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Multipolar fuzzy KU-ideals in KU-algebras

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Abstract. This article presents the idea of an m-polar fuzzy KU-ideal and investigates its characteristics. The relationship between an m-polar fuzzy KU-subalgebra and an m-polar fuzzy KU-ideal is presented in the discussion

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1. Introduction

A fuzzy set is a useful tool developed by Zadeh [11] for dealing with probabilistic uncertainty related to perceptions, state inaccuracies, and preferences. Since that time, fuzzy set theory has gained much attention in a variety of disciplines, including graph theory, statistics, life and medical sciences, engineering, social sciences, decision-making, computer networks, robotics automata theory, artificial intelligence, pattern recognition, and many others. In [8] and [9], a new algebraic structure called KU-algebras was constructed. Mostafa et al. [7] introduced the notion of fuzzy KU-ideals of KU-algebras and then investigated several basic properties related to fuzzy KU-ideals. Recently, Akram and Sarwar [3] applied the notion of m-polar fuzzy set theory to the graph theory. Also, Al-Masarwah and Ahmad [4] discussed the notion of m-polar fuzzy sets with an application to BCK/BCI-algebras. In this study, we will introduce the notions of m-polar fuzzy subalgebras and m-polar fuzzy (closed, commutative) ideals and investigate several properties. This manuscript aims to apply the notion of an m-polar fuzzy set to fuzzy KU-ideal in KU-algebras. The notions of an m-polar fuzzy KU-ideal were introduced and their properties were investigated. The relationship between an m-polar fuzzy KU-subalgebra and an m-polar fuzzy KU-ideal was examined. Moreover, the relationship between an m-polar fuzzy KU-ideal and the ideal of BCK/BCI-algebras was presented.

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2. Preliminaries

We first recall some elementary aspects which are used in the present paper. Throughout this paper, X always denotes a KU-algebra without any specifications.

Definition 1 (8). *Let X be a nonempty set with a binary operation $*$ and a constant 0, then $(X, *, 0)$ is called a KU -algebra, if for all $u, v, w \in X$ the following axioms are hold:*
(ku 1) $(u * v) * [(v * w)) * (u * w)] = 0$,
(ku 2) $u * 0 = 0$,
(ku 3) $0 * u = u$,
(ku 4) $u * v = 0$ *and* $v * u = 0$ *implies* $u = v$,
(ku 5) $u * u = 0$

On a KU-algebra $(X, *, 0)$ we can define a binary relation $\leq$ on X by putting: $u \leq v \iff v * u = 0$. Then $(X, \leq)$ is a partially ordered set and 0 is its smallest element. Thus $(X, *, 0)$ satisfies the following conditions: for all $u, v, w \in X$.
(1) : $(v * w) * (u * w) \leq (u * v)$
(2): $0 \leq u$
(3): $u \leq v, v \leq u$ implies $u = v$,
(4): $v * u \leq u$.

A subset S of a KU-algebra X is called KU-subalgebra of X, if $u, v \in S$, implies $(u * v) \in S$. A non- empty subset I of a KU-algebra X is said to be a KU-ideal of X if it satisfies:
(K1) $0 \in I$,
(K2) $u * (v * w) \in I$ and $v \in I$ imply $u * w \in I$ for all u, v and $w \in X$.

Theorem 1 (7). *In a KU-algebra $(X, *, 0)$, the following axioms are satisfied: for all $u, v, w \in X$,*
(1) $u \leq v$ *imply* $v * w \leq u * w$,
(2) $u * (v * w) = v * (u * w)$ *,for all* $u, v, w \in X$,
(3) $((v * u) * u) \leq v$

Definition 2 (7). *Let μ be a fuzzy set on a KU-algebra X, then μ is called a fuzzy KU-subalgebra of X if $\mu(u * v) \geq min\{\mu(u), \mu(v)\}$, for all $u, v \in X$.*

Definition 3 (7). *Let X be a KU-algebra. A fuzzy set μ in X is called a fuzzy KU-ideal of X if it satisfies:*
(FK1) $\mu(0) \geq \mu(u)$, *(FK2)* $\mu(u * w) \geq min\{\mu(u * (v * w)), \mu(v)\}$, *for all u, v and $w \in X$.*

Lemma 1 (7). *If A fuzzy KU-subalgebra of X, then $\mu(0) \geq \mu(u)$, for all $u \in X$.*

Proposition 1 (7). *If A fuzzy KU-ideal of X and $u \leq v$, then $\mu(u) \geq \mu(v)$, for all $u, v \in X$.*

Theorem 2 (7). *A fuzzy KU-ideal of X is a fuzzy KU-subalgebra of X.*

By an m-polar fuzzy set of a set X (see [5]), we mean a function $\hat{O} : X \to [0,1]^m$. The membership value of every element $u \in X$ is denoted by

$$\hat{O}(u) := \{(\pi_1 \circ \hat{O})(u), (\pi_2 \circ \hat{O})(u), ..., (\pi_m \circ \hat{O})(u)\},$$

