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Double Fuzzy δ-Continuous Functions in Double Fuzzy Topological Spaces

Wadei AL-Omeri

Department of Mathematics, Faculty of Science, Jadara University, Irbid, Jordan

Abstract. In this paper, we introduce $(\mathfrak{r}, \mathfrak{s})$-$\delta$-fuzzy closed sets in double fuzzy topological spaces and investigate some of their properties. Moreover, we introduce the concept of double fuzzy δ-continuous functions. Several interesting properties and characterizations are introduced and discussed. Furthermore, the relationships among the new concepts are introduced and established with some interesting counterexamples.

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1. Preliminaries

The concept of fuzzy sets was introduced by Zadeh in his classical paper [1]. In 1968, Chang [2] used fuzzy sets to introduce the notion of fuzzy topological spaces. Çoker [3, 4] defined the intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. Later on, Demirci and Çoker [5] defined intuitionistic fuzzy topological spaces which is a generalization of fuzzy topological spaces and intuitionistic fuzzy topological spaces. Mondal and Samanta [6] succeeded to make the topology itself intuitionistic. The resulting structure is given the new name ”intuitionistic gradation of openness”. The name ”intuitionistic” did not continue due to some doubts that were thrown about the suitability of this term. These doubts were quickly ended in 2005 by Gutiérrez García and Rodabaugh [7]. They proved that this term is unsuitable in mathematics and applications. Therefore, they replaced the word ”intuitionistic” by ”double” and renamed its related topologies. The notion of intuitionistic gradation of openness is given the new name ”double fuzzy topological spaces” see [8–10].

The fuzzy type of the notion of topology can be studied in the fuzzy mathematics see [11–13], which has many applications in different branches of mathematics and physics theory. For example, fuzzy topological spaces can be applied in the modeling of spatial objects such as rivers, roads, trees, and buildings. Since double fuzzy topology forms an

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Email addresses: wadeimoon1@hotmail.com (W. F. AL-Omeri)

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extension of fuzzy topology and general topology, we think that our results can be applied in modern physics and GIS Problems.

In this paper, we define and study $(\mathfrak{r},\mathfrak{s})$-$\delta$-fuzzy closed sets in double fuzzy topological spaces and investigate some of their properties. Moreover, we introduce the concepts of double fuzzy δ-continuous functions. Several interesting properties and characterizations are introduced and discussed.

Throughout this paper, let X be a nonempty set and I be the closed unit interval $[0,1]$, $I_0=(0,1]$ and $I_1=[0,1)$. The family of all fuzzy subsets on X is denoted by I^X. By $\underline{0}$ and $\underline{1}$, we denote the smallest and the greatest fuzzy subsets on X. For a fuzzy subset $\rho\in I^X$, $\underline{1}-\rho$ denotes it's complement. Given a function $f:X\to Y$, $f(\rho)$ and $f^{-1}(\rho)$ define the direct image and the inverse image of f, by $f(\rho)(y)=\bigvee_{f(x)=y}\rho(x)$ and $f^{-1}(\nu)(x)=\nu(f(x))$, for each $\rho\in I^X$, $\nu\in I^Y$ and $x\in X$, respectively. For fuzzy subsets ρ and φ in X, we write $\rho q\varphi$ to mean that ρ is quasi coincident (q-coincident) with φ that is, there exists at least one point $x\in X$ such that $\rho(x)+\varphi(x)>1$. Negation of such a statement is denoted as $\rho\overline{q}\varphi$. Notions and notations not described in this paper are standard and usual.

Definition 1.1. [8, 14, 15] The pair of functions $\mathcal{T}$, $\mathcal{T}^*:I^X\to I$ is called a double fuzzy topology on X if it satisfies the following conditions:

(i) $\mathcal{T}(\rho)+\mathcal{T}^*(\rho)\leq 1$,

(ii) $\mathcal{T}^*(\rho_1\bigwedge\rho_2)\geq\mathcal{T}^*(\rho_1)\bigwedge\mathcal{T}^*(\rho_2)$ and $\mathcal{T}(\rho_1\bigwedge\rho_2)\leq\mathcal{T}(\rho_1)\bigvee\mathcal{T}(\rho_2)$,

(iii) $\mathcal{T}(\bigvee_{i\in I}\rho_i)\geq\bigwedge_{i\in I}\mathcal{T}(\rho_i)$ and $\mathcal{T}^*(\bigvee_{i\in I}\rho_i)\leq\bigvee_{i\in I}\mathcal{T}^*(\rho_i)$ for each $\rho_I\in I^X$,$i\in I$.

The triplet $(X,\mathcal{T},\mathcal{T}^*)$ is called a double fuzzy topological space (DFTS, for short). $\mathcal{T}(\rho)$ and $\mathcal{T}^*(\rho)$ may be interpreted as a gradation of openness and gradation of non-openness for ρ. A function $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is said to be double fuzzy continuous if $\mathcal{T}_1(f^{-1}(\nu))\geq\mathcal{T}_2(\nu)$ and $\mathcal{T}_1^*(f^{-1}(\nu))\leq\mathcal{T}_2^*(\nu)$ for each $\nu\in I^Y$.

Theorem 1.2. *[8, 14] Let $(X,\mathcal{T},\mathcal{T}^*)$ be an DFTS. Then, for each $\mathfrak{r}\in I_0$, $\mathfrak{s}\in I_1$, and $\rho\in I^X$, we define an operator $C_{\mathcal{T},\mathcal{T}^*}:I^X\times I_1\times I_0\longrightarrow I^X$ as follows:*

$$C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})=\bigwedge\{\varphi\in I^X|\rho\leq\varphi,\mathcal{T}(\underline{1}-\varphi)\geq\mathfrak{r},\ \mathcal{T}^*(\underline{1}-\varphi)\leq\mathfrak{s}\}.$$

For each ρ, $\varphi\in I^X$, $\mathfrak{r}$, $\mathfrak{r}_1\in I_0$ and s, $\mathfrak{s}_1\in I_1$, the operator $C_{\mathcal{T},\mathcal{T}^*}$ satisfies the following statements:

(i) $C_{\mathcal{T},\mathcal{T}^*}(\underline{0},\mathfrak{r},\mathfrak{s})=\underline{0}$.

(ii) $\rho\leq C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$.

(iii) $C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})\bigwedge C_{\mathcal{T},\mathcal{T}^*}(\varphi,\mathfrak{r},\mathfrak{s})=C_{\mathcal{T},\mathcal{T}^*}(\rho\bigwedge\varphi,\mathfrak{r},\mathfrak{s})$.

(iv) $C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})\leq C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r}_1,\mathfrak{s}_1)$, *if* $\mathfrak{r}\leq\mathfrak{r}_1$ *and* $\mathfrak{s}\geq\mathfrak{s}_1$.

(v) $C_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}),\mathfrak{r},\mathfrak{s}) = C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$.

