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Submaximality on bigeneralized topological spaces

Yasser Farhat[1], Muthumari Krishnan[2,*], Vadakasi Subramanian[2],
M. R. Ahmadi Zand[3]

[1] *Academic Support Department, Abu Dhabi Polytechnic, P. O. Box 111499, Abu Dhabi, UAE*
[2] *Department of Mathematics, A.K.D.Dharma Raja Women's College, Rajapalayam*
[3] *Department of Mathematics, Yazd University, Yazd, Iran*

Abstract. In this article, in a bigeneralized topological space, we introduce a new space namely, (s, v)-bigeneralized submaximal space, and analyze its nature. Also, the characterization theorem for a (s, v)-bigeneralized submaximal space, image and preimage of (s, v)-bigeneralized submaximal is a (s, v)-bigeneralized submaximal space under (μ, η)-open, (μ, η)-continuous map, respectively are proved. Further, the relationship between hyperconnected space and submaximal space in a pairwise bigeneralized submaximal space is given.

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1. Introduction

The new most interesting tool namely, generalized topological space was founded by Császár in [2]. Most of them studied the nature of this space and some, researchers have defined some new tools in this space and examined their significance in generalized topological space. Particularly, submaximal space was introduced by Ekici in a generalized topological space. In generalized topological space, he launched some characterization theorems for submaximal space. Based on this, some mathematicians established some new results for generalized submaximal space e.g. [9, 19]. In 2016, Ahmadi Zand et.al gave few results for submaximal space and defined a space namely, generalized G_δ-submaximal space, and studied the nature of this space [19].

In [11], J.C. Kelly introduced the notion of bitopological space. Motivated by this, Boonpok defined the concept of bigeneralized topological space in 2010 [5]. In bigeneralized topological space, he proved some results for (m, n)-closed sets. In [13, 14, 17, 18], some

*Corresponding author.
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Email addresses: farhat.yasser.1@gmail.com (Y. Farhat),
muthumarikrishnan060196@gmail.com (K.Muthumari)

new properties of different types of sets are proved. Based on this, here we prove some interesting results for nowhere-dense sets in bigeneralized topological space.

From the previous observations, it has been motivated, in section 3, (s, v)-bigenralized submaximal space is defined and the equivalent conditions for this space are given and some properties for nowhere dense sets are proved, also, the significance of an image and pre-image of (s, v)-bigeneralized submaximal space are analyzed. In a pairwise bigeneralized submaximal space, the relationship between generalized submaximal space and generalized hyperconnected space is discussed.

In section 4, we define the notion $(s, v)^{\star}$-bigeneralized submaximal space and the relationship between (s, v) and $(s, v)^{\star}$ bigeneralized submaximal space is proven. In the last section, the necessary conditions for a bigeneralized topological space is a $(s, v)^{\star\star}$-bigeneralized topological space are given. In this space, few results for closed sets are proven.

With the results given in the mentioned sections, it is possible to easily check whether a given space is (s, v) or $(s, v)^{\star}$ or $(s, v)^{\star\star}$ bigeneralized submaximal space or not, is obtain some tricks to check in a (s, v) ($(s, v)^{\star}$ and $(s, v)^{\star\star}$) bigeneralized submaximal space, a given set is (v, s)-nowhere dense or not, are given the necessary conditions for check whether a given space is submaximal space or not.

2. Preliminaries

Let X be any non-null set. A collection μ of subsets of X is a *generalized topology* [2] in X if it contains the empty set and is closed under arbitrary union. The pair (X, μ) is called a *generalized topological space* (GTS). The pair (X, μ) is called a *strong generalized topological space* (sGTS) [20] if $X \in \mu$.

In [3], if $Q \in \mu$, then Q is called a *μ-open set* ; if $X - Q \in \mu$, then Q is said to be a *μ-closed set.* Let D be a subset of a GTS (X, μ). The *interior of* D [20] denoted by iD, is the union of all μ-open sets contained in D and the *closure of* D [20] denoted by cD, is the intersection of all μ-closed sets containing D. For simplicity of notation, we will write $i(D)$ and $c(D)$ when no confusion can arise.

Next, we present some definitions and lemmas that are useful for the development of the following sections.

In [10], notated by;

$$\tilde{\mu} = \{D \in \mu \mid D \neq \emptyset\},$$
$$\mu(x) = \{D \in \mu \mid x \in D\}.$$

A subset Q of a GTS (X, μ) is said to be;
- *μ-nowhere dense* [8] if $icQ = \emptyset$.
- *μ-dense* [8] if $cQ = X$.
- *μ-codense* [9] if $c(X - Q) = X$.

A GTS (X, μ) is called as a ;
- *hyperconnected* space [8] if $c_{\mu}(Q) = X$ whenever $Q \in \mu$.
- *generalized submaximal* [9] if $Q \in \tilde{\mu}$ whenever $c_{\mu}(Q) = X$.

Let μ_1 and μ_2 be two generalized topologies defined on a non-null set X. Then the triple (X, μ_1, μ_2) is called as *bigeneralized topological space* (briefly, BGTS) [5].

Let Q be a subset of a BGTS (X, μ_1, μ_2). Then the *closure of* D and the *interior of* D with respect to μ_s are denoted by $c_s(D)$ and $i_s(D)$, respectively, for $s = 1, 2$ [5].

A subset Q of a BGTS (X, μ_1, μ_2) is called (s, v)-closed if $c_s(c_v(Q)) = Q$, where $s, v = 1$ or 2 ; $s \neq v$. If $X - J$ is (s, v)-closed, then J is called as (s, v)-open where $s, v = 1$ or 2 ; $s \neq v$ [5].

Let (X, μ) and (Y, η) be two GTSs. A function $h : (X, \mu) \to (Y, \eta)$ is called as ;
- (μ, η)*-continuous* [6] if $h^{-1}(Q) \in \mu$ for each $Q \in \eta$.
- (μ, η)*-open* [12] if $h(Q) \in \eta$ for each $Q \in \mu$.

Lemma 1. [19, Lemma 2.6] Let (X, μ) be a GTS and $Q \subset X$. Then $i_\mu(c_\mu(Q)) - Q = \emptyset$.

Lemma 2. [7] A mapping $h : (X, \mu) \to (Y, \eta)$ is (μ, η)-continuous if and only if $c(h^{-1}(Q)) \subset h^{-1}(cQ)$ for any $Q \subset Y$.

Lemma 3. [20, Lemma 7.3] A mapping $h : (X, \mu) \to (Y, \eta)$ is (μ, η)-open if and only if $h^{-1}(cQ) \subset ch^{-1}(Q)$ for any $Q \subset Y$.

3. (s, v)-bigeneralized submaximal Space

Previously, few debilitated forms of open sets are studied. Császár was introduced and studied some new open sets, namely, semi-open, pre-open, etc.... in [2, 3, 6]. Using these tools, the necessary condition for a BGTS is a (s, v)-bigeneralized submaximal space is proved.

In this section, some shortcuts for examining whether a bigeneralized topological space is (s, v)-bigeneralized submaximal space or not are given. In a (s, v)-bigeneralized submaximal space, we prove some easier way to check whether the given set is nowhere dense or not.

We begin by remembering some needed definitions.

In [3], let (X, μ) be a GTS and $Q \subset X$ is called as;
i). μ*-semi-open* if $Q \subset c_\mu(i_\mu(Q))$.
ii). μ*-pre-open* if $Q \subset i_\mu(c_\mu(Q))$.
iii). μ*-*α*-open* if $Q \subset i_\mu(c_\mu(i_\mu(Q)))$.
iv). μ*-*β*-open* if $Q \subset c_\mu(i_\mu(c_\mu(Q)))$.
v). μ*-b-open* [4] if $Q \subset c_\mu(i_\mu(Q)) \cup i_\mu(c_\mu(Q))$.

Let Q be a subset of a GTS (X, μ). Then *frontier of* Q [15] is denoted by $Fr_\mu(Q)$ and defined by $Fr_\mu(Q) = c_\mu Q \cap c_\mu(X - Q)$.

A subset Q of a BGTS (X, μ_1, μ_2) is said to be (s, v)*-nowhere dense* [1] set in X if $i_s(c_v(Q)) = \emptyset$ where $s, v = 1, 2$ and $s \neq v$.