Where $\pi_i : [0,1]^m \to [0,1]$ is the i-th projection for all $i = 1, 2, ..., m$.
Given an m-polar fuzzy set on a set X, we consider the set

$$U(\hat{O}; \hat{r}) := \{u \in X | \hat{O}(u) \geq \hat{r}\}$$

that is,

$$U(\hat{O}; \hat{r}) := \{u \in X | (\pi_i \circ \hat{O})(u) \geq r_i, i = 1, 2, ..., m\},$$

which is called an m-polar $\hat{r}$-level cut set of $\hat{O}$.

$$\hat{O}(v) = \begin{cases} \hat{t} = (t_1, t_2, ..., t_m) \in (0,1]^m & ; u = v \\ \hat{0} = (0, 0, ...0) & ; u \neq v \end{cases}$$

and it is denoted by $u_{\hat{t}}$. We say that u is the support of $u_{\hat{t}}$, and $\hat{t}$ is the value of $u_{\hat{t}}$.

We say that an m-polar fuzzy point $u_{\hat{t}}$ is contained in an m-polar fuzzy set $\hat{O}$ denoted by $u_{\hat{t}} \in \hat{O}$, if $\hat{O}(u) \geq \hat{t}$, that is, $(\pi_i \circ \hat{O})(u) \geq t_i$ for all $i = 1, 2, ..., m$.

Definition 4 (4). *An m-polar fuzzy set $\hat{O}$ of BCK/BCI-algebra X is called an m-polar fuzzy subalgebra if the following assertion is valid:*

$$\hat{O}(u * v) \geq min\{\hat{O}(u), \hat{O}(v)\}$$

that is,

$$(\pi_i \circ \hat{O})(u * v) \geq min\{(\pi_i \circ \hat{O})(u), (\pi_i \circ \hat{O})(v)\}$$

for all $u, v \in X, i = 1, 2, ..., m$.

Definition 5 (4). *An m-polar fuzzy set $\hat{O}$ of BCK/BCI-algebra X is called an m-polar fuzzy ideal if the following assertion is valid:*

$$\hat{O}(0) \geq \hat{O}(u) \geq min\{\hat{O}(u * v), \hat{O}(v)\}$$

that is,

$$(\pi_i \circ \hat{O})(0) \geq (\pi_i \circ \hat{O})(u) \geq min\{(\pi_i \circ \hat{O})(u * v), (\pi_i \circ \hat{O})(v)\}$$

for all $u, v \in X, i = 1, 2, ..., m$.

3. m-Polar fuzzy KU-subalgebras and KU-ideals

In this section, we introduce the notions of an m-polar fuzzy KU-subalgebras, an m-polar fuzzy KU-ideals in KU-algebras and investigate some of their related properties.

Definition 6. *An m-polar fuzzy set $\hat{O}$ of X is called an m-polar fuzzy KU-subalgebra if the following assertion is valid for all $u, v \in X$.*

$$\hat{O}(u * v) \geq min\{\hat{O}(u), \hat{O}(v)\} \tag{1}$$

that is,

$$(\pi_i \circ \hat{O})(u * v) \geq min\{(\pi_i \circ \hat{O})(u), (\pi_i \circ \hat{O})(v)\}$$

for all $u, v \in X, i = 1, 2, ..., m$.

Example 1. *Let $X = \{0, 1, 2, 3, 4\}$ be KU-algebra with a binary operation $*$ defined by the following table*

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	3
2	0	0	0	1	4
3	0	0	0	0	3
4	0	0	0	0	0

Define a 3-polar fuzzy set $\hat{O} = X \to [0, 1]^3$ by:

$$\hat{O}(u)=\begin{cases} (0.3, 0.4, 0.6) & ; u = 0 \\ (0.2, 0.3, 0.2) & ; u = 1 \\ (0.1, 0.2, 0.3) & ; u = 2 \\ (0.2, 0.3, 0.4) & ; u = 3 \\ (0.2, 0.3, 0.5) & ; u = 4 \end{cases}$$

It is routine to verify that $\hat{O}$ is a 3-polar fuzzy KU-subalgebra of X.