Theorem 1.3. *(see [12-14]) Let $(X,\mathcal{T},\mathcal{T}^*)$ be an DFTS. Then for each $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$, and $\rho \in I^X$, we define an operator $\mathcal{I}_{\mathcal{T},\mathcal{T}^*} : I^X \times I_1 \times I_0 \longrightarrow I^X$ as follows:*

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \bigvee\{\varphi \in I^X : \varphi \leq \rho, \mathcal{T}(\varphi) \geq r, \mathcal{T}^*(\varphi) \leq s\}.$$

For each ρ, $\varphi \in I^X$, r, $\mathfrak{r}_1 \in I_0$ and s, $\mathfrak{s}_1 \in I_1$, the operator $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}$ satisfies the following statements:

(i) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\rho,\mathfrak{r},\mathfrak{s}) = \underline{1} - C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$.

(ii) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{1},\mathfrak{r},\mathfrak{s}) = \underline{1}$.

(iii) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) \leq \rho$.

(iv) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) \bigvee \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi,\mathfrak{r},\mathfrak{s}) = \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho \bigvee \varphi,\mathfrak{r},\mathfrak{s})$.

(v) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho \bigvee \varphi,\mathfrak{r},\mathfrak{s}) \geq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho \bigvee \varphi,\mathfrak{r}_1,\mathfrak{s}_1)$ *if* $\mathfrak{r} \leq \mathfrak{r}_1$ *and* $\mathfrak{s} \geq \mathfrak{s}_1$.

(vi) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}),\mathfrak{r},\mathfrak{s}) = \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$.

(vii) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}),\mathfrak{r},\mathfrak{s}) = \rho$, *then* $C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\rho,\mathfrak{r},\mathfrak{s}),\mathfrak{r},\mathfrak{s}) = \underline{1} - \rho$.

Definition 1.4. [16] Let $(X,\mathcal{T},\mathcal{T}^*)$ be an DFTS. $\rho \in I^X$, $x_i \in FP(X)$, $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$, a fuzzy set ρ is called $(\mathfrak{r},\mathfrak{s})$-$Q$-neighborhood of x_i, if $\mathcal{T}(\rho) \geq \mathfrak{r}, \mathcal{T}^*(\rho) \leq s$, and $x_i q \rho$.

Definition 1.5. [17] Let $(X,\mathcal{T},\mathcal{T}^*)$ be an DFTS. Then, for each $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$ and $\rho \in I^X$ is called $(\mathfrak{r},\mathfrak{s})$-fuzzy regular open ($(\mathfrak{r},\mathfrak{s})$-FRO, for short) if $\rho = \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}),\mathfrak{r},\mathfrak{s})$. A fuzzy set ρ is called a $(\mathfrak{r},\mathfrak{s})$-fuzzy regular closed ($(\mathfrak{r},\mathfrak{s})$-FRC, for short) iff $\underline{1} - \rho$ is a $(\mathfrak{r},\mathfrak{s})$-FRO set.

2. $(\mathfrak{r},\mathfrak{s})$-fuzzy δ-closed sets

Definition 2.1. Let $(X,\mathcal{T},\mathcal{T}^*)$ be a Double Fuzzy Topological Spaces. Then for each $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$, and $\rho \in I^X$, $x_{(\alpha,\beta)} \in FP(X)$, where $FP(X)$ is the family of all fuzzy points in X. A double fuzzy point $x_{(\alpha,\beta)}$ is said to be a double fuzzy δ-cluster point of a fuzzy set ρ in an DFTS X iff every (r, s)-fuzzy regular open set containing a double fuzzy point $x_{(\alpha,\beta)}$ (having same support as $x_{(\alpha,\beta)}$) has non-null intersection with ρ, or if for every $(\mathfrak{r},\mathfrak{s})$-fuzzy regular open Q-neighborhood φ of $x_{(\alpha,\beta)}$ is q-coincident with ρ, by other words $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\varphi,\mathfrak{r},\mathfrak{s}),\mathfrak{r},\mathfrak{s}) q \rho$.

Definition 2.2. Let $(X,\mathcal{T},\mathcal{T}^*)$ be an DFTS. Then, for each $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$, and $\rho \in I^X$, where ρ be a fuzzy subset of an DFTS X. Let φ be a fuzzy subset of X satisfying the following conditions:
(a) Every double fuzzy point $x_{(\alpha,\beta)}$ in φ is a double fuzzy δ-cluster point of ρ,
(b) If ν is a double fuzzy set, such that $\varphi \leq \nu$, then there is a double fuzzy point.
$x_{(\alpha,\beta)}$ in ν which is not a double fuzzy δ-cluster point of ρ.

Any double fuzzy subset of X having same support as φ is defined to be $(\mathfrak{r},\mathfrak{s})$-fuzzy δ-closure of ρ.

Definition 2.3. The set of all double fuzzy δ-cluster points of ρ is called the $(\mathfrak{r},\mathfrak{s})$-fuzzy δ-closure of ρ and denoted by $\delta C_{\mathcal{T},\mathcal{T}^*}$. Thus a property related with the notation $\delta C_{\mathcal{T},\mathcal{T}^*}$, will always imply that the property holds for all $(\mathfrak{r},\mathfrak{s})$-fuzzy δ-closure of ρ. A double fuzzy set ρ is said to be a $(\mathfrak{r},\mathfrak{s})$-fuzzy δ-closed set if $\rho = \delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$. The complement of a $(\mathfrak{r},\mathfrak{s})$-fuzzy δ-closed set is said to be a $(\mathfrak{r},\mathfrak{s})$-fuzzy δ-open set.

Definition 2.4. Let $(X,\mathcal{T},\mathcal{T}^*)$ be an DFTS. Then, for each $\mathfrak{r}\in I_0$, $\mathfrak{s}\in I_1$, and $\rho\in I^X$, the $\delta C_{\mathcal{T},\mathcal{T}^*}$ and $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}$ operators are define as follows:

(i) $\delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \bigwedge\{\varphi\in I^X|\rho\le\varphi,\ \varphi\, is\,(r,s)-FRC\}$.

(ii) $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \bigvee\{\varphi\in I^X|\rho\ge\varphi,\ \varphi\, is\,(r,s)-FRO\}$.

From the above definition, we have the following relations:

(i) $\delta C_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\rho,\mathfrak{r},\mathfrak{s}) = \underline{1}-\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$,

(ii) $\underline{1}-\delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})) = \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\rho,\mathfrak{r},\mathfrak{s})$.

Note: $(\mathfrak{r},\mathfrak{s})$-fuzzy δ-closure of a fuzzy set in an DFTS is not unique. However, it is unique up to it's support, and any $(\mathfrak{r},\mathfrak{s})$-FRO set of an DFTS X is a $(\mathfrak{r},\mathfrak{s})$-fuzzy open set of X. Therefore, the set of all $(\mathfrak{r},\mathfrak{s})$-FRO sets is subset of the set of all $(\mathfrak{r},\mathfrak{s})$-fuzzy open sets of an DFTS. So, if a double fuzzy point $x_{(\alpha,\beta)}$ of a double fuzzy set ρ in an DFTS X is a double fuzzy δ-cluster point of ρ, then $x_{(\alpha,\beta)}$ is also a double fuzzy cluster point of ρ.

