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Second order Expansions for Extreme Quantiles of Burr Distributions and Asymptotic Theory of Record Values

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Abstract. In this paper we investigate the Burr distributions family which contains twelve members. Second order expansions of quantiles of the Burr's distributions are provided on which may be based statistical methods, in particular in extreme value theory. Beyond the proper interest of these expansions, we apply them to characterize the asymptotic laws of their records of Burr's distributions, lead to new statistical tests.

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1. Introduction

This papers deals of asymptotic laws of records values, extreme value theory when applied to the important family of Burr distributions ([7], [8] and [9]). Let us introduce to each of these three elements of the paper.

Records theory. The notion of records is present in real life at any corner. It is said that the year 2021 is the hottest one in History. In general, Records of many natural phenomena are monitored on a regularly basis: the coldest or hottest day, month, year, etc,; the rainiest month, year, etc.; the month or year with the greatest or smallest number of car or planes crashes, of biggest or smallest gross domestic product (GDP) [four countries]. At least any

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superlative corresponds to a record (lower record for positive superlatives, upper record for naive ones). Generally, records are associated to catastrophes or to big successes. Sports is punctuated by beaten records, in Olympic games, World championships. For example, the (lower records) of 100m in Athletics is particularly followed. Fr upper records in Sports, we can cite the upper apnea record (time spent under water). So, the notion of record is extremely present in real file and it's modeling is highly valuable.

Univariate Extreme value Theory (UEVT). That theory is strongly connected to Records theory. Given a series of data $(X_j)_{j\geq 1}$, the $UEVT$ mainly studies the behavior of the partial maxima $M_n = max(X_1, \cdots, X_n)$, $n \geq 1$. Of course, each M_{n+1} is a strong upper record value if and only if it exceeds the preceding one, that is $M_{n+1} > M_n$. The importance of $UEVT$ resides in the following paradox. It happens that some extreme events, for example $p = \mathbb{P}(X > x)$ for large values of x, are not observed in samples and so, any plug-in estimator is exactly zero. In that context, how can we estimate the probability of occurrence of such events. The probability is usually given in the form $1/T$, where T is defined as the temps needed to see a new occurrence of the event of exceedance of x. For large values of x, T is counted in thousands or more. In conclusion, the $UEVT$ is the theory of rare events. It's applications are countless and very important to circumvent catastrophes.

As expected, the asymptotic law of records values is strongly influenced by the asymptotic behavior of the extreme value. On that basis, we will give a more detailed account for these two theories but still concise in Subsections 1.1 and 1.2, respectively.

These two asymptotic theories are applied to the family of Burr's statistics whose elements have very interesting statistical properties and have important applications in a significant number of disciplines as in Telecommunications, Reliability, Actuarial Sciences, Survival analysis, etc. So, we have to introduce to that family in Subsection 1.3.

1.1. Univariate Extreme value theory

Let X, X_1, X_2, $\cdots$ be a sequence of independent real-valued randoms, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with common cumulative distribution function F, which has the lower and upper endpoints, the first asymptotic moment function and the generalized inverse function respectively defined by

$$lep(F) = \inf\{x \in \mathbb{R},\ F(x) > 0\},\ uep(F) = \sup\{x \in \mathbb{R},\ F(x) < 1\}$$

$$R(x,F) = \frac{1}{1-F(x)} \int_x^{uep(F)} (1 - F(y))\ dy,\ x \in]lep(F), uep(F)[$$

and

$$F^{-1}(u) = \inf\{x \in \mathbb{R},\ F(x) \geq u\}\ for\ u \in]0,1[\ and\ F^{-1}(0) = F^{-1}(0+).$$

F is said to be in the extreme value domain of attraction of a non-degenerate *df* M whenever there exist real and nonrandom sequences $(a_n > 0)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that for any continuity point x of M,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_{n,n} - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = M(x). \tag{1}$$

It is known that M is necessarily of the family of the Generalized Extreme Value (GEV) *df* :

$$H_\gamma(x) = \exp(-(1+\gamma x)^{-1/\gamma}),\ 1+\gamma x \geq 0, \tag{2}$$

parametrized by $\gamma \in \mathbb{R}$, with $H_0(x) = 1 - \exp(-e^{-x})$, $x \in \mathbb{R}$, for $\gamma = 0$. The parameter γ is called the extreme value index.

In this paper, we use some important facts from EVT that we can summarize below, especially regarding functional representation of *cdf*'s and their quantile functions in the extreme domain of attraction as well as their quantile functions.More details can be found in [15], as a quick introduction on EVT and to have abroad view on how to find the domain of attraction of a *cdf*. We will need the two following propositions.

Proposition 1. *(see [**?**]) We have the following equivalences.*

Let $G(x) = F(e^x)$ *the cdf of the log-transformation. We have :*

(1) If $\gamma > 0$,

$$F \in D(H_\gamma) \ \Leftrightarrow\ (G \in D(H_0) \text{ and } R(x,G) \to \gamma \text{ as } x \to uep(G)).$$

(2) If $\gamma = 0$,

$$F \in D(H_0) \ \Leftrightarrow\ (G \in D(H_0) \text{ and } R(x,G) \to 0 \text{ as } x \to uep(G)).$$

(3) If $\gamma < 0$,

$$F \in D(H_\gamma) \ \Leftrightarrow\ G \in D(H_\gamma).$$

The next proposition provides interesting functional representations of quantile functions of *cdf*'s in the extreme domain of attraction.

