widetilde{q} O^c_{h_E}$ and so, $h_E \widetilde{q} O^c_{h_E}$. Since (U, δ, E) is FS-GR_3, there are FSO-sets $O^*_{h_E}$ and $O_{O^c_{h_E}}$ such that $O^*_{h_E} \tilde{q} O_{O^c_{h_E}}$ *and so*, $O^*_{h_E} \sqsubseteq O^c_{O^c_{h_E}}$ *and* $cl(O^*_{h_E}) \sqsubseteq O^c_{O^c_{h_E}}$. Since $O^c_{h_E} \sqsubseteq O_{O^c_{h_E}}$ we get $O^c_{O^c_{h_E}} \sqsubseteq O_{h_E}$ and $cl(O^*_{h_E}) \sqsubseteq O^c_{O^c_{h_E}} \sqsubseteq O_{\ h_E}$. Hence the result holds.
Conversely, It follows directly from hypothesis.

Theorem 9. *An $FSTS$ (U, δ, E) is FS-GR_3 if and only if for any FSg-closed sets f_E and g_E with $f_E \tilde{q} g_E$, there are FSO-sets O_{f_E} and O_{g_E} containing f_E and h_E respectively, such that $cl(O_{f_E}) \tilde{q}\ cl(O_{g_E})$.*

Proof. Let (U, δ, E) be FS-GR_3 and f_E, g_E be any FSg-closed sets with $f_E \tilde{q} g_E$, there exist $O^{\#}_{f_E}, O_{g_E} \in \delta$ such that $O^{\#}_{f_E} \tilde{q} O_{g_E} \Longrightarrow O^{\#}_{f_E} \tilde{q} cl(O_{g_E})$ (by Lemma 1). A gain, since (U, δ, E) is FS-GR_3, then there are $O^*_{f_E}, O_{cl(O_{g_E})} \in \delta$ such that $O^*_{f_E} \tilde{q} O_{cl(O_{g_E})} \Longrightarrow cl(O^*_{f_E}) \tilde{q} O_{cl(O_{g_E})}$ (by Lemma 1). Now we put $O_{f_E} = O^{\#}_{f_E} \sqcap O^*_{f_E}$. Since (U, δ, E) is FS-GR_3 and $O^{\#}_{f_E} \in \delta$, by the above theorem there is $O_{f_E} \in \delta$ such that $cl(O_{f_E}) \sqsubseteq O^{\#}_{f_E}$. Since $O^{\#}_{f_E} \tilde{q} cl(O_{g_E})$, we get $cl(O_{f_E}) \tilde{q} cl(O_{g_E})$.
Conversely, It follows directly from hypothesis.

Definition 14. *An $FSTS(U, \delta, E)$ is called FSG_4 iff it is FS-GR_3 and FS-symmetric.*

Theorem 10. *Every FSG_4 space is FSG_3.*

Proof. Let (U, δ, E) be FSG_4, then it is FS-GR_3 and FS-symmetric. Let h_E be an FSg-closed set with $x^e_\alpha \tilde{q} h_E$. Then x^e_α is an FSg-closed set, because U is FS-symmetric. Since (U, δ, E) is FS-GR_3, there are FSO-sets $O_{x^e_\alpha}$ and O_{h_E} such that $O_{x^e_\alpha} \tilde{q} O_{h_E}$. Thus U is FS-GR_2 and so, (U, δ, E) is FSG_3.

Corollary 2. *Every FS-GR_3 and FS-symmetric space is FS-GR_2 .*

Proposition 5. *An $FSTS(U, \delta, E)$ is FS-GR_3 if and only if it is FSR_3 and $FST_{\frac{1}{2}}$.*

Proof. By similar way as that of Theorem 2.

Theorem 11. *An $FSTS(U, \delta, E)$ is FSG_4 if and only if it is FST_4.*

Proof. By similar way as that of Theorem 7.

4. Some properties and relations

Here we shall investigate some preservation theorems and relationships of FS-GR_2 and FS-GR_3 spaces.