Proposition 2.5. Let $(X,\mathcal{T},\mathcal{T}^*)$ be an DFTS. Then, for each $r,\mathfrak{r}_1\in I_0$, $s,\mathfrak{s}_1\in I_1$, and $\rho\in I^X$, the operator $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}$ satisfies the following statements:

(i) $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{0},\mathfrak{r},\mathfrak{s}) = \underline{0}$, $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{1},\mathfrak{r},\mathfrak{s}) = \underline{1}$.

(ii) $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) \le \rho$.

(iii) $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) \le \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r}_1,\mathfrak{s}_1)$, if $\mathfrak{r}\ge\mathfrak{r}_1$ and $\mathfrak{s}\le\mathfrak{s}_1$.

(iv) $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho\bigwedge\varphi,\mathfrak{r},\mathfrak{s}) = \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})\bigwedge\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi,\mathfrak{r},\mathfrak{s})$.

(v) $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}((\rho,\mathfrak{r},\mathfrak{s}),\mathfrak{r},\mathfrak{s})) = \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$.

Proof. (i), (ii), (v) are obvious by using the definition 2.4.
(iii) Let $\mathfrak{r}\ge\mathfrak{r}_1$ and $\mathfrak{s}\le\mathfrak{s}_1$. Then for each $(\mathfrak{r},\mathfrak{s})$-FRO and $(\mathfrak{r}_1,\mathfrak{s}_1)$-FRO sets, we have:

$$\begin{aligned}\delta\mathcal{I}(\rho,\mathfrak{r},\mathfrak{s}) &= \bigvee\{\varphi\in I^X|\rho\ge\varphi,\ \varphi\, is\,(\mathfrak{r},\mathfrak{s})-FRO\}\\ &= \bigvee\{\varphi\in I^X|\rho\ge\varphi,\ \varphi\, is\,(\mathfrak{r}_1,\mathfrak{s}_1)-FRO\}\end{aligned}$$

$$= \delta\mathcal{I}(\rho, \mathfrak{r}_1, \mathfrak{s}_1).$$

(iv) Since $\rho \wedge \varphi \le \rho$ and $\rho \wedge \varphi \le \varphi$, then we have

$$\delta\mathcal{I}(\rho \wedge \varphi, \mathfrak{r}, \mathfrak{s}) \le \delta\mathcal{I}(\rho, \mathfrak{r}, \mathfrak{s}),$$
$$\textit{and}$$
$$\delta\mathcal{I}(\rho \wedge \varphi, \mathfrak{r}, \mathfrak{s}) \le \delta\mathcal{I}(\varphi, \mathfrak{r}, \mathfrak{s}).$$

Thus, $\delta\mathcal{I}(\rho \wedge \varphi, \mathfrak{r}, \mathfrak{s}) \le \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \wedge \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s})$.
Conversely, it is clear that

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \wedge \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}) \le \rho \wedge \varphi.$$

Also,

$$\begin{aligned}&\mathcal{T}(\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \wedge \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}))\\ &\ge \mathcal{T}(\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) \wedge \mathcal{T}(\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}))\\ &\ge \mathfrak{r} \wedge \mathfrak{r} = \mathfrak{r},\end{aligned}$$

and

$$\begin{aligned}&\mathcal{T}^*(\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \wedge \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}))\\ &\le \mathcal{T}^*(\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) \wedge \mathcal{T}^*(\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}))\\ &\le \mathfrak{s} \wedge \mathfrak{s} = \mathfrak{s}.\end{aligned}$$

By the definition of $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}$, we get

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \wedge \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}) \le \delta\mathcal{I}(\rho \wedge \varphi, \mathfrak{r}, \mathfrak{s}).$$

Hence,

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho \wedge \varphi, \mathfrak{r}, \mathfrak{s}) = \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \wedge \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}).$$

Proposition 2.6. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an DFTS. Then, for each $\mathfrak{r}, \mathfrak{r}_1 \in I_0$, $\mathfrak{s}, \mathfrak{s}_1 \in I_1$, and $\rho \in I^X$, the operator $\delta C_{\mathcal{T},\mathcal{T}^*}$ satisfies the following statements:

(i) $\delta C_{\mathcal{T},\mathcal{T}^*}(\underline{0}, \mathfrak{r}, \mathfrak{s}) = \underline{0}$, $\delta C_{\mathcal{T},\mathcal{T}^*}(\underline{1}, \mathfrak{r}, \mathfrak{s}) = \underline{1}$.

(ii) $\delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \ge \rho$.

(iii) $\delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \le \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}_1, \mathfrak{s}_1)$, if $\mathfrak{r} \le \mathfrak{r}_1$ and $\mathfrak{s} \ge \mathfrak{s}_1$.

(iv) $\delta C_{\mathcal{T},\mathcal{T}^*}(\rho \vee \varphi, \mathfrak{r}, \mathfrak{s}) = \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \vee \delta C_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s})$.

(v) $\delta C_{\mathcal{T},\mathcal{T}^*}(\delta C_{\mathcal{T},\mathcal{T}^*}((\rho, \mathfrak{r}, \mathfrak{s}), \mathfrak{r}, \mathfrak{s})) = \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$.

Proof. Similar to proposition 2.5.

Lemma 2.7. *(i) For any double fuzzy set ρ in an double fuzzy topology $(\mathcal{T}, \mathcal{T}^*)$ on X, $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))$ is a $(\mathfrak{r}, \mathfrak{s})$-FRO set.*

(ii) double fuzzy set ρ in an double fuzzy topology $(\mathcal{T}, \mathcal{T}^*)$ on X, such that $x_i q\rho$, $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))$ is a $(\mathfrak{r}, \mathfrak{s})$-FRO Q-neighborhood of x_i.

Proof. (i) It's enough to show that $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) = \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})))$. *Since*

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) \leq C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))),$$

and we have

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))).$$

Thus

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))).$$

Conversely, since

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) \leq C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}),$$

and we have

$$C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) \leq C_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})) = C_{\mathcal{T},\mathcal{T}^*}((\rho, \mathfrak{r}, \mathfrak{s}), \mathfrak{r}, \mathfrak{s}).$$

Thus

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})).$$

Hence $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))$ is a $(\mathfrak{r}, \mathfrak{s})$-FRO set.

(ii) Clearly,

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})).$$

Since ρ is a $(\mathfrak{r}, \mathfrak{s})$-FRO set, we have

$$\rho = \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})).$$

By (1), $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))$ is a $(\mathfrak{r}, \mathfrak{s})$-FRO set. Therefore $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$ is a $(\mathfrak{r}, \mathfrak{s})$-FRO Q-neighborhood of x_i.

A $(\mathfrak{r}, \mathfrak{s})$-FRC set is not always $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-closed set. For example,

Example 2.8. Let $X = \{a, b\}$ and the fuzzy set μ_1 define as follows:

$$\mu_1(a) = 0.5, \ \mu_1(b) = 0.6,$$

The double fuzzy space $(\mathcal{T}, \mathcal{T}^*)$ is define on X as follows::

$$\mathcal{T}(\rho) = \begin{cases} 1, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \rho = \mu_1, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{T}^*(\rho) = \begin{cases} 0, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \rho = \mu_1, \\ 1, & \text{otherwise.} \end{cases}$$

Since $\mu_1 = C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\mu_1, \mathfrak{r}, \mathfrak{s}), \mathfrak{r}, \mathfrak{s})$. Then, μ_1 is $(\frac{1}{2}, \frac{1}{2})$-FRC set, but is not $(\frac{1}{2}), \frac{1}{2})$-fuzzy δ-closed set.