Proposition 2. *([13] and [11]) We have the following characterizations for the three extremal domains.*

(a) $F \in D(H_\gamma)$, $\gamma > 0$, *if and only if there exist a constant* c *and functions* $a(u)$ *and* $\ell(u)$ *of* $u \in]0,1[$ *satisfying*

$$(a(u), \ell(u)) \to (0,0) \text{ as } u \to 0,$$

such that F^{-1} *admits the following representation of Karamata*

$$F^{-1}(1-u) = c(1+a(u))u^{-\gamma} \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{3}$$

(b) $F \in D(H_\gamma)$, $\gamma < 0$, *if and only if* $uep(F) < +\infty$ *and there exist a constant* c *and functions* $a(u)$ *and* $\ell(u)$ *of* $u \in]0,1[$ *satisfying*

$$(a(u), \ell(u)) \to (0,0) \text{ as } u \to 0,$$

such that F^{-1} *admit the following representation of Karamata*

$$uep(F) - F^{-1}(1-u) = c(1+a(u))u^{-\gamma} \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{4}$$

(c) $F \in D(H_0)$ *if and only if there exist a constant* d *and a slowly varying function* $s(u)$ *such that*

$$F^{-1}(1-u) = d + s(u) + \int_u^1 \frac{s(t)}{t} dt, 0 < u < 1, \tag{5}$$

and there exist a constant c *and functions* $a(u)$ *and* $\ell(u)$ *of* $u \in]0,1]$ *satisfying*

$$(a(u), \ell(u)) \to (0,0) \text{ as } u \to 0,$$

such that the function $s(u)$ *of* $u \in]0,1[$ *admits the representation*

$$s(u) = c(1+a(u)) \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{6}$$

Moreover, if $F^{-1}(1-u)$ *is differentiable for small values of* u *such that* $r(u) = -u(F^{-1}(1-u))' = -u\ dF^{-1}(1-u)/du$ *is slowly varying at zero, then (5) may be replaced by*

$$F^{-1}(1-u) = d + \int_u^{u_0} \frac{r(t)}{t} dt, 0 < u < u_0 < 1, \tag{7}$$

which will be called a reduced de Haan representation of F^{-1}.

On top of these representations, we may use the simple criteria (See more criteria in [11] and [**?**], recalled in theorem 1).

Proposition 3. *We have:*

(a) $F \in D(G_\gamma)$, $\gamma > 0$ *if and only if* $uep(F) = +\infty$ *and*

$$\forall \lambda > 0,\ u \in]0,1[,\ \lim_{u \to 0} \frac{F^{-1}(1-\lambda u)}{F^{-1}(1-u)} = \lambda^{-\gamma}$$

(b) $F \in D(G_\gamma)$, $\gamma < 0$ *if and only if* $uep(F) < \infty$ *and*

$$\forall \lambda > 0, u \in]0,1[,\ \lim_{u \to 0} \frac{uep(F) - F^{-1}(1-\lambda u)}{uep(F) - F^{-1}(1-u)} = \lambda^{-\gamma}$$

(c) $F \in D(G_\gamma)$, $\gamma = 0$ *if and only there exists a function* $s(u)$ *of* $u \in]0,1[$ *which is slowly varying function at zero and such that*

$$\forall \lambda > 0, u \in]0,1[,\ \lim_{u \to 0} \frac{F^{-1}(1-\lambda u) - F^{-1}(1-u)}{s(u)} = -\log \lambda$$

if and only if

$$\forall \lambda > 0, \forall\, 0 < \mu \neq 1, u \in]0,1[,\ \lim_{u \to 0} \frac{F^{-1}(1-\lambda u) - F^{-1}(1-u)}{F^{-1}(1-\mu u) - F^{-1}(1-u)} = \frac{\log \lambda}{\log \mu}.$$

1.2. Gaussian asymptotic laws of record values

Let us begin by define records values and records times for a sequence of real random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$: Y_1, Y_2, $\cdots$. Record times and record values are defined as follows.

Strong upper record times. Let us put $u(1) = 1$ as the first strong upper record time. For any $n \geq 2$, we define, by induction, whenever the $(n-1)^{th}$ upper record time $u(n-1)$ exists,

$$U_n = \left\{ j > u(n-1),\ Y_j > Y_{u(n-1)} \right\}.$$

Hence, for $n \geq 2$, the $(n)^{th}$ upper record time is $u(n) = +\infty$ if U_n is empty and, otherwise

$$u(n) = \inf U_n.$$

Strong lower record times. Let us put $\ell(1) = 1$ as the first strong lower record time. For any $n \geq 2$, we define, by induction, whenever the $(n-1)^{th}$ lower record time $\ell(n-1)$ exists,

$$L_n = \left\{j > \ell(n-1),\ Y_j < Y_{\ell(n-1)}\right\}$$

Hence, for $n \geq 2$, the *n-th* lower record time is $\ell(n) = +\infty$ if L_n is empty and, otherwise

$$\ell(n) = \inf L_n.$$

Strong record values. For each $n \geq 1$ such that $u(n)$ is finite, we have a sequence of strong upper record values

$$(Y^{(k)} = Y_{u(k)},\ 1 \leq k \leq n).$$

For each $n \geq 1$ such that $\ell(n)$ is finite, we have a sequence of strong lower record values

$$Y_{(k)} = Y_{\ell(k)},\ 1 \leq k \leq n.$$

There are many results on probability laws of record values and record times, especially for iid random variables with common cdf F, eventually associated with the pdf f with respect to the Lebesgue measure λ[or iid random variables with common mass probability functions p]. Important books introduce the the study of records in the *iid* scheme, such as [17], [1], [2], [3], [4], [6], etc. Also, statistical applications are largely available and Characterization problems involving functional equations (See [1], [17], [19], etc.).

In a recent paper, we are interesting in finding the asymptotic laws of records values of elements of Burr's family. In that extent, we will mainly follow [15] who provided practical methods of finding the asymptotic law of record values depending, in general, on the extreme value attraction domain of a distribution F. We intend to use such results to the Burr's family. Let us broaden the notation in order to be able to expose the main theorem of the cited authors. Given the sequence defined above, we consider the sequence of strong record values $X^{(1)} = X_1$, $X^{(n)}$, $\cdots$ and the sequence of record times $U(1) = 1$, $U(2)$, $\cdots$. Some conditions depend on a sequence of finite sum of n standard exponential random variables, $n \geq 1$,

$$S_{(n)} = E_{1,n} + \cdots + E_{n,n},$$

and we denote

$$V_n = \exp(-S_{(n)}) \text{ and } v_n = \exp(-n),\ n \geq 1.$$

From this, we may set

$$(Ha)\ :\ \sup\left\{\left|\frac{s(u)}{s(v)} - 1\right|,\ \min(v_n, V_n) \leq u, v \leq \max(v_n, V_n)\right\} \to_{\mathbb{P}} 0 \text{ as } n \to +\infty,$$

$$(Hb)\ :\ (\exists \alpha > 0),\ \sqrt{n}\ s(v_n) \to \alpha\ as\ n \to +\infty,$$

where $\to_{\mathbb{P}}$ stands for the convergence in probability.