Theorem 3. *Let $\hat{O}$ be an m-polar fuzzy set of X. Then $\hat{O}$ is an m-polar fuzzy KU-subalgebra of X if and only if $U(\hat{O}; r) \neq \phi$ is a KU-subalgebra of X for all $\hat{r} = (r_1, r_2, ..., r_m) \in [0, 1]^m$.*

Proof. Assume that $\hat{O}$ is an m-polar fuzzy subalgebra of X and let $\hat{r} \in [0, 1]^m$ be such that $U(\hat{O}; r) \neq \phi$. Let $u, v \in U(\hat{O}; \hat{r})$. Then $\hat{O}(u) \geq \hat{r}$ and $\hat{O}(v) \geq \hat{r}$. It follows from definition 6 that $\hat{O}(u * v) \geq min\{\hat{O}(u), \hat{O}(v)\} \geq \hat{r}$, so that $(u * v) \in U(\hat{O}; \hat{r})$. Hence $U(\hat{O}; \hat{r})$ is a subalgebra of X.

Conversely, assume that $U(\hat{O}; \hat{r})$ is a subalgebra of X. Suppose that there exist $u, v \in X$ such that $\hat{O}(u * v) < min\{\hat{O}(u), \hat{O}(v)\}$. Then there exists $\hat{r} = (r_1, r_2, ..., r_m) \in [0, 1]^m$ such taht $\hat{O}(u * v) < \hat{r} \leq min\{\hat{O}(u), \hat{O}(v)\}$. It follows that $u, v \in U(\hat{O}; \hat{r})$, but $u * v \notin U(\hat{O}; \hat{r})$. This is a contradiction, and so $\hat{O}(u * v) \geq min\{\hat{O}(u), \hat{O}(v)\}$, $\forall u, v \in X$. Therefore $\hat{O}$ is an m-polar fuzzy KU-subalgebra of X.

Lemma 2. *Every m-polar fuzzy subalgebra $\hat{O}$ of X satisfies the following inequality:*

$$(\forall u \in X)(\hat{O}(0) \geq \hat{O}(u)) \tag{2}$$

that is,

$$(\pi_i \circ \hat{O})(0) \geq (\pi_i \circ \hat{O})(u)$$

for all $u \in X, i = 1, 2, ..., m$.

Proof. Note that $u * u = 0$ for all $u \in X$. Using definition 6, we have

$$\hat{O}(0) = \hat{O}(u * u) \geq min\{\hat{O}(u), \hat{O}(u)\} = \hat{O}(u).$$

for all $u \in X$.

Proposition 2. *If every an m-polar fuzzy subalgebra $\hat{O}$ of X satisfies the following inequality:*

$$(\forall u, v \in X)(\hat{O}(u * v) \geq \hat{O}(v)) \tag{3}$$

Then, $\hat{O}(0) = \hat{O}(u)$
that is,

$$(\pi_i \circ \hat{O})(u * v) \geq (\pi_i \circ \hat{O})(v)$$

Then, $(\pi_i \circ \hat{O})(0) = (\pi_i \circ \hat{O})(u)$, for all $u, v \in X, i = 1, 2, ..., m$.

Proof. Let $u \in X$. Using (ku 2) and (3), we have $\hat{O}(u) = \hat{O}(u * 0) \geq \hat{O}(0)$.
It follows from Lemma 2 that $\hat{O}(0) = \hat{O}(u)$.

Definition 7. *An m-polar fuzzy set $\hat{O}$ of X is called an m-polar fuzzy KU-ideal if the following conditions are valid:*

$$(\forall u \in X)(\hat{O}(0) \geq \hat{O}(u))$$

$$(\forall u, v, w \in X)(\hat{O}(u * w) \geq min\{\hat{O}(u * (v * w)), \hat{O}(v)\} \tag{4}$$

that is,

$$(\forall u \in X)((\pi_i \circ \hat{O})(0) \geq (\pi_i \circ \hat{O})(u))$$

$$(\forall u, v, w \in X)((\pi_i \circ \hat{O})(u * w) \geq min\{(\pi_i \circ \hat{O})(u * (v * w)), (\pi_i \circ \hat{O})(v)\}$$

for all $i = 1, 2, ..., m$.

Proposition 3. *If $\hat{O}$ is an m-polar fuzzy KU-ideal of X and $u \leq v$, then*

$$(\hat{O}(u) \geq \hat{O}(v))(\forall u, v \in X) \tag{5}$$

that is,

$$((\pi_i \circ \hat{O})(u) \geq (\pi_i \circ \hat{O})(v))(\forall u, v \in X, i = 1, 2, ..., m)$$

Proof. If $u \leq v$, then $v * u = 0$ and (ku 3) $0 * u = u$. Since $\hat{O}$ is an m-polar fuzzy KU-ideal of X, we get

$$\hat{O}(0 * u) = \hat{O}(u) \geq min\{\hat{O}(0 * (u * v)), O(v)\} = min\{\hat{O}(0 * 0), O(v)\} = min\{\hat{O}(0), O(v)\} = \hat{O}(v).$$

for all $u, v \in X$.