Theorem 2.9. *For any DFS ρ in an DFTS $(X, \mathcal{T}, \mathcal{T}^*)$, we have:*
(i) $C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \leq \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$.
(ii) $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$.
Proof. It is Obvious.

Theorem 2.10. *Let $\rho, \varphi \in I^X$. The finite union of $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-closed sets is also $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-closed, where $\mathfrak{r} \in I_0$ and $\mathfrak{s} \in I_1$. That is, if $\rho = \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$ and $\varphi = \delta C_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s})$, then $\rho \vee \varphi = \delta C_{\mathcal{T},\mathcal{T}^*}(\rho \vee \varphi, \mathfrak{r}, \mathfrak{s})$.*

Proof. Clearly $(\rho \vee \varphi) \leq \delta C_{\mathcal{T},\mathcal{T}^*}(\rho \vee \varphi, \mathfrak{r}, \mathfrak{s})$. We will show that

$$\delta C_{\mathcal{T},\mathcal{T}^*}((\rho \vee \varphi), \mathfrak{r}, \mathfrak{s}) \leq \rho \vee \varphi.$$

Let x_i for each $i \in I$ be a double fuzzy point. Suppose that $x_i \in \delta C_{\mathcal{T},\mathcal{T}^*}((\rho \vee \varphi), \mathfrak{r}, \mathfrak{s})$. Then for any $(\mathfrak{r}, \mathfrak{s})$-fuzzy regular Q-neighborhood γ of $x_\alpha \in \gamma q(\rho \vee \varphi)$. Thus $\gamma q \rho$ or $\gamma q \varphi$. Hence $x_\alpha \in \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \vee \delta C_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s})$. That is, $x_\alpha \in (\rho \vee \varphi)$.

Corollary 2.11. *If ρ is a $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-closed set in an DFTS $(X, \mathcal{T}, \mathcal{T}^*)$, then ρ is double fuzzy closed. The converse don't holds, the following example shows it.*

Example 2.12. Let $X = \{a, b\}$ and the fuzzy set μ_1 is fuzzy set define by $\mu_1(a) = 0.5$ and $\mu_1(b) = 0.3$,

Take the $(\mathcal{T}, \mathcal{T}^*)$ on X as in Example 2.8.
Since $\mu_1 = \delta C_{\mathcal{T},\mathcal{T}^*}(\mu_1, \mathfrak{r}, \mathfrak{s})$. Then, μ_1 is $(\frac{1}{2}), \frac{1}{2})$-fuzzy δ-closed set, but is not $(\frac{1}{2}, \frac{1}{2})$-FRC set.

Theorem 2.13. *If ρ is a $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-open set in an DFTS $(X, \mathcal{T}, \mathcal{T}^*)$, then the double fuzzy closure and $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-closure are the same, i.e. $C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) = \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$.*

Proof. By Theorem 2.9, it is sufficient to show that $\delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) \leq C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$. Take any $x_i \in \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$. Suppose that $x_i \notin C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$. Then there exists a $(\mathfrak{r}, \mathfrak{s})$-fuzzy open Q-neighborhood φ of x_i such that $\varphi \tilde{q} \rho$. Since $\varphi \tilde{q} \rho$, we have $\varphi \leq \underline{1} - \rho$. Since $(\underline{1} - \rho)$ is a $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-closed set,

$$C_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}) \leq C_{\mathcal{T},\mathcal{T}^*}(\underline{1} - \rho, \mathfrak{r}, \mathfrak{s}) = \underline{1} - \rho.$$

Therefore,

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s})) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{1} - \rho, \mathfrak{r}, \mathfrak{s}) \leq \underline{1} - \rho,$$

i.e. $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}))\tilde{q}\rho$. By Lemma 2.7, $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}))$ is a $(\mathfrak{r}, \mathfrak{s})$-FRO set Q-neighborhood of x_i such that $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\varphi, \mathfrak{r}, \mathfrak{s}))\tilde{q}\rho$. Hence $x_i \notin \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s})$.

Theorem 2.14. *For any $(\mathfrak{r},\mathfrak{s})$-fso set ρ, $C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$.*

Proof. *It's enough to show that $\delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) \leq C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$. Take any $x_i \in \delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$ and suppose that $x_i \notin C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$. Now, $x_i \notin C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$, then there exists a $(\mathfrak{r},\mathfrak{s})$-fuzzy open Q-neighborhood ν of x_i such that $\varphi\tilde{q}\rho$. By definition of $(\mathfrak{r},\mathfrak{s})$-fso set, there exists a $\mathcal{T}(\varphi) \geq r$ and $\mathcal{T}^*(\varphi) \leq s$ such that $\varphi \leq \rho \leq C_{\mathcal{T},\mathcal{T}^*}(\varphi,\mathfrak{r},\mathfrak{s})$ whenever, $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$. Thus*

$$\nu \leq \underline{1} - \rho \leq \underline{1} - \varphi.$$

Hence,

$$C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s}) \leq C_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\rho,\mathfrak{r},\mathfrak{s}) \leq C_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\varphi,\mathfrak{r},\mathfrak{s}) = \underline{1} - \varphi.$$

Also,

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s})) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\rho,\mathfrak{r},\mathfrak{s})) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\varphi,\mathfrak{r},\mathfrak{s}))$$
$$= \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\varphi,\mathfrak{r},\mathfrak{s}) \leq \underline{1} - \varphi,$$

i.e.

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s})) \leq \underline{1} - \varphi.$$

Therefore,

$$\varphi \leq (\underline{1} - \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s}))).$$

Hence,

$$\rho \leq C_{\mathcal{T},\mathcal{T}^*}(\varphi,\mathfrak{r},\mathfrak{s}) \leq C_{\mathcal{T},\mathcal{T}^*}(\underline{1} - \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s}))) = (\underline{1} - \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s})).$$

Because of $(\underline{1} - (\underline{1} - \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s})))) \geq r$ and $(\underline{1} - (\underline{1} - \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s}))) \leq s$. Thus $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s}))\tilde{q}\rho$. By Lemma 2.7, $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s}))$ is a $(\mathfrak{r},\mathfrak{s})$-fuzzy regular open Q-neighborhood ν of x_i such that $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\nu,\mathfrak{r},\mathfrak{s}))\tilde{q}\rho$. Hence, $x_i \notin \delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$.

Proposition 2.15. Let $(X,\mathcal{T},\mathcal{T}^*)$ be an DFTS. Then, for each $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$, and $\rho \in I^X$. Let $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*} : I^X \times I_0 \times I_1 \longrightarrow I^X$ be a function satisfy the conditions (1-5) in proposition 2.5 such that $\mathcal{T}$, $\mathcal{T}^* : I^X \to I$ be functions defined by
$\mathcal{T}(\rho) = \bigvee\{\mathfrak{r} \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \rho\}$ and
$\mathcal{T}^*(\rho) = \bigwedge\{\mathfrak{s} \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \rho\}$. Then $\zeta = (\mathcal{T}, \mathcal{T}^*)$ is double fuzzy topology in X.