[15] obtained the results below that cover the whole extreme value domain of attraction. Let us begin by asymptotic laws for $F \in \mathcal{D} = D(G_\gamma)$, $\gamma \in \mathbb{R}$.

Theorem 1. *Let $F \in \mathcal{D}$. We have :*

(a) If $\gamma > 0$, the asymptotic law of $X^{(n)}$ is lognormal, precisely

$$\left(\frac{X^{(n)}}{F^{-1}\left(1-e^{-n}\right)}\right)^{n^{-1/2}} \rightsquigarrow LN(0,\gamma^2),$$

where $LN(m,\sigma^2)$ is the lognormal law of parameters m and $\sigma > 0$.

(b) If $\gamma > 0$ and $X > 0$, $Y = \log X \in D(G_\gamma)$ and $R(x,G) \to \gamma$ as $x \to uep(G)$ and we have

$$\frac{Y^{(n)} - G^{-1}\left(1-e^{-n}\right)}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0,\gamma^2).$$

(c) If $\gamma < 0$, the asymptotic law of $X^{(n)}$ is lognormal, precisely

$$\left(\frac{uep(F) - X^{(n)}}{uep(F) - F^{-1}\left(1-e^{-n}\right)}\right)^{n^{-1/2}} \rightsquigarrow \exp(\mathcal{N}(0,\gamma^2)).$$

(d) Suppose that $\gamma = 0$ and $R(x,G) \to 0$ as $x \to uep(G)$. If (Ha) and (Hb) hold both, we have

$$X^{(n)} - F^{-1}\left(1-e^{-n}\right) \rightsquigarrow \mathcal{N}(0,\alpha^2).$$

More precisely, we have : Given $\gamma = 0$, $R(x,G) \to 0$ as $x \to uep(G)$ and (Ha), the above asymptotic normality is valid if and only if (Hb) holds.

The theorem can be extended outside the extreme domain of attraction as follows. Suppose that:

(Ga) $uep(F) = +\infty$ and F is differentiable in some neighborhood of $]x_0,\ +\infty[$.

(Gb) The function

$$s(x) = e^{-x}F^{-1}(1-t)\Big|_{t=e^{-x}},\ e^x < u_0 < 1,\ for\ some\ u_0 \in]0,1[$$

decreases to 0 as $x \to +\infty$ and is : for any sequence $(x_n, y_n)_{n\geq 1}$ such that

$$\limsup_{n\to+\infty} |x_n - y_n|/\sqrt{n} < +\infty,$$

we have, for some $\alpha > 0$,

$$\lim_{n\to+\infty} \sqrt{n}\ s\left(\exp(\min(x_n, y_n))\right) = \lim_{n\to+\infty} \sqrt{n}\ s\left(\exp(\max(x_n, y_n))\right) = \alpha.$$

We have the following generalization.

Theorem 2. *If F satisfies Assumptions (Ga) and (Gb), we have*

$$X^{(n)} - F^{-1}\left(1 - e^{-n}\right) \rightsquigarrow \mathcal{N}(0, \alpha^2).$$

Important result. For $\gamma \neq 0$, we need no condition for the asymptotic law to hold true.

Rule of working . Suppose that F lies in $\mathcal{D}$, the extreme domain of attraction.

(e) If $F \in D(H_\gamma)$, $\gamma \neq 0$, we apply Points (a) or (c) of theorem 1 without any further condition.

(f) If $F \in D(H_0)$ and $\exp(X) \in D(H_\gamma)$ for some $\gamma > 0$, we apply Point (b) without any further condition.

(g) If $F \in D(H_0)$ and $s(u) \to 0$ as $u \to 0$ and if (Ha) and (Hb) hold, we conclude by applying Point (d). If not (as it is for a lognormal law), we search whether $X_1 = \exp(X) \in D(H_\gamma)$ for some $\gamma > 0$ or $X_1 = \exp(X)$ fulfills (Ha) and (Hb). If yes, we conclude by Point (b) or Point (d). If not, we consider $X_2 = \exp(X_1)$, and we continue the process until we reach $X_p = \exp(X_{p-1}) \in D(G_\gamma)$ for some $\gamma > 0$ or $X_p = \exp(X_{p-1})$ for some $p \geq 1$.

Other results in [15] are expressed as representations as in the following theorem.

Theorem 3. *Let $F \in D(H_\gamma)$, $\gamma \in \mathbb{R}$. Then, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ holding a sequence of independent standard exponential random variables $(E_n)_{n\geq 1}$ and a Brownian Process $\{W(t),\ t \geq 0\}$ such that the record values $X^{(n)}$, $n \geq 1$, of the sequence $X_j = F^{-1}\left(1 - e^{E_j}\right)$, $j \geq 1$, satisfy the following representations below under the appropriate conditions. Here, $S_n = E_1 + ... + E_n$, $n \geq 1$, are the partial sums of the sequence $(E_n)_{n\geq 1}$, $S_n^* = n^{-1/2}(S_n - n)$, $v_n = e^{-n}$ and $V_n = e^{-S_n}$. Below, the function $a(u)$, $b(u)$ and $s(u)$ of $u \in]0, 1[$ are those in the representations in Proposition 2.*

By denoting $W_n^* = n^{-1/2}W(n)$ and $c_n = n^{-1/2}\log n$, we have

$$W_n^* \sim \mathcal{N}(0, 1) \quad \textit{and} \quad |S_n^* - W_n^*| = O_{\mathbb{P}}(c_n).$$

Further, we have the following results.