We denoted by, $(s, v) - \mathcal{N}(X) = \{Q \subset X \mid Q \text{ is a } (s, v)\text{-nowhere dense set in } X\}$ where $s, v = 1, 2$; $s \neq v$ [16].

A subset Q of a BGTS (X, μ_1, μ_2) is called (s, v)-dense [16] if $c_s(c_v(Q)) = X$, where $s, v = 1, 2$; $s \neq v$. Notated by, $(s, v) - \mathcal{D}(X) = \{Q \subset X \mid Q$ is a (s, v)-dense set in $X\}$ where $s, v = 1, 2$; $s \neq v$ [16].

Definition 1. Let (X, μ_1, μ_2) be a bigeneralized topological space. A space X is called (μ_s, μ_v)-*bigeneralized submaximal* (briefly, (s, v)-bigeneralized submaximal) if $Q \in \mu_s$ whenever $c_{\mu_v}(Q) = X$ where $s, v = 1, 2$; $s \neq v$.

Example 2. (a). Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}; \mu_1 = \{\emptyset, \{p, q\}, \{p, s\}, \{q, r\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{p, r\}, \{q, s\}, \{p, r, s\}, X\}$. Then $\{p, q\}, \{p, s\}, \{q, r\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are μ_2-dense subsets of X. Also, every μ_2-dense subset of X is a μ_1-open set in X so that X is a $(1, 2)$-bigeneralized submaximal space.

(b). Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}; \mu_1 = \{\emptyset, \{p, s\}, \{q, r\}, \{q, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{p, q\}, \{p, r\}, \{p, s\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$. For that, $\{p, q\}, \{p, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are μ_1-dense subsets of X. Also, every μ_1-dense subset of X is a μ_2-open set in X. Thus, X is a $(2, 1)$-bigeneralized submaximal space.

Definition 3. Let (X, μ_1, μ_2) be a bigeneralized topological space. If X is $(1, 2)$-bigeneralized submaximal and $(2, 1)$-bigeneralized submaximal, then X is called *pairwise bigeneralized submaximal* space.

Example 4. Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}$; $\mu_1 = \{\emptyset, \{p, q\}, \{p, r\}, \{p, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{p\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$.

(a). So that $\{p, q, r\}, \{p, q, s\}, \{p, r, s\}$ and X are μ_2-dense subsets of X, also, every μ_2-dense subset of X is a μ_1-open set of X for that X is $(1, 2)$-bigeneralized submaximal.

(b). Since $\{p\}, \{p, q\}, \{p, r\}, \{p, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are μ_1-dense subset of X and also, every μ_1-dense subset of X is a μ_2-open set in X we have X is $(2, 1)$-bigeneralized submaximal.
Therefore, X is pairwise bigeneralized submaximal.

The following Theorem 5 is the characterization of (s, v)-bigeneralized submaximality for a BGTS.

Theorem 5. *Let (X, μ_1, μ_2) be a BGTS. Then the following are equivalent.*
(a) (s, v)-bigeneralized submaximal space
(b) Every $Q \subset X$ with $i_v(Q) = \emptyset$, is a μ_s-closed set.
(c) $Q \subset X, c_v(Q) - Q$ is a μ_s-closed set where $s, v = 1, 2$ and $s \neq v$.

Proof. We give the detailed proof only for $s = 1, v = 2$.

$(a) \Rightarrow (b)$ Assume that, X is a $(1,2)$-bigeneralized submaximal space and take $Q \subset X$ with $i_2(Q) = \emptyset$ so that $c_{\mu_2}(X - Q) = X$ which implies $X - Q \in \mu_1$, by hypothesis. Therefore, Q is a μ_1-closed set.

$(b) \Rightarrow (c)$ By Lemma 1, $i_2(c_2(Q) - Q) = \emptyset$ whereby by (b) $c_2(Q) - Q$ is a μ_1-closed set.
$(c) \Rightarrow (a)$ Consider, $c_{\mu_2}(Q) = X$ for that $c_2(Q) - Q = X - Q$ so that $X - Q$ is a μ_1-closed set, by assumption which implies $Q \in \mu_1$. Therefore, X is a $(1,2)$-bigeneralized submaximal space.

Proposition 6. *Let (X, μ_1, μ_2) be a BGTS. If X is a (s,v)-bigeneralized submaximal space, then (X, μ_s) is a sGTS where $s, v = 1, 2$; $s \neq v$.*

Proof. If $s = 1, v = 2$ and X is a $(1,2)$-bigeneralized submaximal space, then every μ_2-dense subset of X is μ_1-open and so X is μ_1-open whereby (X, μ_1) is a sGTS. Similarly, we can prove the result for $s = 2; v = 1$.

Proposition 7. *Let (X, μ_1, μ_2) be a (s,v)-bigeneralized submaximal space. If $Q \in (v,s) - \mathcal{N}(X)$, then it is a μ_s-closed set where $s, v = 1, 2$ and $s \neq v$.*

Proof. Fix $s = 2, v = 1$, by hypothesis, X is a $(2,1)$-bigeneralized submaximal space and let $Q \in (1,2) - \mathcal{N}(X)$ so $i_1(c_2(Q)) = \emptyset$ for that $i_1(Q) = \emptyset$ whereby by Theorem 5, Q is a μ_2-closed set in X. By similar considerations, we get the proof for $s = 1; v = 2$.

The following Example 8 shows that the reverse implication of Proposition 7 need not be true.

Example 8. Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}$; $\mu_1 = \{\emptyset, \{p, q\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{q, r\}, \{q, s\}, \{r, s\}, \{q, r, s\}\}$.Fix $s = 1$ and $v = 2$. Here every μ_2-dense is μ_1-open so that X is $(1,2)$-bigeneralized submaximal space. Choose $L = \{r, s\}$ we get L is μ_1-closed but not in $(2,1) - \mathcal{N}(X)$. Because, $i_2(c_1(L)) = i_2(L) = L \neq \emptyset$.

Choose $s = 2$ and $v = 1$. Take $\mu_1 = \{\emptyset, \{r\}, \{q, r\}, \{q, s\}, \{q, r, s\}\}$; $\mu_2 = \{\emptyset, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, r, s\}, \{q, r, s\}, X\}$. Since every μ_1-dense set is μ_2-open we have X is $(2,1)$-bigeneralized submaximal space. Consider $K = \{p, r\}$. Then K is μ_2-closed but not in $(1,2) - \mathcal{N}(X)$, because, $i_1(c_2(K)) = i_1(K) = \{r\} \neq \emptyset$.

Proposition 9. *Let (X, μ_1, μ_2) be a (s,v)-bigeneralized submaximal space. If $\mu_s \subset \mu_v$, then (X, μ_v) is a generalized submaximal space where $s, v = 1, 2$; $s \neq v$.*

Proof. Assume that, X is a (s,v)-bigeneralized submaximal space. Take $s = 2, v = 1$ then we get X is $(2,1)$-bigeneralized submaximal and $\mu_2 \subset \mu_1$. Let K be a μ_1-dense set in X we have $i_{\mu_1}(X - K) = \emptyset$ so that $X - K$ is μ_2-closed, by hypothesis and Theorem 5. Thus, K is μ_2-open for that K is μ_1-open, since $\mu_2 \subset \mu_1$. Hence (X, μ_1) is generalized submaximal. By similar arguments, we can prove this result for the case $s = 1, v = 2$.

In a BGTS (X, μ_1, μ_2), if $\mu_1 = \mu_2 = \mu$, then every generalized submaximal space is a pairwise bigeneralized submaximal space, and conversely.

Theorem 10. *Let (X, μ_1, μ_2) and (X, η_1, η_2) be two BGTSs. If $\mu_i \subset \eta_i$ where $i = 1, 2$, and if X is a (μ_s, μ_v)-bigeneralized submaximal space, then X is a (η_s, η_v)-bigeneralized submaximal space for $s, v = 1, 2$ and $s \neq v$.*

Theorem 11. *Let (X, μ_1, μ_2) be a BGTS and Q be a non-null subset of X. If X is a (s, v)-bigeneralized submaximal space, then $c_v(Q) - Q \in (v, s) - \mathcal{N}(X)$ where $s, v = 1, 2$; $s \neq v$.*

Proof. It is enough to the case only for $s = 1, v = 2$. Assume that,

$$X \quad is \quad a \quad (1,2) - bigeneralized \quad submaximal \quad space \tag{1}$$

and Q is a non-null subset of X, consider, $G = c_2(Q) - Q$. By Lemma 1, $i_2(G) = \emptyset$ and so G is a μ_1-closed set, by (1). Thus, $i_2(c_1(G)) = \emptyset$. Therefore, $c_2(Q) - Q \in (2, 1) - \mathcal{N}(X)$.