Proof. (a) Since $(\mathfrak{r},\mathfrak{s}) \in I_0 \times I_1$ and we have $\mathfrak{r}+\mathfrak{s} \leq 1$. Hence $\mathfrak{s} \leq 1-\mathfrak{r}$, Thus,

$$\begin{aligned}
&\underline{1} - \mathcal{T}(\rho) = \underline{1} - \bigvee\{\mathfrak{r} \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \rho\} \\
&= \bigwedge\{1-\mathfrak{r} \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \rho\} \\
&\geq \bigwedge\{\mathfrak{s} \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}) = \rho\} \\
&= \mathcal{T}^*
\end{aligned}.$$

(b) Suppose that $\mathcal{T}(\rho_1 \bigwedge \rho_2) \leq \mathcal{T}(\rho_1) \bigwedge \mathcal{T}(\rho_2)$. Then there is a $\xi \in I$ such that

$$\mathcal{T}(\rho_1 \bigwedge \rho_2) \leq \xi \leq \mathcal{T}(\rho_1) \bigwedge \mathcal{T}(\rho_2).$$

Since,

$$\xi \leq \mathcal{T}(\rho_i) = \bigvee\{\mathfrak{r} \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_\xi, \mathfrak{r}, \mathfrak{s}) = \rho_i\},$$

for each $i = 1, 2$ there exist $(\mathfrak{r}_1, \mathfrak{s}_1), (\mathfrak{r}_2, \mathfrak{s}_2) \in I_0 \times I_1$ such that $\xi \leq \mathfrak{r}_i \leq \mathcal{T}(\rho_i)$ and $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}_i, \mathfrak{s}_i) = \rho_i$. Now, for each $i = 1, 2$, let $\mathfrak{r} = \mathfrak{r}_1 \bigwedge \mathfrak{r}_2$ and $\mathfrak{s} = \mathfrak{s}_1 \bigvee \mathfrak{s}_2$.
Since $(\mathfrak{r}_1, \mathfrak{s}_1), (\mathfrak{r}_2, \mathfrak{s}_2) \in I_0 \times I_1$, we have:

$$\begin{aligned} 1 - \mathfrak{r} &= 1 - (\mathfrak{r}_1 \bigwedge \mathfrak{r}_2) \\ &= (1 - \mathfrak{r}_1) \bigvee (1 - \mathfrak{r}_2) \geq \mathfrak{s}_1 \bigvee \mathfrak{s}_2 = \mathfrak{s} \end{aligned}$$

Hence, $(\mathfrak{r}, \mathfrak{s}) \in I_0 \times I_1$. Since $\mathfrak{r} \leq \mathfrak{r}_i$ and $\mathfrak{s} \geq \mathfrak{s}_i$ for each $i = 1, 2$ we have,

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_\xi, \mathfrak{r}, \mathfrak{s}) \geq \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}_i, \mathfrak{s}_i) = \rho_\xi.$$

Hence, $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}, \mathfrak{s}) = \rho_i$ for each $i = 1, 2$, we get

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_1 \bigwedge \rho_2, \mathfrak{r}, \mathfrak{s}) = \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_1, \mathfrak{r}, \mathfrak{s}) \bigwedge \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_2, \mathfrak{r}, \mathfrak{s}) = \rho_1 \bigwedge \rho_2.$$

Thus,

$$\xi \geq \mathcal{T}(\rho_1 \bigwedge \rho_2) = \bigvee\{r^* \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_1 \bigwedge \rho_2, \mathfrak{r}, \mathfrak{s}) = \rho_1 \bigwedge \rho_2\} \geq r = \mathfrak{r}_1 \bigwedge \mathfrak{r}_2 \geq \xi.$$

Which is a contradiction. Hence, $\mathcal{T}(\rho_1 \bigwedge \rho_2) \geq \mathcal{T}(\rho_1) \bigwedge \mathcal{T}(\rho_2)$.
Next, suppose that

$$\mathcal{T}^*(\rho_1 \bigwedge \rho_2) \geq \mathcal{T}^*(\rho_1) \bigvee \mathcal{T}^*(\rho_2).$$

Then, there is a $\xi \in I$ such that

$$\mathcal{T}^*(\rho_1 \bigwedge \rho_2) \geq \xi \geq \mathcal{T}^*(\rho_1) \bigvee \mathcal{T}^*(\rho_2).$$

Since $\xi \geq \mathcal{T}^*(\rho_i) = \bigwedge\{\mathfrak{s} \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}, \mathfrak{s}) = \rho_\xi\}$ for each $i = 1, 2$ there are $(\mathfrak{r}_1, \mathfrak{s}_1), (\mathfrak{r}_2, \mathfrak{s}_2) \in I_0 \times I_1$ such that $\xi \geq \mathfrak{s}_\xi \geq \mathcal{T}(\rho_i)$ and $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}_i, \mathfrak{s}_i) = \rho_i$.
Now, let $\mathfrak{r} = \mathfrak{r}_1 \bigwedge \mathfrak{r}_2$ and $\mathfrak{s} = \mathfrak{s}_1 \bigvee \mathfrak{s}_2$. Then $(\mathfrak{r}, \mathfrak{s}) \in I_0 \times I_1$ and $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, r, \mathfrak{s}_i) = \rho_i$ for each $i = 1, 2$.

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_1 \bigwedge \rho_2, \mathfrak{r}, \mathfrak{s}) = \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_1, \mathfrak{r}, \mathfrak{s}) \bigwedge \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_2, \mathfrak{r}, \mathfrak{s}) = \rho_1 \bigwedge \rho_2.$$

Thus,

$$\xi \geq \mathcal{T}*(\rho_1 \bigwedge \rho_2) = \bigwedge\{r^* \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_1 \bigwedge \rho_2, \mathfrak{r}, \mathfrak{s}) = \rho_1 \bigwedge \rho_2\} \leq s = \mathfrak{s}_1 \bigwedge \mathfrak{s}_2 \leq \xi.$$

Which is a contradiction. Hence,

$$\mathcal{T}^*(\rho_1 \bigwedge \rho_2) \leq \mathcal{T}^*(\rho_1) \bigvee \mathcal{T}^*(\rho_2).$$

(c) First suppose that, $\mathcal{T}(\bigvee \rho_i) \leq \bigwedge \mathcal{T}(\rho_i)$. Then, there exist ξ such that $\mathcal{T}(\bigvee \rho_i) \leq \xi \leq \bigwedge \mathcal{T}(\rho_i)$. Since $\xi \leq \bigvee\{\mathfrak{r} \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}, \mathfrak{s}) = \rho_i\}$ for each i there exist $(\mathfrak{r}_i, \mathfrak{s}_i) \in I_0 \times I_1$

such that $\xi \leq \mathfrak{r}_i \leq \mathcal{T}(\rho_i)$ and $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}_i, \mathfrak{s}_i) = \rho_i$.
Now, let $\mathfrak{r} = \bigwedge \mathfrak{r}_i$ and $\mathfrak{s} = \bigvee \mathfrak{s}_i$. Since $(\mathfrak{r}_i, \mathfrak{s}_i) \in I_0 \times I_1$ for each i we have,

$$1 - \mathfrak{r} = 1 - \bigwedge(\mathfrak{r}_i) = \bigvee(1 - \mathfrak{r}_i) \geq \bigvee \mathfrak{s}_i = s.$$