(a) Let $\gamma > 0$. Suppose that

$$1 - \frac{1 + a(V_n)}{1 + a(v_n)} = O_{\mathbb{P}}(a_n), \ \sup\{|b(t)|, \ 0 \leq t \leq v_n \vee V_n\} = O_{\mathbb{P}}(b_n) \ . \tag{8}$$

Then, we have

$$\begin{aligned} \left(\frac{X^{(n)}}{F^{-1}(1 - e^{-n})}\right)^{n^{-1/2}} &= \exp(\gamma S_n^*) + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \exp(\gamma W_n^*) + O_{\mathbb{P}}(a_n \vee b_n \vee c_n). \end{aligned}$$

(b) Let $\gamma > 0$ and $X > 0$, $Y = \log X \in D(G_0)$ and $R(x, G) \to \gamma$ as $x \to uep(G)$ and we have

$$\begin{aligned} \frac{Y^{(n)} - G^{-1}(1 - e^{-n})}{\sqrt{n}} &= \gamma S_n^* + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \gamma W_n^* + O_{\mathbb{P}}(a_n \vee b_n \vee c_n). \end{aligned}$$

(c) Let $\gamma < 0$. Then, by using the rates of convergence in Formula (8), we have

$$\begin{aligned} \left(\frac{uep(F) - X^{(n)}}{uep(F) - F^{-1}(1 - e^{-n})}\right)^{n^{-1/2}} &= \exp(\gamma S_n^*) + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \exp(\gamma W_n^*) + O_{\mathbb{P}}(a_n \vee b_n \vee c_n). \end{aligned}$$

(d) Suppose that $\gamma = 0$ and $R(x, G) \to 0$ as $x \to uep(G)$. Suppose that (Ha) and (Hb) hold both. If

$$\sup\left\{\left|\frac{s(u)}{s(v)} - 1\right|, \ \min(v_n, V_n) \leq u, v \leq \max(v_n, V_n)\right\} = O_{\mathbb{P}}(d_n), \ \textit{and} \ \sqrt{n}s(v_n) - \alpha = O(e_n),$$

we have

$$\begin{aligned} X^{(n)} - F^{-1}\left(1 - e^{-n}\right) &= \alpha S_n^* + O_{\mathbb{P}}(d_n \vee e_n) \\ &= \alpha W_n^* + O_{\mathbb{P}}(c_n \vee d_n \vee e_n). \end{aligned}$$

Rules of working . In the domain of extremal attraction, most of the *cdf*'s which are used in applications are differentiable in a left-neighborhood of the upper endpoint. In such a case, we may take $a \equiv 0$ in Representation (3) and (4) in Proposition 2. By solving easy differential equations, we have the representation for

$$b(u) = -u\biggr(G^{-1}(1-u)\biggr)' - \gamma,\ u \in]0,1[\ \ and\ \ a \equiv 0 \tag{9}$$

for $\gamma > 0$ and

$$b(u) = -\gamma - \frac{u}{F'\biggr(F^{-1}(1-u)\biggr)\biggr(uep(F) - F^{-1}(1-u)\biggr)},\ u \in]0,1[. \tag{10}$$

for $\gamma < 0$, whenever we have $b(u) \to 0$ as $u \to 0$. Consequently, the rate of convergence reduces to $O_{\mathbb{P}}(b_n \vee c_n)$.

For $\gamma = 0$, Representation (7) in Proposition 2 holds for

$$s(u) = -u\biggr(F^{-1}(1-u)\biggr)',\ 0 < u < 1,$$

whenever it is slowly varying at zero and the rate of convergence d_n becomes useless. In such case, the rate of convergence reduces $O_{\mathbb{P}}(d_n \vee c_n)$.

Furthermore, based on the limit $S_n/n \to 1$ as $n \to +\infty$, we get that, for any $\eta \in]0,1[$,

$$\liminf_{n\to+\infty} \mathbb{P}\biggr(e^{-n/\eta} \leq e^{-S_n} \leq e^{-\eta n}\biggr) = 1. \tag{11}$$

So we may replace the rates of convergence d_n *and* b_n by $d_n(\eta)$ and $b_n(\eta)$ defined as follows, for $\eta \in]0,1[$

$$\sup\{|b(t)|,\ 0 \leq t \leq e^{-\eta n}\} = O(b_n(\eta)) \tag{12}$$

and

$$\sup\left\{\left|\frac{s(u)}{s(v)} - 1\right|,\ e^{-n/\eta} \leq u,v \leq e^{-\eta n}\right\} = O(d_n(\eta)). \tag{13}$$

1.3. Burr's Family

We may quote [12], "In a serie of papers, Burr ([7], [8] and [9]) has proposed a versatile family of densities". That class has twelve (12) statistical distributions given in Table 1, in which k, c and r are positive parameters and the support or domain of each element of the family are precised. We added the distribution (Xa) as a version of the distribution (X).

Since, some elements of that family have taken important roles in many parts of Statistics and have been extended a great number of times. Let us denote any Burr distribution by $\mathcal{B}(T, a, b, c)$ where T stands of (I), $\cdots$, (XII), the last cited parameter is the final power of the *cdf*, if such a power exists, and the first parameters are cited in the order of appearance.

Also, that family intersects with celebrated other families of distribution : Pearson, Dagum and Singh-Magdalla families to cite a few. For example the Sing-Madalla (See [21], [10], [18], etc) defined by

$$F_{sm(a,b,c)}(x) = 1 - (1 + ax^c)^{-r}, \ x \geq 0, \quad a > 0, \ b > 0, \ c > 0$$

reduces to the $\mathcal{B}(XII, 1, c, r)$ law. If $X \sim sm(a, b, c)$, the variable $1/X$ becomes a Dagum law with

$$F_{D(a,b,c)}(x) = (1 + ax^{-b})^{-c}, \ x \geq 0, \quad a > 0, \ b > 0, \ c > 0.$$

That element of the Dagum class has been generalized in [18] as the *Topp-Leone Dagum Distribution* of parameters $a > 0$, $b > 0$, $c > 0$, $d > 0$, $f > 0$,

$$F(x) = \left(1 - \left(\left\{1 - (1 + ax^{-b})^{-c}\right\}\right)^d\right)^f, \ x \geq 0, \tag{14}$$

where $f = 2$ is [18]. But we let $f > 0$ and we still have a *cdf.*

Especially, the $\mathcal{B}(XII, a, b, c)$ distribution is an instrumental tool in extreme value distribution (See for example [20]). Also the $\mathcal{B}(III, a, b, c)$ distribution is also a member of the Dagum distributions system ([14]) which Dagum himself named as a *generalized Burr system.*

1.4. Motivations and organization of the paper

Our achievements in that paper is that we entirely characterized the asymptotic laws for record values of *cdf*'s in the Burr family based on the second order expansions of their