The below Theorem 12 easily follows from similar considerations in the above Theorem 11.

Theorem 12. *Let (X, μ_1, μ_2) be a BGTS and H be a non-null subset of X. If X is a (v, s)-bigeneralized submaximal space, then $c_sH - H \in (s, v) - \mathcal{N}(X)$ where $s, v = 1, 2$; $s \neq v$.*

Theorem 13 provides tricks to effortlessly check, in a (s, v)-bigeneralized submaximal space, whether the frontier of a given set is (s, v)-nowhere dense or not.

Theorem 13. *Let (X, μ_1, μ_2) be a (s, v)-bigeneralized submaximal space and $Q \in \tilde{\mu}_v$. Then $Fr_v(Q) \in (v, s) - \mathcal{N}(X)$ where $s, v = 1, 2$; $s \neq v$.*

Proof. We give the detailed proof only for $s = 1, v = 2$. Assume that, X is a $(1, 2)$-bigeneralized submaximal space and

$$Q \in \tilde{\mu}_2. \tag{2}$$

Take $E = Fr_2(Q)$. Then $E = c_2Q \cap c_2(X - Q)$. From (2), we have $E = c_2Q - Q$. By Lemma 1, $i_2(E) = \emptyset$ and so E is a μ_1-closed set, by our assumption. Thus, $i_2(c_1(E)) = \emptyset$. Therefore, $Fr_2(Q) \in (2, 1) - \mathcal{N}(X)$.

Example 14 shows that the condition "$Q \in \tilde{\mu}_v$" can not dropped in Theorem 13. Example 15 shows that the condition "X is a (s, v)-bigeneralized submaximal space" is necessary for Theorem 13.

Example 14. (a) Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}$; $\mu_1 = \{\emptyset, \{s\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{q, s\}, \{r, s\}, \{q, r, s\}\}$. Here $\{s\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are μ_2-dense subsets of X. Also, every μ_2-dense is μ_1-open. Therefore, X is a $(1, 2)$-bigeneralized topological space. Let $J = \{q, r\}$. Then

$J \notin \tilde{\mu}_2$. Here $Fr_2(J) = c_2(J) \cap c_2(X - J)$. This implies $Fr_2(J) = X$ which implies that $i_2(c_1(Fr_2(J))) = i_2(X) = \{q, r, s\}$. Thus, $i_2(c_1(Fr_2(J))) \neq \emptyset$.

(b) Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}; \mu_1 = \{\emptyset, \{p, s\}, \{q, s\}, \{p, q, s\}\}$ and $\mu_2 = \{\emptyset, \{s\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$. Here $\{s\}, \{p, q\}, \{p, s\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are μ_1-dense subsets of X. Also, every μ_1-dense set is μ_2-open. Therefore, X is a $(2, 1)$-bigeneralized topological space. Let $K = \{p, q\}$. Then $K \notin \tilde{\mu}_1$. Here $Fr_1(K) = c_1(K) \cap c_1(X - K)$. This implies $Fr_1(K) = X$ which implies that $i_1(c_2(Fr_1(K))) = i_1(X) = \{p, q, s\}$. Thus, $i_1(c_2(Fr_1(K))) \neq \emptyset$.

Example 15. (a) Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}; \mu_1 = \{\emptyset, \{p, s\}, \{r, s\}, \{p, r, s\}\}$ and $\mu_2 = \{\emptyset, \{q, r\}, \{r, s\}, \{q, r, s\}\}$. Take $P = \{q, s\}$. Then $c_{\mu_2}(P) = X$. But $P \notin \mu_1$. Thus, X is not a $(1, 2)$-bigeneralized topological space. Let $K = \{q, r\}$. Then $K \in \tilde{\mu}_2$. Here $Fr_2(K) = c_2(K) \cap c_2(X - K)$. This implies $Fr_2(K) = \{p, s\}$ which implies that $i_2(c_1(Fr_2(K))) = i_2(X) = \{q, r, s\}$. Thus, $i_2(c_1(Fr_2(K))) \neq \emptyset$.

(b) Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s, t\}; \mu_1 = \{\emptyset, \{p, q, s\}, \{p, q, t\}, \{q, s, t\}, \{p, q, s, t\}\}$ and $\mu_2 = \{\emptyset, \{q, r, s\}, \{q, r, t\}, \{r, s, t\}, \{q, r, s, t\}\}$. Take $M = \{q, r\}$. Then $c_{\mu_1}(M) = X$. But $M \notin \tilde{\mu}_2$. Thus, X is not a $(2, 1)$-bigeneralized topological space. Let $J = \{p, q, t\}$. Then $J \in \tilde{\mu}_1$. Here $Fr_1(J) = c_1(J) \cap c_1(X - J)$. This implies $Fr_1(J) = \{r, s\}$ which implies that $i_1(c_2(Fr_1(J))) = i_1(X) = \{p, q, s, t\}$. Thus, $i_1(c_2(Fr_1(J))) \neq \emptyset$.

The below two theorems (Theorem 16 and Theorem 17) reduce the complexity of checking whether the frontier of a given set is in $(s, v) - \mathcal{N}(X)$ or not.

Theorem 16. *Let (X, μ_1, μ_2) be a BGTS. If $Q \in (s, v) - \mathcal{N}(X)$, then $Fr_v(Q) \in (s, v) - \mathcal{N}(X)$ where $s, v = 1, 2$; $s \neq v$.*

Proof. *Fix $s = 1, v = 2$ and let $Q \in (1, 2) - \mathcal{N}(X)$ for that $i_1(c_2(Q)) = \emptyset$, also, $Fr_2(Q) \subseteq c_2(Q)$ which implies that $i_1(Fr_2(Q)) = \emptyset$ which turn implies that $Fr_2(Q) \in (1, 2) - \mathcal{N}(X)$, because, $Fr_2(Q)$ is μ_2-closed. Similarly, we can prove that the result is true for $s = 2, v = 1$.*

Theorem 17. *Let (X, μ_1, μ_2) be a BGTS. If $c_{\mu_v}Q = X$ and if Q is a (s, v)-open set, then $Fr_v(Q) \in (v, s) - \mathcal{N}(X)$ where $s, v = 1, 2$; $s \neq v$.*

Proof. *Consider, $s = 1, v = 2$ and given*

$$c_2(Q) = X. \tag{3}$$

Assume that, Q is a $(1, 2)$-open set for that, $i_1(i_2(Q)) = Q$, also, $Fr_2(Q) = c_2(Q) \cap c_2(X - Q)$ by which $Fr_2(Q) = X - i_2(Q)$ which implies that $c_1(Fr_2(Q)) = c_1(X - i_2(Q))$ so $c_1(Fr_2(Q)) = X - i_1(i_2(Q))$, thus, $c_1(Fr_2(Q)) = X - Q$, whereby by (3), we get $i_2(c_1(Fr_2(Q))) = \emptyset$ and hence $Fr_2(Q) \in (2, 1) - \mathcal{N}(X)$. Similarly, we can prove that the result is true for $s = 2, v = 1$.

The below Corollary 18 directly follows from Theorem 5 and Theorem 17 so the trivial proof is neglected.

Corollary 18. *Let (X, μ_1, μ_2) be a (s, v)-bigeneralized submaximal space and $c_{\mu_v}Q = X$. If Q is a (s, v)-open set, then $Fr_v(Q)$ is μ_s-closed and hence $Fr_v(Q)$ is a (s, v)-closed in X where $s, v = 1, 2$; $s \neq v$.*

The following two theorems (Theorem 19 and Theorem 20) are gives the necessary condition for a BGTS is a (s, v)-bigeneralized submaximal space.