Hence, $(\mathfrak{r}, \mathfrak{s}) \in I_0 \times I_1$. And since $\mathfrak{r} \leq \bigwedge \mathfrak{r}_i$ and $\mathfrak{s} \geq \bigvee \mathfrak{s}_i$ for each i,

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}, \mathfrak{s}) \geq \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}_i, \mathfrak{s}_i) = \rho_i.$$

Hence, $\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\bigvee \rho_i, \mathfrak{r}, \mathfrak{s}) \geq \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\rho_i, \mathfrak{r}, \mathfrak{s}) \geq \rho_i$ for each i. So,

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\bigvee \rho_i, \mathfrak{r}, \mathfrak{s}) \geq \bigvee \rho_i,$$

And hence,

$$\delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\bigvee \rho_i, \mathfrak{r}, \mathfrak{s}) = \bigvee \rho_i.$$

Thus,

$$\begin{aligned}
&\xi \geq \mathcal{T}(\bigvee \rho_i) \\
&= \bigvee\{\mathfrak{r}^* \in I | \delta\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\bigvee \rho_i, \mathfrak{r}^*, \mathfrak{s}^*) = \bigvee \rho_i\} \\
&\geq \mathfrak{r} = \bigwedge \mathfrak{r}_i \geq \xi.
\end{aligned}$$

Which is a contradiction. Hence, $\mathcal{T}(\bigvee \rho_i) \geq \bigwedge \mathcal{T}(\rho_i)$. Similar for $\mathfrak{s}$.
Therefor, $\zeta = (\mathcal{T},\, \mathcal{T}^*)$ is double fuzzy topology in X.

Proposition 2.16. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an DFTS. Then, for each $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$, and $\rho \in I^X$. Let $\delta C_{\mathcal{T},\mathcal{T}^*} : I^X \times I_0 \times I_1 \longrightarrow I^X$ be a function satisfy the conditions (1-5) in proposition 2.6 such that $\mathcal{T},\, \mathcal{T}^* : I^X \to I$ be a function defined by:
$\mathcal{T}(\rho) = \bigvee\{\mathfrak{r} \in I | \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) = \rho\}$ and
$\mathcal{T}^*(\rho) = \bigwedge\{\mathfrak{s} \in I | \delta C_{\mathcal{T},\mathcal{T}^*}(\rho, \mathfrak{r}, \mathfrak{s}) = \rho\}$. Then $\zeta = (\mathcal{T},\, \mathcal{T}^*)$ is a double fuzzy family of closed sets in X.

Proof. Similar to proposition 2.15

3. $(\mathfrak{r}, \mathfrak{s})$-Fuzzy δ-continuous functions

Definition 3.1. A function f FROm an DFTS $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ into an DFTS $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is said to be double fuzzy δ-continuous at a double fuzzy point $x_{\alpha,\beta}$ in X, if $f^{-1}(\upsilon)$ is a $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-open set, for each $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$, and υ is a $(\mathfrak{r}, \mathfrak{s})$-FRO set in I^Y such that $\mathcal{T}_2(\upsilon) \geq r$ and $\mathcal{T}_2^*(\upsilon) \leq s$. i.e for each double fuzzy point $x_\alpha inX$ and for any $(\mathfrak{r}, \mathfrak{s})$-FRO set Q-neighborhood γ of $f(x_\alpha)$ in I^Y, there exists a $(\mathfrak{r}, \mathfrak{s})$-FRO set Q-neighborhood β of x_α such that $f(\beta) \leq \gamma$.

Theorem 3.2. *Let f be a function FROm an DFTS $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ into an DFTS $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$. Then the following two conditions are equivalent* *(i)* f *is double fuzzy* δ*-continuous,*

(ii) *for each double fuzzy point* x_i *in* X *and each* $(\mathfrak{r},\mathfrak{s})$*-FRO set* ρ *containing* $f(x_i)$*, there exists a* $(\mathfrak{r},\mathfrak{s})$*-FRO set* φ *containing* x_i *where* $\mathfrak{r}\in I_0$*,* $\mathfrak{s}\in I_1$ *such that* $f(\varphi)\le\rho$*.*

Proof. (1) $\Rightarrow$ (2) *Let* x_i *be a fuzzy point in* $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ *and* ρ *be a* $(\mathfrak{r},\mathfrak{s})$*-FRO set containing* $f(x_i)$*, where* $\mathfrak{r}\in I_0$*,* $\mathfrak{s}\in I_1$*. Since every* $(\mathfrak{r},\mathfrak{s})$*-FRO set is a fuzzy open set, it follows FROm (1) that there exists a double fuzzy open nbd* φ *of* x_i *such that,*

$$f(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\varphi,\mathfrak{r},\mathfrak{s})))\le\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\rho,\mathfrak{r},\mathfrak{s}))=\rho.$$

Taking $\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\varphi_1,\mathfrak{r},\mathfrak{s})))=\varphi$ *(being a* $(\mathfrak{r},\mathfrak{s})$*-FRO set containing* x_i*, we obtain (2).*
(2) $\Rightarrow$ (1) *Let* x_i *be a double point of* X *and* ρ *be a double fuzzy open nbd containing* $f(x_i)$*. Then* $\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\rho,\mathfrak{r},\mathfrak{s}))$ *is a* $(\mathfrak{r},\mathfrak{s})$*-FRO set containing* $f(x_i)$ *and* $\rho\le\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\rho,\mathfrak{r},\mathfrak{s}))$*. By (2), there is a* $(\mathfrak{r},\mathfrak{s})$*-FRO set* φ *containing* x_i *such that* $f(\varphi)\le\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\rho,\mathfrak{r},\mathfrak{s}))$ *(i.e., since a* $(\mathfrak{r},\mathfrak{s})$*-FRO set is a double fuzzy open set) and* $\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\varphi,\mathfrak{r},\mathfrak{s}))=\varphi$ *there is a double fuzzy open set* φ *containing* x_i *such that ,*$f(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\varphi,\mathfrak{r},\mathfrak{s})))\le\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\rho,\mathfrak{r},\mathfrak{s}))$*.*

Theorem 3.3. *For a function* $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\longrightarrow(Y,\mathcal{T}_2,\mathcal{T}_2^*)$*. The following statements are equivalent:*

(i) f *is double fuzzy* δ*-continuous,*

(ii) $f(\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(\rho,\mathfrak{r},\mathfrak{s}))\le\delta C_{\mathcal{T}_2,\mathcal{T}_2^*}(f(\rho),\mathfrak{r},\mathfrak{s})$*, for each* $\mathfrak{r}\in I_0$*,* $\mathfrak{s}\in I_1$*, and* $\rho\in I^X$*.*

(iii) $\delta C_{\mathcal{T}_2,\mathcal{T}_2^*}(f(\varphi)^{-1},\mathfrak{r},\mathfrak{s})\le f^{-1}(\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(\varphi,\mathfrak{r},\mathfrak{s}))$*, for each* $\mathfrak{r}\in I_0$*,* $\mathfrak{s}\in I_1$*, and* $\varphi\in I^Y$*.*