Number	$F(x)$	Domain	$F^{-1}(u)$
I	$F(x) = x$	$(0, 1)$	$F^{-1}(u) = u$
II	$\left(1 + e^{-x}\right)^{-r}$	$\mathbb{R}$	$\log\left(\frac{u^{1/r}}{1-u^{1/r}}\right)$
III	$\left(1 + x^{-k}\right)^{-r}$	$\mathbb{R}_+$	$\left(\frac{u^{1/r}}{1-u^{1/r}}\right)^{1/k}$
IV	$\left(1 + \left(\frac{c-x}{x}\right)^{1/c}\right)^{-r}$	$(0, c)$	$\frac{c}{1+\left(\frac{1-u^{1/r}}{u^{1/r}}\right)^{c}}$
V	$\left(1 + ke^{-\tan x}\right)^{-r}$	$(-\pi/2,\ \pi/2)$	$\arctan\left(\log\left(\frac{k}{(u^{-1/r}-1)}\right)\right)$
VI	$\left(1 + ke^{-\sinh x}\right)^{-r}$	$\mathbb{R}$	$\arg\sinh\left(\log\left(\frac{k}{u^{-1/r}-1}\right)\right)$
VII	$2^{-r}\left(1 + \tanh x\right)^{r}$	$\mathbb{R}$	$\arg\tanh\left(2u^{1/r} - 1\right)$
VIII	$\left(\frac{2}{\pi}\arctan(e^{x})\right)^{r}$	$\mathbb{R}$	$\log\left(\tan\left(\frac{\pi}{2}u^{1/r}\right)\right)$
IX	$1-\left(\frac{2}{2+k((1+e^{x})^{r}-1)}\right)$	$\mathbb{R}$	$\log\left(\left(1 + k^{-1}\left(\frac{1+u}{1-u}\right)\right)^{1/r} - 1\right)$
X	$\left(1 + e^{-x^{2}}\right)^{-r}$	$\mathbb{R}_+$	$\left(\log\left(\frac{1}{u^{-1/r}-1}\right)\right)^{1/2}$
XI	$\left(x - \frac{1}{2\pi}\sin(2\pi x)\right)^{r}$	$(0, 1)$	non explicit
XII	$1-\left(1 + x^{c}\right)^{-r}$	$\mathbb{R}_+$	$\left(\left(1 - u\right)^{-1/r} - 1\right)^{1/c}$
Xa	$\left(1 - e^{-x^{2}}\right)^{r}$	$\mathbb{R}_+$	$\left(\log\left(\frac{1}{1-u^{1/r}}\right)\right)^{1/2}$

Table 1: Burr's distributions

quantile functions. Statistical tests are derived from these results. Simulations are given to back the results. But adaptive tests will not be considered here. They will be the object of an applied statistics paper.

The rest of the paper is organized as follows. In Section 2, we expose the second order expansions of their quantile functions of *cdf*'s in the Burr family the extremal domains of attraction and the asymptotic law of their record values. In Section 3, we proceed to a simulation study. The very technical part on the second order expansions of quantile functions in given in Section 4. A conclusion part (Section 5) finishes the paper.

2. Our main results

For each *cdf* element of the Burr family, we give the expansion of the quantile function, determine the extreme value domain $D(G_\gamma)$ and next apply Theorems 1, 2 and 3 in [15] to find the asymptotic law of the record values. However, we need to complete their theorem 3 by Theorem 4 below.

Theorem 4. *Suppose that $\gamma = 0$ and $R(x, G) \to 0$ as $x \to uep(G)$. Suppose that (Ha) holds. If*

$$\sup\left\{\left|\frac{s(u)}{s(v)} - 1\right|,\ \min(v_n, V_n) \leq u, v \leq \max(v_n, V_n)\right\} = O_{\mathbb{P}}(d_n),$$

then we have

$$\frac{X^{(n)} - F^{-1}(1 - e^{-n})}{s(v_n)\sqrt{n}} = S_n^* + O_{\mathbb{P}}(d_n) = W_n^* + O_{\mathbb{P}}(c_n \vee d_n).$$

Proof. In the proof of Theorem 3 in [15], in Formula (25), we have

$$\begin{aligned} X^{(n)} - F^{-1}(1 - e^{-n}) &= s(V_n) - s(v_n) + \int_{v_n}^{V_n} \frac{s(u)}{u}\, du \\ &= s(v_n)\left(\frac{s(V_n)}{s(v_n)} - 1\right) + s(v_n)(S_n - n)(1 + O_{\mathbb{P}}(d_n)). \end{aligned}$$

Hence

$$\frac{X^{(n)} - F^{-1}(1 - e^{-n})}{s(v_n)\sqrt{n}} = \frac{1}{n^{1/2}}\left(\frac{s(V_n)}{s(v_n)} - 1\right) + (1 + O_{\mathbb{P}}(d_n))\ S_n^*$$

$$= \quad S_n^* + \frac{O_{\mathbb{P}}(d_n)}{n^{1/2}} + S_n^* O_{\mathbb{P}}(d_n) = S_n^* + O_{\mathbb{P}}(d_n).$$

So the conclusion

$$\frac{X^{(n)} - F^{-1}(1-e^{-n})}{\sqrt{n}s(v_n)} = S_n^* + O_{\mathbb{P}}\left(d_n\right) = W_n^* + O_{\mathbb{P}}(c_n \vee d_n),$$

is straightforward. Of course if $n^{1/2}s(v_n) \to \alpha > 0$, we get the result of Theorem 2 again. ■

Now, let us state the asymptotic laws of record values for all members of the Burr family and the distribution (Xa) in the theorem below. All the proofs of the quantile function of second order expansions are given in Section 4. For every asymptotic law of record values, we precise the arguments (from Theorems 1 and/or 3, and/or 4) to be applied. We do not need to give the detailed proof of all of them. However, after the statement of all the results, for each argument in Theorems 1 and/or 3, and/or 4, we will give the proof of a case on which it is applied.

Theorem 5. *Let X follows $\mathcal{B}(T,a,b,c)$ with cdf F. We have the following results for the distribution in the table 1.*

Burr I.

(i) The quantile function is expanded as follows.

$$uep(F) - F^{-1}(1-u) = u. \tag{15}$$

(ii) $F \in D(G_\gamma)$, $\gamma = -1$, $uep(F) = 1$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, is given as follows.

$$\begin{aligned}\left(e^n(1-X^{(n)})\right)^{n^{-1/2}} &= \exp(-S_n^*) = \exp(-W_n^*) + O_{\mathbb{P}}(c_n)\\ &\to \exp(\mathcal{N}(0,1)). \end{aligned} \tag{16}$$

Burr II.