Theorem 19. *Let (X, μ_1, μ_2) be a BGTS and $Q \subset X$. If $c_s(X - Q) \subset c_vQ - Q$, then X is a (s, v)-bigeneralized submaximal space for $s, v = 1, 2$; $s \neq v$.*

$$(a) \longrightarrow X \quad is \quad a \quad (s,v) - bigeneralized \quad submaximal \quad space \longleftarrow (b)$$

$$\uparrow$$

$$(c)$$

where, (a) Every μ_v-pre-open is μ_s-open.
(b) Every μ_v-β-open is μ_s-open.
(c) Every μ_v-b-open is μ_s-open.

The following Theorem 20 describes the above diagram.

Theorem 20. *Let (X, μ_1, μ_2) be a BGTS and μ_v is a sGT. Then X is a (s, v)-bigeneralized submaximal space if any one of the following hold;*
(a) Every μ_v-pre-open is μ_s-open.
(b) Every μ_v-β-open is μ_s-open.
(c) Every μ_v-b-open is μ_s-open where $s, v = 1, 2$ and $s \neq v$.

Proof. We give the detailed proof for only $s = 1, v = 2$.

(a) Assume that, μ_2 is a sGT, every μ_2-pre-open is μ_1-open and let $c_2(Q) = X$ by which $i_2(c_2(Q)) = X$, thus, $Q \subset i_2(c_2(Q))$ so for Q is μ_2-pre-open, whereby by hypothesis, Q is a μ_1-open set and hence X is a $(1, 2)$-bigeneralized submaximal space.

(b) Consider, μ_2 is a sGT, every μ_2-β-open is μ_1-open, we take $c_2(P) = X$ so for $i_2(c_2(P)) = X$, by hypothesis, this implies $P \subset c_2(i_2(c_2(P)))$ which implies P is μ_2-β-open which turn implies that P is a μ_1-open set by which X is a $(1, 2)$-bigeneralized submaximal space.

(c) Given μ_2 is a sGT, every μ_2-b-open is μ_1-open and consider, $c_2(J) = X$ implies $i_2(c_2(J)) = X$, by our assumption so for $J \subset c_2(i_2(J)) \cup i_2(c_2(J))$, this implies J is μ_2-b-open which implies that J is a μ_1-open set, by hypothesis. Therefore, X is a $(1, 2)$-bigeneralized submaximal space.

Next, in the rest of this section with the series of theorems in a BGTS, the significance of (s, v)-bigeneralized submaximality is analyzed via functions.

Theorem 21. *Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two BGTS and $h : (X, \mu_i) \to (Y, \eta_i)$ be a (μ_i, η_i)-open map for $i = 1, 2$. If h is surjective, then image of a (s, v)-bigeneralized submaximal space is (s, v)-bigeneralized submaximal space where $s, v = 1, 2$; $s \neq v$.*

Proof. *It is enough to prove the case only for $s = 1, v = 2$. Assume that, X is a $(1, 2)$-bigeneralized submaximal space. Given $h : (X, \mu_i) \to (Y, \eta_i)$ be a (μ_i, η_i)-open map for $i = 1, 2$. Then*

$$h \quad is \quad a \quad (\mu_1, \eta_1) - open \quad map \tag{4}$$

$$h \quad is \quad a \quad (\mu_2, \eta_2) - open \quad map \tag{5}$$

Let Q be a η_2-dense subset of Y. From (5) and Lemma 3, $c_{\mu_2}(h^{-1}(Q)) = X$. By our assumption, $h^{-1}(Q) \in \mu_1$. By (4), $h(h^{-1}(Q)) \in \eta_1$. Thus, $Q \in \eta_1$, by hypothesis. Hence Y is a $(1, 2)$-bigeneralized submaximal space

Example 22 shows that the hypothesis cannot be dropped in Theorem 21.

Example 22. Consider the bigeneralized topological spaces (X, μ_1, μ_2) and (Y, η_1, η_2) where $X = \{p, q, r, s\}; Y = \{p_1, q_1, c_1, s_1\}$. Define a map $h : (X, \mu_i) \to (Y, \eta_i)$ for $i = 1, 2$ as follows $h(p) = q_1; h(q) = p_1; h(r) = s_1; h(s) = r_1$. Clearly, h is a surjective map.

(a) Let $\mu_1 = \{\emptyset, \{p, q\}, \{p, s\}, \{q, r\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, X\}; \mu_2 = \{\emptyset, \{p\}, \{s\}, \{p, s\}, \{p, q, r\}, \{q, r, s\}, X\}; \eta_1 = \{\emptyset, \{q_1\}, \{p_1, q_1\}, \{p_1, s_1\}, \{q_1, r_1\}, \{p_1, q_1, r_1\}, \{p_1, q_1, s_1\}, \{q_1, r_1, s_1\}, Y\}$ and $\eta_2 = \{\emptyset, \{p_1, s_1\}, \{q_1, s_1\}, \{p_1, q_1, s_1\}\}$. Here $h(P) \in \eta_1$ whenever $P \in \mu_1$. Therefore, h is a (μ_1, η_1)-open map. Let $J = \{p\}$. Then $J \in \mu_2$. But $h(J) \notin \eta_2$. Thus, h is not a (μ_2, η_2)-open map. Here $\{p, s\}, \{p, q, s\}, \{p, r, s\}$ and X are μ_2-dense subsets of X. Also, every μ_2-dense set is μ_1-open. Therefore, X is a $(1, 2)$-bigeneralized submaximal space. Let $K = \{s_1\}$. Then $c_{\eta_2}(K) = Y$. But $K \notin \eta_1$. Thus, Y is not a $(1, 2)$-bigeneralized submaximal space.

(b) Let μ_1, η_1 and η_2 are generalized topologies defined as in (a). Take $\mu_2 = \{\emptyset, \{p\}, \{s\}, \{p, s\}, \{p, q\}, \{p, r\}, \{q, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$. Here $h(P) \in \eta_1$ whenever $P \in \mu_1$. Therefore, h is a (μ_1, η_1)-open map. Let $Q = \{p\}$. Then $Q \in \mu_2$. But $h(Q) \notin \eta_2$. Thus, h is not a (μ_2, η_2)-open map. Here $\{p, q\}, \{p, r\}, \{q, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are μ_1-dense subsets of X. Also, every μ_1-dense set is μ_2-open. Therefore, X is a $(2, 1)$-bigeneralized submaximal space. Let $J = \{p_1, q_1\}$. Then $c_{\eta_1}(J) = Y$. But $J \notin \eta_2$. Thus, Y is not a $(2, 1)$-bigeneralized submaximal space.

(c) Let $\mu_1 = \{\emptyset, \{p\}, \{s\}, \{p, q\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}; \mu_2 = \{\emptyset, \{p, s\}, \{q, s\}, \{p, q, s\}\}; \eta_1 = \{\emptyset, \{p_1, r_1\}, \{r_1, s_1\}, \{p_1, r_1, s_1\}\}$ and $\eta_2 = \{\emptyset, \{p_1, r_1\}, \{q_1, r_1\}, \{p_1, q_1, r_1\}\}$. Here $h(P) \in \eta_2$ whenever $P \in \mu_2$. Therefore, h is a (μ_2, η_2)-open map. Let $J = \{p\}$. Then $J \in \mu_1$. But $h(J) \notin \eta_1$. Thus, h is not a (μ_1, η_1)-open map. Here $\{s\}, \{p, q\}, \{p, s\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X

are μ_2-dense subsets of X. Also, every μ_2-dense set is μ_1-open. Therefore, X is a $(1,2)$-bigeneralized submaximal space. Let $M = \{p_1, q_1, r_1\}$. Then $c_{\eta_2}(M) = Y$. But $M \notin \eta_1$. Thus, Y is not a $(1,2)$-bigeneralized submaximal space.