(iv) *for every* $(\mathfrak{r},\mathfrak{s})$*-fuzzy* δ*-closed set* φ *in* I^Y*,* $\mathfrak{r}\in I_0$*,* $\mathfrak{s}\in I_1$*,* $f^{-1}(\varphi)$ *is* $(\mathfrak{r},\mathfrak{s})$*-fuzzy* δ*-closed set in* I^X*.*

(v) *for every* $(\mathfrak{r},\mathfrak{s})$*-fuzzy* δ*-open set* φ *in* I^Y*,* $\mathfrak{r}\in I_0$*,* $\mathfrak{s}\in I_1$*,* $f^{-1}(\varphi)$ *is* $(\mathfrak{r},\mathfrak{s})$*-fuzzy* δ*-open set in* I^X*.*

Proof. (1) $\Rightarrow$ (2) *Let* $\rho\in I^X$*,* $x_\alpha\in\delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$ *and* γ *be a* $(\mathfrak{r},\mathfrak{s})$*-regular open* Q*-neighborhood of* $f(x_\alpha)$*. Then there exists a* $(\mathfrak{r},\mathfrak{s})$*-regular open* Q*-neighborhood* β *of* x_α *such that* $f(\gamma)\le\beta$*. Since* $x_\alpha\in\delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s})$*, we have* $\beta q\rho$*. Then* $f(\beta)qf(\rho)$*. Thus* $\gamma qf(\rho)$ *and hence* $f(x_\alpha)\in\delta C_{\mathcal{T},\mathcal{T}^*}(f(\rho),\mathfrak{r},\mathfrak{s})$*. So* $f(\delta C_{\mathcal{T},\mathcal{T}^*}(\rho,\mathfrak{r},\mathfrak{s}))\le\delta C_{\mathcal{T},\mathcal{T}^*}(f(\rho),\mathfrak{r},\mathfrak{s})$*.*

(2) $\Rightarrow$ (3) *Let* $\varphi\in I^Y$*. By using* (2)*,*

$$f(\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(f^{-1}(\varphi),\mathfrak{r},\mathfrak{s}))\le\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(f(f^{-1}(\varphi)),\mathfrak{r},\mathfrak{s})\le\delta C_{\mathcal{T}_2,\mathcal{T}_2^*}(\varphi,\mathfrak{r},\mathfrak{s}).$$

Hence,

$$\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(f^{-1}(\varphi),\mathfrak{r},\mathfrak{s})\le f^{-1}(\delta C_{\mathcal{T}_2,\mathcal{T}_2^*}(\varphi,\mathfrak{r},\mathfrak{s})).$$

(3) ⇒ (4) *We have* $\varphi = \delta C_{\mathcal{T}_2,\mathcal{T}_2^*}(\varphi, \mathfrak{r}, \mathfrak{s})$. *Now by (3),*

$$\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(f^{-1}(\varphi), \mathfrak{r}, \mathfrak{s}) \leq f^{-1}(\delta C_{\mathcal{T}_2,\mathcal{T}_2^*}(\varphi, \mathfrak{r}, \mathfrak{s})) = f^{-1}(\varphi).$$

Therefore, $\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(f^{-1}(\varphi), \mathfrak{r}, \mathfrak{s}) = f^{-1}(\varphi)$. *Hence* $f^{-1}(\varphi)$ *is a* $(\mathfrak{r}, \mathfrak{s})$*-fuzzy* δ*-closed.*

(4) ⇒ (5) *Let* $\varphi \in I^Y$, *and* φ *be a* $(\mathfrak{r}, \mathfrak{s})$*-fuzzy* δ*-open. Then,* $\underline{1} - \varphi$ *is a* $(\mathfrak{r}, \mathfrak{s})$*-fuzzy* δ*-closed in* I^Y. *By (4),* $f^{-1}(\underline{1} - \varphi)$ *is a* $(\mathfrak{r}, \mathfrak{s})$*-fuzzy* δ*-closed in* I^X. *Since* $f^{-1}(\underline{1} - \varphi) = 1 - f^{-1}(\varphi)$, $f^{-1}(\varphi)$ *is* $(\mathfrak{r}, \mathfrak{s})$*-fuzzy* δ*-open in* I^X.

(5) ⇒ (1) *The proof is clear.*

Theorem 3.4. *For a function* $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$. *The following statements are equivalent:*

(i) f *is double fuzzy* δ*-continuous function,*

(ii) $f^{-1}(\mu)$ *is an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy* δ*-closed set in* I^X*for each* $\mu \in I^Y$, $\mathfrak{r}_0 \in I_0$, $s_1 \in I_1$,

(iii) $f^{-1}(\mu)$ *is an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy* δ*-open set in* I^X*for each* $\mu \in I^Y$,$r0 \in I_0$, $s_1 \in I_1$.

Proof. (1) ⇒ (2) *Let* μ *be an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy* δ*-closed set in* I^X*for each* $\mu \in I^Y$, $r0 \in I_0$, $s_1 \in I_1$. *Then* $\mu = \delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu, \mathfrak{r}_0, \mathfrak{s}_1)$. *But, by Theorem 3.3 we have* $\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(f^{-1}(\mu), \mathfrak{r}_0, \mathfrak{s}_1) \leq f^{-1}(\delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu, \mathfrak{r}_0, \mathfrak{s}_1)) = f^{-1}(\mu)$. *Hence* $f^{-1}(\mu) = \delta C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu, \mathfrak{r}_0, \mathfrak{s}_1)$. *So we get,* $f^{-1}(\mu)$ *is an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy* δ*-closed set in* I^X.

(2) ⇒ (3) *Trivial by taking the complement of* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy* δ*-closed set to be an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy* δ*-open set.*

(3) ⇒ (1) *Let* $x_{(\alpha,\beta)}$ *be a fuzzy point in* I^X *and let* γ *be an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy regular-open q- neighborhood of* $f(x_{(\alpha,\beta)})$ *for each* $\gamma \in I$, $r0 \in I_0$, $\mathfrak{s}_1 \in I_1$. *But,* γ *is an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy* δ*-open set in* I^Y *and by hypothesis,* $f^{-1}(\gamma)$ *is an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy* δ*-open set in* I^X. *Since* $x_{(\alpha,\beta)}$ q $f^{-1}(\gamma)$ *so we get,* $f^{-1}(\gamma)$ *is* $(\mathfrak{r}_0, \mathfrak{s}_1)$*- fuzzy* δ*-neighborhood of* $x_{(\alpha,\beta)}$. *Therefore, there exists an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*- fuzzy regular-neighborhood of* μ *of* $x_{(\alpha,\beta)}$*such that* $\mu \leq f^{-1}(\gamma, \mathfrak{r}_0, \mathfrak{s}_1)$. *Hence,*$f(\mu) \leq \gamma$.