Definition 15. *[31] An FS-map $f_{up} : (U, \delta, E) \longrightarrow (V, \sigma, K)$ is said to be:*

(i) FSg-closed if $f_{up}(h_E)$ is FSg-closed in (V, σ, K) for any FSC-set h_E in (U, δ, E).

(ii) FSg-open if $f_{up}(h_E)$ is FSg-open in (V, σ, K) for any FSO-set h_E in (U, δ, E).

Lemma 2. *If $f_{up} : (U, \delta, E) \longrightarrow (V, \sigma, K)$ is an FS-open, FSg-continuous bijection map, then f_{up} is $FSgc$-irresolute.*

Proof. Let $h_E \in FSgC(V)$ and $f^{-1}_{up}(h_E) \sqsubseteq g_E$, where $g_E \in FSO(U)$, then $h_E \sqsubseteq f_{up}(g_E)$. Since f_{up} is FS-open, we have $f_{up}(g_E) \in FSO(V)$. Since h_E is an FSg-closed set on V, we obtain $cl(h_E) \sqsubseteq f_{up}(g_E)$. Hence $f^{-1}_{up}(cl(h_E) \sqsubseteq g_E$ (because f_{up} is one-one). Since f_{up} is FSg-continuous, we have $f^{-1}_{up}(cl(h_E)$ is an FSg-closed set in U and so, $cl(f^{-1}_{up}(h_E\ )) \sqsubseteq cl(f^{-1}_{up}\ (cl\ (h_E))) \sqsubseteq g_E$. Hence $f^{-1}_{up}(h_E)$ is an FSg-closed set on V.

Theorem 12. *If $f_{up} : (U, \delta, E) \longrightarrow (V, \sigma, K)$ is an FS-open, FSg-continuous bijection map and (U, δ, E) is FS-GR_2, then (V, σ, K) is FS-GR_2.*

Proof. Let $h_E \in FSgC(V)$ and $y^e_\alpha \tilde{q} h_E$. Since f_{up} is FS-open, FSg-continuous bijective, by the above lemma, f_{up} is $FSgc$-irresolute and so, $f^{-1}_{up}(h_E)$ is FSg-closed. Take $f_{up}\,(x^e_\alpha) = y^e_\alpha$, then $x^e_\alpha \tilde{q} f^{-1}_{up}(h_E))$. Since U is FS-GR_2, there are FSO-sets $O_{x^e_\alpha}$ and $O_{f^{-1}_{up}(h_E)}$ such that $O_{x^e_\alpha} \tilde{q} O_{f^{-1}_{up}(h_E)}$. Since f_{up} is FS-open and bijective, we have $y^e_\alpha \widetilde{\in} f_{up}(O_{x^e_\alpha})$, $h_E \sqsubseteq f_{up}(O_{f^{-1}_{up}(h_E)})$ and $f_{up}(O_{x^e_\alpha}) \tilde{q} f_{up}(O_{f^{-1}_{up}(h_E)})$. The result holds.

Theorem 13. *If $f_{up} : (U, \delta, E) \longrightarrow (V, \sigma, K)$ is an FSg-continuous, FSg-closed one-one map and (V, σ, K) is FS-GR_2, then (U, δ, E) is FS-GR_2.*