(d) Let μ_1 and η_1 are generalized topologies defined as in (c). Take $\mu_2 = \{\emptyset, \{p,r\}, \{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{q,r,s\}, X\}$ and $\eta_2 = \{\emptyset, \{q_1, s_1\}, \{p_1, q_1, r_1\}, \{p_1, q_1, s_1\}, \{p_1, r_1, s_1\}, \{q_1, r_1, s_1\}, Y\}$. Here $h(P) \in \eta_2$ whenever $P \in \mu_2$. Therefore, h is a (μ_2, η_2)-open map. Let $K = \{p\}$. Then $K \in \mu_1$. But $h(K) \notin \eta_1$. Thus, h is not a (μ_1, η_1)-open map. Here $\{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{q,r,s\}$ and X are μ_1-dense subsets of X. Also, every μ_1-dense set is μ_2-open. Therefore, X is a $(2,1)$-bigeneralized submaximal space. Let $K = \{r_1\}$. Then $c_{\eta_1}(K) = Y$. But $K \notin \eta_2$. Thus, Y is not a $(2,1)$-bigeneralized submaximal space.

Consider the bigeneralized topological spaces (X, μ_1, μ_2) and (Y, η_1, η_2) where $X = \{p,q,r,s\}; Y = \{p_1, q_1, r_1, s_1, t_1\}$. Define a map $h : (X, \mu_i) \to (Y, \eta_i)$ for $i = 1,2$ as follows $h(p) = p_1; h(q) = q_1; h(r) = r_1; h(s) = s_1$. Clearly, h is not a surjective map.

(e) Let $\mu_1 = \{\emptyset, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{q,s\}, \{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{q,r,s\}, X\}; \mu_2 = \{\emptyset, \{p,q\}, \{q,r\}, \{p,q,r\}\}; \eta_1 = \{\emptyset, \{q_1\}, \{p_1, q_1\}, \{p_1, r_1\}, \{q_1, r_1\}, \{q_1, s_1\}, \{p_1, q_1, r_1\}, \{p_1, q_1, s_1\}, \{p_1, r_1, s_1\}, \{q_1, r_1, s_1\}, \{p_1, q_1, r_1, s_1\}\}$ and $\eta_2 = \{\emptyset, \{p_1, q_1\}, \{q_1, r_1\}, \{p_1, q_1, r_1\}\}$. Here $h(P) \in \eta_1$ whenever $P \in \mu_1$. Therefore, f is a (μ_1, η_1)-open map. Also, $h(M) \in \eta_2$ whenever $M \in \mu_2$. Hence h is a (μ_2, η_2)-open map. Here $\{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{q,s\}, \{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{q,r,s\}$ and X are μ_2-dense subsets of X. Also, every μ_2-dense set is μ_1-open. Therefore, X is a $(1,2)$-bigeneralized submaximal space. Let $J = \{p_1, q_1, r_1, t_1\}$. Then $c_{\eta_2}(J) = Y$. But $J \notin \eta_1$. Thus, Y is not a $(1,2)$-bigeneralized submaximal space.

(f) Let μ_1 and η_1 are generalized topologies defined as in (e). Take $\mu_2 = \{\emptyset, \{p,q\}, \{q,r\}, \{p,q,r\}, \{p,q,s\}, \{q,r,s\}, X\}; \eta_2 = \{\emptyset, \{p_1, q_1\}, \{q_1, r_1\}, \{p_1, q_1, r_1\}, \{p_1, q_1, s_1\}, \{q_1, r_1, s_1\}, \{p_1, q_1, r_1, s_1\}\}$. Here $h(K) \in \eta_1$ whenever $K \in \mu_1$. Therefore, h is a (μ_1, η_1)-open map. Also, $h(Q) \in \eta_2$ whenever $Q \in \mu_2$. Hence h is a (μ_2, η_2)-open map. Here $\{p,q\}, \{q,r\}, \{p,q,r\}, \{p,q,s\}, \{q,r,s\}$ and X are μ_1-dense subsets of X. Also, every μ_1-dense set is μ_2-open. Therefore, X is a $(2,1)$-bigeneralized submaximal space. Let $P = \{q_1, r_1, s_1, t_1\}$. Then $c_{\eta_1}(P) = Y$. But $P \notin \eta_2$. Thus, Y is not a $(2,1)$-bigeneralized submaximal space.

Theorem 23. *Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two BGTSs and $h : (X, \mu_i) \to (Y, \eta_i)$ be a (μ_i, η_i)-continuous map for $i = 1,2$. Then inverse image of a (s,v)-bigeneralized submaximal space is (s,v)-bigeneralized submaximal space, $s, v = 1,2$; $s \neq v$.*

Proof. It follows from Lemma 2 and the similar arguments in Theorem 21.

The hypothesis of Theorem 23 is necessary as shown by Example 24.

Example 24. Consider the bigeneralized topological spaces (X, μ_1, μ_2) and (Y, η_1, η_2) where $X = \{p,q,r,s\}; Y = \{p_1, q_1, r_1, s_1\}$. Define a map $h : (X, \mu_i) \to (Y, \eta_i)$ for $i = 1,2$ as follows $h(p) = p_1; h(q) = q_1; h(r) = r_1; h(s) = s_1$. Clearly, h is a surjective map.

(a) Let $\mu_1 = \{\emptyset, \{p\}, \{r\}, \{p,q\}, \{p,r\}, \{p,s\}, \{q,r\}, \{q,s\}, \{r,s\}, \{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{q,r,s\}, X\}$; $\mu_2 = \{\emptyset, \{p,q\}, \{q,r\}, \{p,q,r\}\}$; $\eta_1 = \{\emptyset, \{p_1\}, \{r_1\}, \{p_1,q_1\}, \{p_1,r_1\}, \{p_1,s_1\}, \{q_1,r_1\}, \{q_1,s_1\}, \{r_1,s_1\}, \{p_1,q_1,r_1\}, \{p_1,q_1,s_1\}, \{p_1,r_1,s_1\}, \{q_1,r_1,s_1\}, Y\}$ and $\eta_2 = \{\emptyset, \{p_1,q_1,r_1\}, \{p_1,q_1,s_1\}, \{p_1,q_1,r_1,s_1\}\}$. Here $h^{-1}(P) \in \mu_1$ whenever $P \in \eta_1$. Therefore, h is a (μ_1,η_1)-continuous map. Let $J = \{p_1,q_1,s_1\}$. Then $J \in \eta_2$. But $h^{-1}(J) \notin \mu_2$. Thus, h is not a (μ_2,η_2)-continuous map. Here $\{p_1\}, \{r_1\}, \{p_1,q_1\}, \{p_1,r_1\}, \{p_1,s_1\}, \{q_1,r_1\}, \{q_1,s_1\}, \{r_1,s_1\}, \{p_1,q_1,r_1\}, \{p_1,q_1,s_1\}, \{p_1,r_1,s_1\}, \{q_1,r_1,s_1\}$ and Y are η_2-dense subsets of Y. Also, every η_2-dense set is η_1-open. Therefore, Y is a $(1,2)$-bigeneralized submaximal space. Let $K = \{q\}$. Then $c_{\mu_2}(K) = X$. But $K \notin \mu_1$. Thus, X is not a $(1,2)$-bigeneralized submaximal space.

(b) Let μ_1, η_1 and μ_2 are generalized topologies defined as in (a). Take $\eta_2 = \{\emptyset, \{p_1,q_1,r_1\}, \{p_1,r_1,s_1\}, Y\}$. Here $h^{-1}(M) \in \mu_1$ whenever $M \in \eta_1$. Therefore, h is a (μ_1,η_1)-continuous map. Let $Q = \{p_1,r_1,s_1\}$. Then $Q \in \eta_2$. But $h^{-1}(Q) \notin \mu_2$. Thus, h is not a (μ_2,η_2)-continuous map. Here $\{p_1,q_1,r_1\}, \{p_1,r_1,s_1\}$ and Y are η_1-dense subsets of Y. Also, every η_1-dense set is η_2-open. Therefore, Y is a $(2,1)$-bigeneralized submaximal space. Let $J = \{p,r,s\}$. Then $c_{\mu_1}(J) = X$. But $J \notin \mu_2$. Thus, X is not a $(2,1)$-bigeneralized submaximal space.