Theorem 3.5. *Let* $f : (X, \tau_x, \tau_{x^*}) \to (Y, \tau_y, \tau_{y^*})$ *be a bijection function. Then the following statements are equivalent:*

(i) f *is double fuzzy* δ*-continuous function,*

(ii) $\delta I\tau_x, \tau_{x^*}(\lambda, \mathfrak{r}_0, \mathfrak{s}_1)$=$f(\delta I\tau_x, \tau_{x^*}(\lambda, \mathfrak{r}_0, \mathfrak{s}_1))$ *for each* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy set* λ *in* I^X,$\mathfrak{r}_0 \in I_0$, $s_1 \in I_1$.

Proof. (1) ⇒ (2) *Let* λ *be an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy set in* I^X,$r0 \in I_0$, $s_1 \in I_1$. *Then* $f(\lambda)$ $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy set* λ *in* I^Y. *But,* f *is one- to-one, then*

$f^{-1}(\delta I\tau_x, \tau_{x^*}(f(\lambda), \mathfrak{r}_0, \mathfrak{s}_1) \leq \delta I\tau_y, \tau_{y^*}(\mathrm{f}-1(f(\lambda)), \mathfrak{r}_0, \mathfrak{s}_1) = \delta I\tau_x, \tau_{x^*}(\lambda, \mathfrak{r}_0, \mathfrak{s}_1)$. *Since* f *is onto, so*

$\delta I\tau_x, \tau_{x^*}(f(\lambda), \mathfrak{r}_0, \mathfrak{s}_1)$ $=f(f^{-1}(I\tau_y, \tau_{y^*}(f(\lambda), \mathfrak{r}_0, \mathfrak{s}_1))) \leq f(I\tau_x, \tau_{x^*}(\lambda, \mathfrak{r}_0, \mathfrak{s}_1)$.

(2) $\Rightarrow$ (1) *Let* λ *be an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy set in* I^Y, $\mathfrak{r}_0 \in I_0$, $\mathfrak{s}_1 \in I_1$. *Then* $f^{-1}(\lambda)$ *is an* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy set* λ *in* I^Y. *But by hypothesis,* $\delta I\tau_y, \tau_{y^*}$ $(\lambda,\ \mathfrak{r}_0, \mathfrak{s}_1) = \delta I\tau_x, \tau_{x^*}(f(f-1(\lambda, \mathfrak{r}_0, \mathfrak{s}_1) \leq f(\delta I\tau_y, \tau_{y^*}(f^{-1}(\lambda), \mathfrak{r}_0, \mathfrak{s}_1))$) *for each* $(\mathfrak{r}_0, \mathfrak{s}_1)$*-fuzzy set* λ *in* I^X, $\mathfrak{r}_0 \in I_0$, $s_1 \in I_1$. *But* f *is one -to-one function, then*
$f^{-1}(\delta I\tau_y, \tau_{y^*}(\lambda, \mathfrak{r}_0, \mathfrak{s}_1) \leq f^{-1}(f(\delta I\tau_x, \tau_{x*}(f^{-1}(\lambda), \mathfrak{r}_0, \mathfrak{s}_1))) = \delta I\tau_y, \tau_{y^*}(f^{-1}(\lambda), \mathfrak{r}_0, \mathfrak{s}_1)$
Hence by Theorem 3.3, f *is double fuzzy* δ*-continuous function.*

Remark 3.6. The concepts of a $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-continuous and a $(\mathfrak{r}, \mathfrak{s})$-fuzzy-continuous are independent to each other for each $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$.

Example 3.7. Let $X = [0, 1]$ and $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (X, \mathcal{T}_2, \mathcal{T}_2^*)$ be the identity function
(1)- define ρ_1 and β_1 as follows:

$$\rho_1(0) = 0.3,\ \rho_1(1) = 0.7,$$
$$\beta_1(0) = 0.8,\ \beta_1(1) = 0.2,$$

And the two spaces $(\mathcal{T}_1, \mathcal{T}_1^*)$ and $(\mathcal{T}_2, \mathcal{T}_2^*)$ are define as follows:

$$\mathcal{T}_1(\rho) = \begin{cases} 1, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \rho = \rho_1, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{T}_1^*(\rho) = \begin{cases} 0, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if } \rho = \rho_1, \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\mathcal{T}_2(\rho) = \begin{cases} 1, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \rho = \rho_1, \\ \frac{8}{10}, & \text{if } \rho = \beta_1, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{T}_2^*(\rho) = \begin{cases} 0, & \text{if } \rho \in \{\underline{0}, \underline{1}\} \\ \frac{2}{3}, & \text{if } \rho = \rho_1, \\ \frac{2}{10}, & \text{if } \rho = \beta_1, \\ 1, & \text{otherwise.} \end{cases}$$

Then, f is double fuzzy δ-continuous function but not double fuzzy continuous function.

(2)- define ρ_1 and β_1 as follows:

$$\rho_1(0) = 0.3,\ \rho_1(1) = 0.7,$$
$$\beta_1(0) = 0.5,\ \beta_1(1) = 0.5,$$

And the two spaces $(\mathcal{T}_1, \mathcal{T}_1^*)$ and $(\mathcal{T}_2, \mathcal{T}_2^*)$ are defined as follows:

$$\mathcal{T}_1(\rho) = \begin{cases} 1, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \rho = \rho_1, \\ \frac{1}{2}, & \text{if } \rho = \beta_1, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{T}_1^*(\rho) = \begin{cases} 0, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if } \rho = \rho_1, \\ \frac{1}{2}, & \text{if } \rho = \beta_1, \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\mathcal{T}_2(\rho) = \begin{cases} 1, & \text{if } \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \rho = \rho_1, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{T}_2^*(\rho) = \begin{cases} 0, & \text{if } \rho \in \{\underline{0}, \underline{1}\} \\ \frac{2}{3}, & \text{if } \rho = \rho_1, \\ 1, & \text{otherwise.} \end{cases}$$

Then, f is double fuzzy continuous function but not double fuzzy δ-continuous function.

Theorem 3.8. *Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$, $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ and $(Z, \mathcal{T}_3, \mathcal{T}_3^*)$ be an DFTSs. For the functions $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ and $g : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Z, \mathcal{T}_3, \mathcal{T}_3^*)$. If f and g are $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-continuous functions, then $g \circ f$ $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-continuous.*

Proof. Straightforward.

Corollary 3.9. *Let $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be double fuzzy regular δ-continuous function, then $\delta C_{\mathcal{T}_1, \mathcal{T}_1^*}(f(\varphi)^{-1}, \mathfrak{r}, \mathfrak{s}) \leq f^{-1}(\delta C_{\mathcal{T}_2, \mathcal{T}_2^*}(\varphi, \mathfrak{r}, \mathfrak{s}))$, for each $\mathcal{T}_2(\varphi) \geq r$, $\mathcal{T}_2^*(\varphi) \leq s$, and $\mathfrak{r} \in I_0$, $\mathfrak{s} \in I_1$.*

4. conclusion

The main purpose of this paper is to introduce a new concept in double fuzzy set theory, namely a $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-closed sets and a double fuzzy δ-continuous functions. On the other hand, the double fuzzy topology on a double fuzzy set is a kind of abstract theory of mathematics. First, we present and study a $(\mathfrak{r}, \mathfrak{s})$-fuzzy δ-closed sets and double fuzzy δ-continuity FROm a double fuzzy topological space on a double fuzzy set into another. Then, we present the relationships between types of double fuzzy δ-continuity functions.

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