Proposition 4. *Let $\hat{O}$ be an m-polar fuzzy KU-ideal of X. If $u * v \leq w$, holds in X then,*

$$(\hat{O}(v) \geq min\{\hat{O}(u), \hat{O}(w)\})(\forall u, v, w \in X) \tag{6}$$

that is,

$$((\pi_i \circ \hat{O})(v) \geq min\{(\pi_i \circ \hat{O})(u), (\pi_i \circ \hat{O})(w)\})(\forall u, v, w \in X, i = 1, 2, ..., m)$$

Proof. Assume that the inequality $u * v \leq w$, holds in X. Then $w * (u * v) = 0$ and (4)

$$\hat{O}(u*v) \geq min\{\hat{O}(u*(w*v)), \hat{O}(w)\} = min\{\hat{O}(w*(u*v)), \hat{O}(w)\} = min\{\hat{O}(0), \hat{O}(w)\} = \hat{O}(w) \tag{7}$$

Now,

$$\hat{O}(0 * v) = \hat{O}(v) = min\{\hat{O}(0 * (u * v)), \hat{O}(u)\} = min\{\hat{O}(u * v), \hat{O}(u)\} \geq min\{\hat{O}(w), \hat{O}(u)\}$$

(by using (7)), i.e. $\hat{O}(v) \geq min\{\hat{O}(u), \hat{O}(w)\}$. This completes the proof.

Theorem 4. *If $\hat{O}$ is an m-polar fuzzy KU-subalgebra of X satisfies the condition in proposition 4, then $\hat{O}$ is an m-polar fuzzy KU-ideal of X.*

Proof. Let $\hat{O}$ be an m-polar fuzzy KU-subalgebra of X satisfies the condition in proposition 4 and by lemma 2. We have $\hat{O}(0) \geq \hat{O}(u)$ for all $u \in X$. By theorem 1(3), we have $(u * (v * w)) * (u * w) \leq v$, for all $u, v, w \in X$. it follows from proposition 4, that $\hat{o}(u * w) \geq min\{\hat{O}(u * (v * w)), \hat{O}(v)\}$ for all $u, v, w \in X$. Therefore, $\hat{O}$ is an m-polar fuzzy KU-ideal of X.

Proposition 5. *Every m-polar fuzzy KU-ideal of X is an m-polar fuzzy ideal.*

Proof. Straightforward.

Proposition 6. *If $\hat{O}$ is an m-polar fuzzy KU-ideal of X, then*

$$(\hat{O}(u * (u * v)) \geq \hat{O}(v))(\forall u, v \in X) \tag{8}$$

that is,

$$((\pi_i \circ \hat{O})(u * (u * v)) \geq (\pi_i \circ \hat{O})(v))(\forall u, v \in X, i = 1, 2, ..., m)$$

Proof. Let $\hat{O}$ be an m-polar fuzzy KU-ideal of a KU-algebra X and let $u, v, w \in X$. Taking $w = u * v$ in (4) and using (ku 2), we get

$$\begin{aligned} \hat{O}(u * (u * v) &\geq min\{\hat{O}(u * (v * (u * v))), \hat{O}(v)\} \\ &= min\{\hat{O}(u * (u * (v * v))), \hat{O}(v)\} \\ &= min\{\hat{O}(u * (u * 0)), \hat{O}(v)\} \\ &= min\{\hat{O}(0)), \hat{O}(v)\} = \hat{O}(v) \end{aligned}$$

Theorem 5. *If $\hat{O}$ is an m-polar fuzzy KU-ideal of X, then the set $B = \{u \in X : \hat{O}(u) = \hat{O}(0)\}$ is an m-polar KU-ideal.*

Proof. Since $0 \in X$, then $\hat{O}(0) = \hat{O}(0)$ implies $0 \in B$, so $B \neq \phi$. Let $u * (v * w) \in B$ and $v \in B$ implies $\hat{O}(u * (v * w)) = \hat{O}(0)$ and $\hat{O}(v) = \hat{O}(0)$. Since $\hat{O}$ is an m-polar fuzzy KU-ideal of X, then

$$(\hat{O}(u * w)) \geq min\{\hat{O}(u * (v * w)), \hat{O}(v)\} = \hat{O}(0).$$

But $\hat{O}(0) \geq \hat{O}(u*w)$. Then $\hat{O}(0) = \hat{O}(u*w)$, it follows that $u*w \in B$, for all $u, v, w \in X$. Hence, the set B is an m-polar KU-ideal.

4. Conclusion

An m-polar fuzzy model is a generalized form of a bipolar fuzzy model. The m-polar fuzzy models provide more precision, flexibility and compatibility to the system when more than one agreement is to be dealt with. This article discussed the KU-ideal of KU-algebras based on m-polar fuzzy sets. The notions of m-polar fuzzy KU-subalgebras and an m-polar fuzzy KU-ideals were introduced, and several properties were investigated.

5. Compliance with ethical standards

Conflict of interest: The author declares that there is no conflict of interest regarding the publication of this paper.

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