(c) Let $\mu_1 = \{\emptyset, \{q,s\}, \{r,s\}, \{q,r,s\}\}$; $\mu_2 = \{\emptyset, \{p\}, \{p,s\}, \{q,s\}, \{p,q,s\}\}$; $\eta_1 = \{\emptyset, \{p_1,q_1\}, \{p_1,s_1\}, \{p_1,q_1,r_1\}, \{p_1,q_1,s_1\}, \{p_1,r_1,s_1\}, Y\}$ and $\eta_2 = \{\emptyset, \{p_1\}, \{p_1,s_1\}, \{q_1,s_1\}, \{p_1,q_1,s_1\}\}$. Here $h^{-1}(P) \in \mu_2$ whenever $P \in \eta_2$. Therefore, h is a (μ_2,η_2)-continuous map. Let $K = \{p_1,q_1\}$. Then $K \in \eta_1$. But $h^{-1}(K) \notin \mu_1$. Thus, h is not a (μ_1,η_1)-continuous map. Here $\{p_1,q_1\}, \{p_1,s_1\}, \{p_1,q_1,r_1\}, \{p_1,q_1,s_1\}, \{p_1,r_1,s_1\}$ and Y are η_2-dense subsets of Y. Also, every η_2-dense set is η_1-open. Therefore, Y is a $(1,2)$-bigeneralized submaximal space. Let $M = \{p,s\}$. Then $c_{\mu_2}(M) = X$. But $M \notin \mu_1$. Thus, X is not a $(1,2)$-bigeneralized submaximal space.

(d) Let μ_1 and η_1 are generalized topologies defined as in (c). Take $\mu_2 = \{\emptyset, \{p\}, \{p,q\}, \{p,r\}, \{p,s\}, \{q,s\}, \{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{q,r,s\}, X\}$ and $\eta_2 = \{\emptyset, \{p_1\}, \{p_1,q_1\}, \{p_1,r_1\}, \{p_1,s_1\}, \{q_1,s_1\}, \{p_1,q_1,r_1\}, \{p_1,q_1,s_1\}, \{p_1,r_1,s_1\}, \{q_1,r_1,s_1\}, Y\}$ Here $h^{-1}(P) \in \mu_2$ whenever $P \in \eta_2$. Therefore, h is a (μ_2,η_2)-continuous map. Let $J = \{p_1,s_1\}$. Then $J \in \eta_1$. But $h^{-1}(J) \notin \mu_1$. Thus, h is not a (μ_1,η_1)-continuous map. Here $\{p_1\}, \{p_1,q_1\}, \{p_1,r_1\}, \{p_1,s_1\}, \{q_1,s_1\}, \{p_1,q_1,r_1\}, \{p_1,q_1,s_1\}, \{p_1,r_1,s_1\}, \{q_1,r_1,s_1\}$ and Y are η_1-dense subsets of X. Also, every η_1-dense set is η_2-open. Therefore, Y is a $(2,1)$-bigeneralized submaximal space. Let $K = \{q,r,s\}$. Then $c_{\mu_1}K = X$. But $K \notin \mu_2$. Thus, X is not a $(2,1)$-bigeneralized submaximal space.

An interesting result has been proved that, in a pairwise bigeneralized submaximal space, every hyperconnected space is submaximal. The proof is a direct consequence of the definitions so the proof is neglected.

Theorem 25. *Let (X, μ_1, μ_2) be a BGTS. If X is pairwise bigeneralized submaximal space, then the followings are true.*

(a) If (X, μ_1) is hyperconnected, then (X, μ_2) is a generalized submaximal space.
(b) If (X, μ_2) is hyperconnected, then (X, μ_1) is a generalized submaximal space.

Example 26 shows that the condition "(X, μ_i) is hyperconnected" for $i = 1, 2$ is necessary in Theorem 25.

Example 26. (a) Let (X, μ_1, μ_2) be a BGTS defined as in 4. Then X is a pairwise bigeneralized topological space. Here $\{p\} \in \mu_2$ but $c_{\mu_2}(\{p\}) \neq X$. Therefore, (X, μ_2) is not a hyperconnected space. Also, $c_{\mu_1}(\{p\}) = X$ but $\{p\} \notin \mu_1$. Thus, (X, μ_1) is not a generalized submaximal space.

(b) Let (X, μ_1, μ_2) be a BGTS where $X = \{p, q, r, s\}$; $\mu_1 = \{\emptyset, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$. Then $\{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are μ_2-dense subsets of X. Also, every μ_2-dense set is μ_2-open. Therefore, X is $(1, 2)$-bigeneralized submaximal space. Here $\{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are μ_1-dense subsets of X. Also, every μ_1-dense set is μ_2-open. Therefore, X is $(2, 1)$-bigeneralized submaximal space. Hence X is pairwise bigeneralized submaximal space. Let $Q = \{p\}$. Then $Q \in \tilde{\mu}_1$. But $c_{\mu_1}(Q) \neq X$. Therefore, (X, μ_1) is not a hyperconnected space. Also, $c_{\mu_2}(\{p, q\}) = X$ but $\{p, q\} \notin \mu_2$. Thus, (X, μ_2) is not a generalized submaximal space.

4. $(s, v)^{\star}$-bigeneralized submaximal space

Here, we define a space namely, $(s, v)^{\star}$-bigeneralized submaximal space and give few results about this space which is helpful to reduce the complexity to check whether the given BGTS is $(s, v)^{\star}$-bigeneralized submaximal space or not.

We begin with a definition of $(s, v)^{\star}$-bigeneralized submaximal space.

Definition 27. Let (X, μ_1, μ_2) be a BGTS. A space X is said to be $(\mu_s, \mu_v)^{\star}$-bigeneralized submaximal (briefly, $(s, v)^{\star}$-bigeneralized submaximal) if $Q \in \mu_s$ whenever $Q \in (s, v) - \mathcal{D}(X)$ where $s, v = 1, 2$; $s \neq v$.

Example 28. (a) Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}$; $\mu_1 = \{\emptyset, \{q\}, \{s\}, \{p, q\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{p, s\}, \{q, s\}, \{p, q, s\}\}$. Here $\{s\}, \{p, q\}, \{p, s\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are $(1, 2)$-dense subsets of X. Thus, $(1, 2) - \mathcal{D}(X) \subset \mu_1$. Hence X is a $(1, 2)^{\star}$-bigeneralized submaximal space.

(b) Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}$; $\mu_1 = \{\emptyset, \{p, q\}, \{q, r\}, \{p, q, r\}\}$ and $\mu_2 = \{\emptyset, \{q\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$. Here $\{q\}, \{p, q\}, \{p, r\}, \{q, r\}, \{q, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are $(2, 1)$-dense subsets of X. Thus, $(2, 1) - \mathcal{D}(X) \subset \mu_2$. Hence X is a $(2, 1)^{\star}$-bigeneralized submaximal space.

Theorem 29. *Let (X,μ_1,μ_2) be a BGTS. If $c_{\mu_s}(Fr_v(J)) = X$, then $J \in (s,v) - \mathcal{D}(X)$ where $s,v = 1,2$ and $s \neq v$.*

Proof. *Fix $s = 1, v = 2$; assume that, $c_{\mu_1}(Fr_2(J)) = X$ by which $c_1(c_2J \cap c_2(X-J)) = X$ and so $c_1(c_2(J)) = X$, thus, $J \in (1,2) - \mathcal{D}(X)$. Similarly, we can prove that the result is true for $s = 2, v = 1$.*

The below Corollary 30 is the direct consequence of the above Theorem 29 and Definition 27, so the easy proof is omitted.

Corollary 30. *Let (X,μ_1,μ_2) be a $(s,v)^\star$-bigeneralized submaximal space. If $c_{\mu_s}(Fr_v(Q)) = X$, then $Q \in \mu_s$ where $s,v = 1,2$; $s \neq v$.*

Theorem 31. *Let (X,μ_1,μ_2) be a BGTS. If X is a $(s,v)^\star$-bigeneralized submaximal space, then X is a (s,v)-bigeneralized submaximal space where $s,v = 1,2$; $s \neq v$.*

Proof. *Consider, $s = 1, v = 2$; assume that, X is a $(1,2)^\star$-bigeneralized submaximal space and let $c_2(J) = X$ for that reason $c_1(c_2(J)) = X$ it turns out $J \in (1,2) - \mathcal{D}(X)$ whereby by our assumption, $J \in \mu_1$ so it result that X is a $(1,2)$-bigeneralized submaximal space. By the same considerations in the above case, we can prove that this result is true for $s = 2, v = 1$.*

Theorem 32 describes the below diagram.

$$(a) \longrightarrow X \quad is \quad a \quad (s,v)^\star - bigeneralized \quad submaximal \quad space \longleftarrow (b)$$

$$\uparrow$$

$$(c)$$

where, (a) Every μ_v-pre-open is μ_s-open.
(b) Every μ_v-β-open is μ_s-open.
(c) Every μ_v-b-open is μ_s-open.