Proof. Let $h_E \in FSgc(U)$ and $x^e_\alpha \tilde{q} h_E$. By FS-continuity and FSg-closedness we have $f_{up}(h_E) \in FSgc(V)$. Indeed, if $f_{up}(h_E) \sqsubseteq g_E$ and g_E is an FSO-set in (V, σ, K), we have $h_E \sqsubseteq f^{-1}_{up}(g_E)$, and so $cl(h_E) \sqsubseteq f^{-1}_{up}(g_E)$. Thus $f_{up}(h_E) \sqsubseteq f_{up}(cl\,(h_E)) \sqsubseteq f_{up} f^{-1}_{up}(g_E) \sqsubseteq g_E$. So $cl(h_E) \sqsubseteq g_E$. Thus $f_{up}(h_E)$ is FSg-closed. Since f_{up} is one-one, we get $f_{up}\,(x^e_\alpha)\,\tilde{q} f_{up}(h_E)$. Since (V, σ, K) is FS-GR_2, there exist FSO-sets $O_{f_{up}(x^e_\alpha)}$ and $O_{f_{up}(h_E)}$ such that $O_{f_{up}(x^e_\alpha)} \tilde{q} O_{f_{up}(h_E)}$. So, we get $x^e_\alpha \tilde{\in} f^{-1}_{up}(O_{f_{up}(x^e_\alpha)})$, $h_E \sqsubseteq f^{-1}_{up}(O_{f_{up}(h_E)})$ and $f^{-1}_{up}(O_{f_{up}(x^e_\alpha)}) \tilde{q} f^{-1}_{up}(O_{f_{up}(h_E)})$. Since f_{up} is FS-continuous, we get $f^{-1}_{up}(O_{f_{up}(x^e_\alpha)})$ and $f^{-1}_{up}(O_{f_{up}(h_E)})$ are FSO-sets in (U, δ, E). The result holds.

Theorem 14. *If $f_{up} : (U, \delta, E) \longrightarrow (V, \sigma, K)$ is FS-continuous, FSg-closed one-one and (V, σ, K) is FS-GR_3, then (U, δ, E) is FS-GR_3.*

Proof. Let $h_E, g_E \in FSgCS(U)$ with $h_E \tilde{q} g_E$. As in the above theorem $f_{up}\,(h_E)$ and $f_{up}\,(g_E) \in FSgC(V)$. Since f_{up} is one-one, we have $f_{up}(h_E) \tilde{q} f_{up}(g_E)$. Since (U, δ, E) is FS-GR_3, there are FSO-sets $O_{f_{up}(h_E)}, O_{f_{up}(g_E)}$ such that $O_{f_{up}(h_E)} \tilde{q}\; O_{f_{up}(g_E)}$. So we get, $h_E \sqsubseteq f^{-1}_{up}(O_{f_{up}(h_E)})$, $g_E \sqsubseteq f^{-1}_{up}(O_{f_{up}(g_E)})$ and $f^{-1}_{up}(O_{f_{up}(h_E)}) \tilde{q} f^{-1}_{up}(O_{f_{up}(g_E)})$. Since f_{up} is FS-continuous, we get $f^{-1}_{up}(O_{f_{up}(h_E)})$ and $f^{-1}_{up}(O_{f_{up}(g_E)})$ are FSO-sets in (U, δ, E). The proof is complete.

Theorem 15. *If $f_{up} : (U, \delta, E) \longrightarrow (V, \sigma, K)$ is FS-open, FSg-continuous bijection, and (U, δ, E) is FS-GR_3, then (V, σ, K) is FS-GR_3.*

Proof. It is analogous to that of the above theorem.

Theorem 16. *If $f_{up} : (U, \delta, E) \longrightarrow (V, \sigma, K)$ is $FSgc$-irresolute, FS-open onto and (U, δ, E) is FS-GR_3, then (V, σ, K) is FS-GR_3.*

Proof. It is similar to that of Theorem 14.

The next two theorems show that FS-GR_2 and FS-GR_3 are hereditary property.

Theorem 17. *Every FS-subspace $(\tilde{V}_E, \delta_V, E)$ of FS-GR_2 is FS-GR_2 .*