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = \log r + \log(1/u) - \frac{r+1}{2r}u + O(u^2). \tag{17}$$

(ii) Here $Z = \exp(X) \in D(G_\gamma)$, $\gamma = 1$, *i.e.,* $X = \log Z$ *and* $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned} \frac{X^{(n)} - n}{\sqrt{n}} &= S_n^* + O_{\mathbb{P}}(d_n(\eta)) = W_n^* + O_{\mathbb{P}}(d_n(\eta) \vee c_n) \\ &\rightarrow \mathcal{N}(0,1). \end{aligned} \tag{18}$$

Burr III.

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = r^{1/k}u^{-1/k}\left(1 - \frac{r+1}{2kr}u + O(u^2)\right). \tag{19}$$

(ii) $F \in D(G_\gamma)$, $\gamma = 1/k$, $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned} \left(\frac{X^{(n)}}{r^{1/k}\exp(n/k)}\right)^{n^{-1/2}} &= \exp\left(\frac{1}{k}S_n^*\right) + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \exp\left(\frac{1}{k}W_n^*\right) + O_{\mathbb{P}}(a_n \vee b_n \vee c_n) \\ &\rightarrow \exp(\mathcal{N}(0, k^{-2})). \end{aligned} \tag{20}$$

Burr IV.

(i) The quantile function is expanded as follows.

$$c - F^{-1}(1-u) = cr^{-c}u^c\left(1 + \frac{c(r+1)}{2r}u + O\left(u^2\right)\right). \tag{21}$$

(ii) $F \in D(G_\gamma)$, $\gamma = -c$, $uep(F) = c$.

(iii) The asymptotic law of the records value $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned}\left(\frac{r^c e^{cn}}{c}\left(c - X^{(n)}\right)\right)^{n^{-1/2}} &= \exp(-cS_n^*) + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \exp(-cW_n^*) + O_{\mathbb{P}}(a_n \vee b_n \vee c_n) \\ &\to \exp(\mathcal{N}(0, c^2)). \end{aligned} \tag{22}$$

Burr V.

(i) The quantile function is expanded as follows.

$$\frac{\pi}{2} - F^{-1}(1-u) = \left(\log\left(\frac{kr}{u}\right)\right)^{-1} - \frac{1}{2}\left(\log\left(\frac{kr}{u}\right)\right)^{-3} + O\left((\log(1/u))^{-5}\right), \tag{23}$$

i.e.,

$$\frac{\pi}{2} - F^{-1}(1-u) = \left(\log\left(\frac{kr}{u}\right)\right)^{-1}\left(1 - \frac{1}{2}\left(\log\left(\frac{kr}{u}\right)\right)^{-2} + O\left((\log(1/u))^{-4}\right)\right). \tag{24}$$

(ii) $F \in D(G_\gamma)$, $\gamma = 0$, $uep(F) = \pi/2$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned}\sqrt{n}\left(\log\left(\frac{\pi}{2} - X^{(n)}\right) + \log n\right) &= -S_n^* + O_{\mathbb{P}}(n^{-1/2}) = -W_n^* + O_{\mathbb{P}}(c_n) \\ &\to \mathcal{N}(0,1). \end{aligned} \tag{25}$$

We also have

$$\begin{aligned}\frac{(\log r + n)^2\left(X^{(n)} - \arctan\left(-\log\left\{\frac{(1-e^{-n})^{-1/r}-1}{k}\right\}\right)\right)}{\sqrt{n}} &= S_n^* + O_{\mathbb{P}}(n^{-1/2}) = W_n^* + O_{\mathbb{P}}(c_n) \\ &\to \mathcal{N}(0,1). \quad \textit{(VAlt)}\end{aligned}$$

Burr VI.

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = \log 2+\log\log kr+\log\log(1/u)+\frac{1}{4}\left(\log\log\frac{kr}{u}\right)^{-2}+O\left(\log\log(1/u)^{-3}\right),\ u<\frac{1}{e}. \tag{26}$$

(ii) $F \in D(G_\gamma)$, $\gamma = 0$, $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned}\sqrt{n}\biggl(\frac{1}{(2n\log(kr))}\exp(X^{(n)})-1\biggr) &= S_n^* + O_{\mathbb{P}}((\log n)^{-3})\\ &= W_n^* + O_{\mathbb{P}}((\log n)^{-3})\\ &\to \mathcal{N}(0,1)\end{aligned} \tag{27}$$

We also have

$$\begin{aligned}\frac{(\log r+n)^2\biggl(X^{(n)} - arcsinh\biggl(-\log\left(\frac{(1-e^{-n})^{-1/r}-1}{k}\right)\biggr)\biggr)}{\sqrt{n}} &= S_n^* + O_{\mathbb{P}}(n^{-1/2})\\ &= W_n^* + O_{\mathbb{P}}(c_n)\\ &\to \mathcal{N}(0,1). \quad (VIAlt)\end{aligned}$$

Burr VII.

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = \log\sqrt{r} + \frac{1}{2}\log(1/u) - \frac{1+r}{4r}u + O\left(u^2\right). \tag{28}$$

(ii) Here $Z = \exp(X) \in D(G_\gamma)$, $\gamma = 1/2$, *i.e.,* $X = \log Z$ *and* $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned}\frac{X^{(n)} - n/2 - \log\sqrt{r}}{\sqrt{n}} &= (1/2)S_n^* + O_{\mathbb{P}}(d_n(\eta)) = (1/2)W_n^* + O_{\mathbb{P}}(c_n)\\ &\to \mathcal{N}(0,1/4).\end{aligned} \tag{29}$$

Burr VIII.

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = \log(2r/\pi) + \log(1/u) - \frac{1-r}{2r}u + O(u^2). \tag{30}$$

(ii) Here $Z = \exp(X) \in D(G_\gamma)$, $\gamma = 1$, *i.e.,* $X = \log Z$ *and* $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned} \frac{X^{(n)} - n - \log(2r/\pi)}{\sqrt{n}} &= S_n^* + O_{\mathbb{P}}(d_n(\eta)) = W_n^* + O_{\mathbb{P}}(c_n) \\ &\to \mathcal{N}(0,1). \end{aligned} \tag{31}$$

Burr IX.