The following Theorem 32 gives a shortcut for finding the relationship between (v,s)-bigeneralized submaximal space and $(s,v)^\star$-bigeneralized submaximal space in a bigeneralized topological space.

Theorem 32. *Let (X,μ_1,μ_2) be a (v,s)-bigeneralized submaximal space. Then X is a $(s,v)^\star$-bigeneralized submaximal space if any one of the following is true;*
(a) Every μ_v-pre-open is μ_s-open.
(b) Every μ_v-β-open is μ_s-open.
(c) Every μ_v-b-open is μ_s-open where $s,v = 1,2$ and $s \neq v$.

Proof. We give the detailed proof for only $s = 1, v = 2$. Suppose that X is $(2,1)$-bigeneralized submaximal space.

(a) Assume that, every μ_2-pre-open is μ_1-open. Let $Q \in (1,2) - \mathcal{D}(X)$. Then $c_1(c_2(Q)) = X$ and so $c_2(Q)$ is a μ_1-dense set in X. By our assumption, $c_2(Q) \in \mu_2$. Thus, $Q \subset i_2(c_2(Q))$. Therefore, Q is μ_2-pre-open. By hypothesis, $Q \in \mu_1$. Hence X is a $(1,2)^\star$-bigeneralized submaximal space.

(b) Suppose every μ_2-β-open is μ_1-open. Let $K \in (1,2) - \mathcal{D}(X)$. Then $c_2(K) \in \mu_2$, by similar arguments in (a). This implies $K \subset c_2(i_2(c_2(K)))$ which implies that K is μ_2-β-open. By our assumption, $K \in \mu_1$. Therefore, X is a $(1,2)^\star$-bigeneralized submaximal space.

(c) Assume that, every μ_2-b-open is μ_1-open. Let $P \in (1,2) - \mathcal{D}(X)$. Then $c_2(P) \in \mu_2$, by similar arguments in (a). Thus, $P \subset c_2(i_2(P)) \cup i_2(c_2(P))$. This implies P is μ_2-b-open which implies that $P \in \mu_1$, by hypothesis. Therefore, X is a $(1,2)^\star$-bigeneralized submaximal space.

The below Theorem 33 gives a characterization of the $(s,v)^\star$-bigeneralized submaximal space in terms of a closed set. This theorem is a direct implication of Definition 27 so the proof is skipped.

Theorem 33. *Let (X, μ_1, μ_2) be a BGTS. Then the following are equivalent.*
(a) $(s,v)^\star$-bigeneralized submaximal space.
(b) Every $Q \subset X$ with $i_s(i_v(Q)) = \emptyset$, is a μ_s-closed set where $s, v = 1, 2$; $s \neq v$.

Corollary 34. *Let (X, μ_1, μ_2) be a $(s,v)^\star$-bigeneralized submaximal space. Then $c_v(Q) - Q$ is μ_s-closed and hence (v,s)-nowhere dense where $Q \subset X$ and $s, v = 1, 2$; $s \neq v$.*

Proof. This proof is directly follows from above Theorem 33.

Theorem 35. *Let (X, μ_1, μ_2) be a pairwise bigeneralized submaximal space. Then the followings are true.*
(a) If (X, μ_2) is hyperconnected and μ_1 is sGT, then X is a $(1,2)^\star$-bigeneralized submaximal space.
(b) If (X, μ_1) is hyperconnected and μ_2 is sGT, then X is a $(2,1)^\star$-bigeneralized submaximal space.

Proof. We will present the detailed proof only for (a). Suppose (X, μ_2) is hyperconnected and μ_1 is a sGT. Let $P \in (1,2) - \mathcal{D}(X)$. Then $c_2(P)$ is a μ_1-dense set in X and so $c_2(P) \in \mu_2$, since X is $(2,1)$-bigeneralized submaximal space. Thus, $c_2(P) \in \tilde{\mu}_2$. By our assumption, $c_2(P)$ is a μ_2-dense set. This implies P is a μ_2-dense set which implies that $P \in \mu_1$, since X is a $(1,2)$-bigeneralized submaximal space. Therefore, X is a $(1,2)^\star$-bigeneralized submaximal space.

5. $(s,v)^{\star\star}$-bigeneralized submaximal space

In this part, we introduce the notion namely, $(s,v)^\star$-bigeneralized submaximal space. Some conditions are proven to examine the given space as either $(s,v)^\star$-bigeneralized submaximal or not.

Definition 36. Let (X, μ_1, μ_2) be a bigeneralized topological space. A space X is said to be $(\mu_s, \mu_v)^{\star\star}$-bigeneralized submaximal (briefly, $(s, v)^{\star\star}$-bigeneralized submaximal) if $Q \in \mu_v$ whenever $Q \in (s, v) - \mathcal{D}(X)$ where $s, v = 1, 2$; $s \neq v$.

Example 37. (a) Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}; \mu_1 = \{\emptyset, \{p, q\}, \{p, r\}, \{p, q, r\}\}$ and $\mu_2 = \{\emptyset, \{p\}, \{s\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$. Here $\{p\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are $(1, 2)$-dense subsets of X. Also, $(1, 2) - \mathcal{D}(X) \subset \mu_2$. Hence X is a $(1, 2)^{\star\star}$-bigeneralized submaximal space.

(b) Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}; \mu_1 = \{\emptyset, \{r\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{p, r\}, \{q, r\}, \{p, q, r\}\}$. Here $\{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}$ and X are $(2, 1)$-dense subsets of X. Also, $(2, 1) - \mathcal{D}(X) \subset \mu_1$. Hence X is a $(2, 1)^{\star\star}$-bigeneralized submaximal space.

The following Theorem 38 describes the below diagram.

$$(a) \longrightarrow X \quad is \quad a \quad (s,v)^{\star\star} - bigeneralized \quad submaximal \quad space \longleftarrow (b)$$
$$\uparrow$$
$$(c)$$

where, (a) Every μ_s-pre-open is μ_v-open.
(b) Every μ_s-β-open is μ_v-open.
(c) Every μ_s-b-open is μ_v-open.

The following Theorem 38 provides an easier way to check the significance of bigeneralized topological space using different types of open sets.

Theorem 38. *Let (X, μ_1, μ_2) be a BGTS and μ_s is sGT. Then X is a $(s, v)^{\star\star}$-bigeneralized submaximal space if any one of the following is true;*
(a) Every μ_s-pre-open is μ_v-open.
(b) Every μ_s-β-open is μ_v-open.
(c) Every μ_s-b-open is μ_v-open where $s, v = 1, 2$ and $s \neq v$.

Proof. It is enough to prove the case for $s = 1, v = 2$.

(a) Assume that, μ_1 is sGT, every μ_1-pre-open is μ_2-open and let $Q \in (1, 2) - \mathcal{D}(X)$, then $c_1(c_2(Q)) = X$ whereby by hypothesis, $c_1(Q) \supset c_2(Q)$, this implies $c_1(Q) \supset c_1(c_2(Q))$ which implies that $c_1(Q) = X$. Also, $i_1(c_1(Q)) = X$, by our assumption. Thus, $Q \subset i_1(c_1(Q))$ so it result that Q is μ_1-pre-open it turns out $Q \in \mu_2$ and hence X is a $(1, 2)^{\star\star}$-bigeneralized submaximal space.

(b) Suppose μ_1 is sGT and every μ_1-β-open is μ_2-open. Consider, $K \in (1, 2) - \mathcal{D}(X)$ so $c_{\mu_1}(K) = X$, by similar arguments in (a). Thus, $i_1(c_1(K)) = X$, by hypothesis. By which $K \subset c_1(i_1(c_1(K)))$ implies that K is μ_1-β-open. By our assumption, $K \in \mu_2$. Therefore,

X is a $(1,2)^{\star\star}$-bigeneralized submaximal space.