Proof. Let (U, δ, E) be FS-GR_2. Suppose that h_E any FSg-closed set in $(\tilde{V}_E, \delta_V, E)$ with $x^e_\alpha \tilde{q} h_E$ for any FS-point in $(\tilde{V}_E, \delta_V, E)$, then there is an FSC-set and so FSg-closed set f_E in (U, δ, E) with $h_E = \tilde{V}_E \sqcap f_E$ and $x^e_\alpha \tilde{q} f_E$. Since (U, δ, E) is FS-GR_2, there are $O_{x^e_\alpha}, O_{f_E} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{f_E}$. Now take $O^*_{x^e_\alpha} = \tilde{V}_E \sqcap O_{x^e_\alpha} \in \delta_V$ and $O^*_{f_E} = \tilde{V}_E \sqcap O_{f_E} \in \delta_V$, then $O^*_{x^e_\alpha}$ and $O^*_{f_E}$ are FSO-sets in $(\tilde{V}_E, \delta_V, E)$ containing x^e_α and f_E respectively, such that $O^*_{x^e_\alpha} \tilde{q} O^*_{f_E}$. The result holds.

Theorem 18. *Every FSC-subspace $(\tilde{V}_E, \delta_V, E)$ of FS-GR_3 is FS-GR_3.*

Proof. It is similar to that of the above theorem.

Theorem 19. *(U, δ_τ, E) is FS-GR_2 if and only if (U, τ) is g-regular.*

Proof. Let (U, δ_τ, E) be FS-GR_2 and B any closed set in (U, τ) with $x \notin B$, then B is a g-closed set and there is an FSC-set f_E such that $f_E = \widetilde{\chi}_B$. Clearly f_E is an FSg-closed set with $x_1^e \tilde{q} f_E$. Since (U, δ_τ, E) is FS-GR_2, there are $O_{x_1^e}, O_{f_E} \in \delta_\tau$ such that $O_{x_1^e} \tilde{q} O_{f_E}$. Thus there are $O_x, O_B \in \tau$ such that $O_{x_1^e} = \widetilde{\chi}_{O_x}, O_{f_E} = \widetilde{\chi}_{O_B}$ and $O_x \cap O_B = \emptyset$. Therefore (U, τ) is g-regular.
Conversely, let (U, τ) be g-regular and h_E any closed set in (U, δ_τ, E) with $x_\alpha^e \tilde{q} h_E$. Then h_E is an FSg-closed set and there is a closed set F in (U, τ) such that $h_E = \widetilde{\chi}_{O_F}$ and $x \notin F$. Clearly F is g-closed and (U, τ) is g-regular, then there are $O_x, O_F \in \tau$ with $O_x \cap O_F = \emptyset$ and so, there are $O_{x_\alpha^e}$ and $O_{h_E} \in \delta_\tau$ such that $O_{x_\alpha^e} = \widetilde{\chi}_{O_x}$, $O_{h_E} = \widetilde{\chi}_{O_F}$ and $O_{x_\alpha^e} \tilde{q} O_{h_E}$. Hence (U, δ_τ, E) is FS-GR_2.

Theorem 20. *(U, δ_τ, E) is FS-GR_3 if and only if (U, τ) is g-normal.*

Proof. It is similar to that of the above theorem.

From the obtained results in section 2, 3. we conclude the next relations.

Corollary 3. *For An $FSTS$ (U, τ, E), the next implications hold.*
1) $FSG_4 \Leftrightarrow FST_4 \Rightarrow FST_3 \Leftrightarrow FSG_3 \Leftrightarrow FS\text{-}GR_2 \wedge FS\text{-symmetric} \Rightarrow FSR_2$.
2)$FSG_3 \Rightarrow FST_{2\frac{1}{2}} \Rightarrow FST_2 \Rightarrow FST_1 \Rightarrow FST_0$.

5. Conclusion

The topological structures play an important role in many applications of complex real-life problems in various field, specially the fields that concerned with handling all cases that contain uncertainties such as medical diagnosis , economic, and decision making,..etc. In this work, we introduced and studied the new classes of spaces namely, FSg-regular and FSg-normal space via fuzzy soft generalized closed sets. We investigated some characterizations for them. Some related theorems and relations are presented with some necessary examples. In addition, the hereditary property and some preservation theorems. In the future work we will try to present some applications for fuzzy soft generalized sets in different aspects.

6. Conflicts of interest

The authors declare no conflict of interest.

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