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = \begin{cases} \frac{1}{r}\log\left(\frac{2}{uk}\right) - \left(\frac{2-k}{2r}\right)u + O(u^2) & \text{if} \quad 0 < r \leq 1/2 \\ \frac{1}{r}\log\left(\frac{2}{uk}\right) - \left(\frac{2-k}{2r}\right)u + O(u^{1/r}) & \text{if} \quad r > 1/2 \end{cases}$$

(ii) Here $Z = \exp(X) \in D(G_\gamma)$, $\gamma = 1/r$, *i.e.,* $X = \log Z$ *and* $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows for both sub-cases.*

$$\begin{aligned} \frac{X^{(n)} - \frac{n}{r} - \frac{1}{r}\log\left(\frac{2}{k}\right)}{\sqrt{n}} &= \frac{1}{r}S_n^* + O_{\mathbb{P}}(d_n(\eta)) = \frac{1}{r}W_n^* + O_{\mathbb{P}}(c_n) \\ &\to \mathcal{N}(0, r^{-2}). \end{aligned} \tag{32}$$

Burr X.

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = (\log(1/u))^{1/2}\left\{1 - \frac{r+1}{4r}\frac{u}{\log(1/u)} + O\left(\frac{u^2}{\log(1/u)}\right)\right\}.$$

(ii) Here $X \in D(G_\gamma)$, $\gamma = 0$, $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned}\sqrt{n}\log\left(\frac{X^{(n)}}{\sqrt{n}}\right) &= \frac{1}{2}S_n^* + O_{\mathbb{P}}(n^{-1/2}) \\ &= \frac{1}{2}W_n^* + O_{\mathbb{P}}(c_n) \\ &\rightarrow \mathcal{N}(0, 1/4). \qquad (33)\end{aligned}$$

We also have

$$\begin{aligned}\frac{(\log r + n)^2\left(X^{(n)} - \left(-\log\left((1-e^{-n})^{-1/r} - 1\right)\right)^{1/2}\right)}{\sqrt{n}} &= S_n^* + O_{\mathbb{P}}(n^{-1/2}) = W_n^* + O_{\mathbb{P}}(c_n) \\ &\rightarrow \mathcal{N}(0,1). \quad \textit{(XAlt)}\end{aligned}$$

Burr XI.

(i) The quantile function is expanded as follows.

$$1 - F^{-1}(1-u) = \alpha^{-1/3}u^{1/3}\left(1 - \frac{\beta}{3\alpha}\alpha^{-1/3}u^{2/3} + O\left(u^{4/9}\right)\right), \qquad (34)$$

where $\alpha = \frac{(2\pi)^2}{6r}$, $\beta = -\frac{(2\pi)^4}{120r}$.

(ii) $F \in D(G_\gamma)$, $\gamma = -1/3$, $uep(F) = 1$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned}\left(\alpha^{1/3}e^{n/3}\left(1 - X^{(n)}\right)\right)^{n^{-1/2}} &= \exp\left(-\frac{1}{3}S_n^*\right) + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \exp\left(-\frac{1}{3}W_n^*\right) + O_{\mathbb{P}}(a_n \vee b_n \vee c_n)\end{aligned}$$

$$\rightarrow \quad \exp\Big(\mathcal{N}(0,1/9)\Big). \tag{35}$$

Burr XII.

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = u^{-1/(rc)}\Big(1 - \frac{1}{c}u^{1/r} + \frac{1-c}{2c^2}u^{2/r} + O(u^{3/r})\Big). \tag{36}$$

(ii) $F \in D(G_\gamma)$, $\gamma = 1/(rc)$, $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned}\left(\frac{X^{(n)}}{e^{n/rc}}\right)^{n^{-1/2}} &= \exp\left(\frac{1}{rc}S_n^*\right) + O_{\mathbb{P}}(a_n \vee b_n)\\ &= \exp\left(\frac{1}{rc}W_n^*\right) + O_{\mathbb{P}}(a_n \vee b_n \vee c_n)\\ &\rightarrow \exp\Big(\mathcal{N}(0,(rc)^{-2})\Big).\end{aligned} \tag{37}$$

Distribution (Xa).

(i) The quantile function is expanded as follows.

$$F^{-1}(1-u) = (\log(1/u))^{1/2}\left(1 + \frac{1-r}{4r}\frac{u}{\log(1/u)} + O\left(\frac{u^2}{\log(1/u)}\right)\right). \tag{38}$$

(ii) Here $\in D(G_\gamma$, $\gamma = 0$, $uep(F) = +\infty$.

(iii) The asymptotic law of the record values $X^{(n)}$, $n \geq 1$, *is given as follows.*

$$\begin{aligned}\sqrt{n}\log\left(\frac{X^{(n)}}{\sqrt{n}}\right) &= \frac{1}{2}S_n^* + O_{\mathbb{P}}(n^{-1/2}) = \frac{1}{2}W_n^* + O_{\mathbb{P}}(c_n)\\ &\rightarrow \mathcal{N}(0,1/4).\end{aligned} \tag{39}$$

We also have

$$\frac{(\log r+n)^2\left(X^{(n)}-\left(-\log\left((1-e^{-n})^{-1/r}-1\right)\right)^{1/2}\right)}{\sqrt{n}} = S_n{}^* + O_{\mathbb{P}}(n^{-1/2})$$
$$= W_n^* + O_{\mathbb{P}}(c_n)$$
$$\rightarrow \mathcal{N}(0,1). \quad \textit{(XaAlt)}$$

Proofs. As announced, we are going to provide the full proof of one case in which one the four arguments is applied. But for cases corresponding to Burr *cdf*'s attracted to $D(G_0)$ with $s(u) \to 0$ as $u \to 0$ (Burr V, Burr VI, Burr X, Burr Xa) , it is easier to draw the asymptotic law of record values directly from the quantile function expansions which are Formulas (25), (27), (33) and (39) respectively. We begin by giving the outlines of the proofs using one of the three points (a,c,d) of Theorems 1 and/or 3.

Next, we give the direct proofs for Burr *cdf*'s attracted to $D(G_0)$ with $s(u) \to 0$ as $u \to 0$.