(c) Assume that, μ_1 is sGT and every μ_1-b-open is μ_2-open. Take $J \in (1,2) - \mathcal{D}(X)$ so for $c_{\mu_1}(J) = X$, by similar arguments in (a). By our assumption, $i_1(c_1(J)) = X$. Thus, $J \subset c_1(i_1(J)) \cup i_1(c_1(J))$. This implies J is μ_1-b-open which implies that $J \in \mu_2$, by hypothesis. Hence X is a $(1,2)^{\star\star}$-bigeneralized submaximal space.

The following Example 39 shows that the hypothesis of Theorem 38 can not be dropped.

Example 39. Consider the bigeneralized topological space (X, μ_1, μ_2) where $X = \{p, q, r, s\}$; $\mu_1 = \{\emptyset, \{p, r\}, \{q, r\}, \{p, q, r\}, X\}$ and $\mu_2 = \{\emptyset, \{p, r\}, \{q, r\}, \{p, s\}, \{q, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\}$. Here, $\mu_1 \subset \mu_2$ and μ_1 is a strong generalized topology.
Fix $s = 1$ and $v = 2$.
• Take $K = \{r\}$. Then K is μ_1-pre-open but not μ_2-open.
• Choose $L = \{p, q\}$. We get L is μ_1-β-open but $L \notin \mu_2$.
• Let $W = \{r, s\}$. Then W is μ_1-b-open but not in μ_2.
Since $\{r\}$ is $(1,2)$-dense but not in μ_2 we have X is not a $(1,2)^{\star\star}$-bigeneralized submaximal space.
Fix $s = 2$ and $v = 1$.
Take $\mu_1 = \{\emptyset, \{p, q\}, \{p, s\}, \{r, s\}, \{p, q, s\}, \{p, r, s\}, X\}$ and $\mu_2 = \{\emptyset, \{p, s\}, \{r, s\}, \{p, r, s\}, X\}$.
Thus, $\mu_2 \subset \mu_1$ and μ_2 is a sGT.
Here $\{s\}$ is μ_2-pre-open, μ_2-β-open and μ_2-b-open but not μ_1-open.
Clearly, X is not a $(2,1)^{\star\star}$-bigeneralized submaximal space. Because, Choose $K = \{p, r\}$. then K is $(2,1)$-dense but not μ_1-open.

Theorem 40. *Let (X, μ_1, μ_2) be a BGTS. Then the following are equivalent.*
(a) $(s, v)^{\star\star}$-bigeneralized submaximal space.
(b) Every $Q \subset X$ with $i_s(i_v(Q)) = \emptyset$, is a μ_v-closed set where $s, v = 1, 2$; $s \neq v$.

Proof. This proof is directly follows from Definition 36 so the easy proof is neglected.

Corollary 41. *Let (X, μ_1, μ_2) be a $(s, v)^{\star\star}$-bigeneralized submaximal space. Then $c_v(Q) - Q$ is a μ_v-closed set where $Q \subset X$ and $s, v = 1, 2$; $s \neq v$.*

Theorem 42. *Let (X, μ_1, μ_2) be a $(s, v)^{\star\star}$-bigeneralized submaximal space. If $Q \in (v, s) - \mathcal{N}(X)$, then $c_s Q$ is μ_v-closed, $s, v = 1, 2$; $s \neq v$.*

Proof. We give the detailed proof only for $s = 1, v = 2$. Assume that, X is a $(1,2)^{\star\star}$-bigeneralized submaximal space. Let $Q \in (2,1) - \mathcal{N}(X)$. Then $i_2(c_1(Q)) = \emptyset$ and so $i_1(i_2(c_1(Q))) = \emptyset$. By hypothesis, $c_1(Q)$ is a μ_2-closed set in X.

$$(a) \longrightarrow X \quad is \quad a \quad (v, s)^{\star\star} - bigeneralized \quad submaximal \quad space \longleftarrow (b)$$

$$\uparrow$$

$$(c)$$

where, (a) Every μ_s-pre-open is μ_s-open.
(b) Every μ_s-β-open is μ_s-open.
(c) Every μ_s-b-open is μ_s-open.

The following Theorem 43 describes the above diagram.

Theorem 43. *Let (X, μ_1, μ_2) be a (s, v)-bigeneralized submaximal space. Then X is a $(v, s)^{\star\star}$-bigeneralized submaximal space if any one of the following is true.*
(a) Every μ_s-pre-open set is μ_s-open.
(a) Every μ_s-β-open set is μ_s-open.
(a) Every μ_s-b-open set is μ_s-open where $s, v = 1, 2$ and $s \neq v$.

Proof. It is enough to prove (a) only and the case $s = 1, v = 2$. Assume that, X is a $(1, 2)$-bigeneralized submaximal space. Let $J \in (2, 1) - \mathcal{D}(X)$. Then $c_1(J)$ is μ_2-dense and so $c_1(J) \in \mu_1$. This implies J is μ_1-pre-open which implies that $J \in \mu_1$-open, by hypothesis. Hence X is a $(2, 1)^{\star\star}$-bigeneralized submaximal space.

References

[1] S. Acharjee, Binod Chandra Tripathy, and Kyriakos Papadopoulos. Two forms of pairwise lindelöfness and some results related to hereditary class in a bigeneralized topological space. *New Mathematics and Natural Computation*, 13(2):181–193, 2017.

[2] Császár Akos. Generalized open sets. *Acta mathematica hungarica*, 75, 1997.

[3] Császár Akos. Generalized open sets in generalized topologies. *Acta mathematica hungarica*, 106, 2005.

[4] D. Andrijević. On b-open sets. *Mat. Vesnik*, 48:59 – 64, 1996.

[5] Boonpok Chawalit. Weakly open functions on bigeneralized topological spaces. *Int. Journal of Math. Analysis*, 4(18):891–897, 2010.

[6] Á. Császár. Generalized topology, generalized continuity. *Acta Math. Hungar.*, 96:351–357, 2002.

[7] Á. Császár. Product of generalized topologies. *Acta Math. Hungar.*, 123:127 – 132, 2009.

[8] Erdal Ekici. Generalized hyperconnectedness. *Acta Mathematica Hungarica*, 133, 2011.

[9] Erdal Ekici. Generalized submaximal spaces. *Acta Math. Hungar.*, 134:132 – 138, 2012.

[10] Korczak-Kubiak Ewa, Loranty Anna, and Pawlak Ryszard J. Baire generalized topological spaces, generalized metric spaces and infinite games. *Acta Mathematica Hungarica*, 140(3):203–231, 2013.

[11] J.C. Kelly. Bitopological spaces. *Pro. London Math. Soc.*, 3(13):71 – 79, 1969.

[12] W. K. Min. Some results on generalized topological spaces and generalized systems. *Acta Math. Hungar.*, 108:171 – 181, 2005.

[13] S.Ale, S.A.Akande, B. Fadipe, A. Tiamiyu, and Q. Rauf. Mathematical model of the public campaign on typhoid fever transmission and control. *International Journal on Recent Trends in Life Science and Mathematics*, 9(3):1–8, 2022.

[14] Vadakasi Subramanian, Yasser Farhat, and Preecha Yupapin. On nowhere dense sets. *European Journal of Pure and Applied Mathematics*, 15(2):403–414, 2022.

[15] Renukadevi V and Vadakasi S. Modifications of strongly nodec spaces. *Communications in Advanced Mathematical Sciences*, 2:99–112, 2018.

[16] S. Vadakasi. Density on bigeneralized topological space. *Communicated*.

[17] V.Yadav, B.K.Chaturvedi, and A.K.Malik. Advantages of fuzzy techniques and applications in inventory control. *International Journal on Recent Trends in Life Science and Mathematics*, 9(3):9–13, 2022.

[18] Dungthaisong Wichai, Boonpok Chawalit, and Viriyapong Chokchai. Generalized closed sets in bigeneralized topological spaces. *International Journal of Mathematical Analysis*, 5(24):1175–1184, 2011.

[19] M. R. Ahmadi Zand and R. Khayyeri. Generalized g_δ-submaximal spaces. *Acta Math. Hungar.*, 149(2):274 – 285, 2016.

[20] Li Zhaowen and Lin Funing. Baireness on generalized topological spaces. *Acta Mathematica Hungarica*, 139(4), 2013.