However, in Appendix (A1), page 651, we will give alternative forms of the asymptotic laws of record values derived form Theorem 4 corresponding to Formulas *(VAlt)*, *(VIAlt)*, *(XAlt)* and *(XaAlt)* respectively.

A - Proofs based on Direct applications of Theorems 1 and/or 3 and/or 4.

Burr I. We have $uep(F) = 1$. Next, by Part (b) of Proposition 3, we have $F \in D(G_\gamma)$ for $\gamma = -1$. The asymptotic law of the record values and the rates of convergence follow from Parts (c) of Theorems 1 and 3.

Burr II. We have $uep(F) = +\infty$ and

$$exp(F^{-1}(1-u)) = ru^{-1}(1+o(u)),\ u \in]0,1[.$$

By Part (a) of Proposition 3, $\exp(X)$ with *cdf* $F \in D(G_\gamma)$ for $\gamma = 1$. So, by Proposition 1, $F \in D(G_0)$. The asymptotic law of the record values and the rates of convergence follow from Theorems 1 and 3.

Burr III. We have $uep(F) = +\infty$. Next, by Part (a) of Proposition 3, we have $F \in D(G_\gamma)$ for $\gamma = 1/k$. The asymptotic law of the record values and the rates of convergence follow from Parts (a) of Theorems 1 and 3.

Burr IV. We have $uep(F) = c$. Next, by Part (b) of Proposition 3, we have $F \in D(G_\gamma)$ for $\gamma = -c$. The asymptotic law of the record values and the rates of convergence follow from Parts (b) of Theorems 1 and 3.

Burr V. See Part B below .

Burr VI. See Part B below .

Burr VII. By Part (a) of Proposition 3, $\exp(X)$ with *cdf* $F \in D(G_\gamma)$ for $\gamma = 1/2$. So, by Proposition 1, $F \in D(G_0)$. The asymptotic law of the record values and the rates of convergence follow from Parts (b) in Theorems 1 and 3.

Burr VIII. By Part (a) of Proposition 3, $\exp(X)$ with *cdf* $F \in D(G_\gamma)$ for $\gamma = 1$. So, by Proposition 1, $F \in D(G_0)$. The asymptotic law of the record values and the rates of convergence follow from Parts (b) in Theorems 1 and 3.

Burr IX. By Part (a) of Proposition 3, $\exp(X)$ with *cdf* $F \in D(G_\gamma)$ for $\gamma = 1/r$. So, by Proposition 1, $F \in D(G_0)$. The asymptotic law of the record values and the rates of convergence follow from Parts (b) in Theorems 1 and 3.

Burr X. See Part B below .

Burr XI. We have $uep(F) = 1$. Next, by Part (b) of Proposition 3, we have $F \in D(G_\gamma)$ for $\gamma = -1/3$. The asymptotic law of the record values and the rates of convergence follow from Parts (c) of Theorems 1 and 3.

Burr XII. $uep(F) = +\infty$. Next, by Part (a) of Proposition 3, we have $F \in D(G_\gamma)$ for $\gamma = 1/(rc)$. The asymptotic law of the record values and the rates of convergence follow from Parts (a) of Theorems 1 and 3.

Burr Xa. See Part B below.

B - Proofs using direct methods.

Burr V. We have to make a little effort to see that $F \in D(G_0)$. We recall the quantile function (23)

$$F^{-1}(1-u) = \frac{\pi}{2} - \left(\log\left(\frac{kr}{u}\right)\right)^{-1} + O\left((\log(1/u))^{-3}\right). \tag{40}$$

Let $\ell(u) = (\log(kr/u)^{-1}$, $u \in]0,\ u_0[$, $u_0 \in]0,\ 1[$. We have $\ell'(u) = (\log(kr/u))^{-2}/u$. So, for $s(u) = -\log(kr/u)^{-2}$, there existe a real number c_0 such that

$$\ell(u) = c_0 - \int_{u_0}^{u} \frac{s(t)}{t}\, dt.$$

So $\ell(\circ)$ is slowly varying at 0 and by the properties of slowly varying functions (See [19],[11] and [16]), we have that for any $\lambda > 0$, for all $u \in]0,\ u_0[$,

$$\lim_{u\to 0} \frac{\ell(\lambda u) - \ell(u)}{s(u)} = -\log \lambda.$$

By remarking that for $g(u) = O\left((\log(1/u))^{-3}\right)$, we have $g(u) = O(s(u))$ as $u \to 0$. Hence, we may replace $\ell(u)$ by $F^{-1}(1-u)$ in the last limit to get

$$\lim_{u\to 0} \frac{F^{-1}(1-\lambda u) - F^{-1}(1-u)}{s(u)} = -\log \lambda.$$

So $F \in D(G_0)$.

To find the asymptotic law of record values, we can use Theorem 4 as in Appendix (A1), page 651. However, it is easier to use a direct method as below. From Expansion 24, we have

$$\frac{\pi}{2} - F^{-1}(1-u) = (\log(kr/u))^{-1}(1 + O((\log(1/u))^{-2}),\ u \in]0,\ 1[, \tag{41}$$

i.e.,

$$\log\left(\frac{\pi}{2} - F^{-1}(1-u)\right) = -(\log\log(kr/u)) + O((\log(1/u))^{-2}),\ u \in]0,\ 1[.$$

We have for $n \geq 1$,

$$\begin{aligned}
\log\left(\frac{\pi}{2} - X^{(n)}\right) &= \log\left(\frac{\pi}{2} - F^{-1}(1 - e^{-S_{(n)}})\right) \\
&= -\log\left(\log(kr) + S_{(n)}\right) + O_{\mathbb{P}}(n^{-2}) \\
&= -\log\left(S_{(n)}\left(1 + \frac{\log kr}{S_{(n)}}\right)\right) + O_{\mathbb{P}}(n^{-2}) \\
&= -\log S_{(n)} + O_{\mathbb{P}}(n^{-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
\log\left(\frac{\pi}{2}-X^{(n)}\right)+\log n &= -\log(S_{(n)}/n)+O_{\mathbb{P}}(n^{-1})\\
&= -\left(\log\left(1+\frac{S_{(n)}}{n}-1\right)\right)+O_{\mathbb{P}}(n^{-1})\\
&= -\frac{S_{(n)}-n}{n}+O_{\mathbb{P}}(n^{-1/2})+O_{\mathbb{P}}(n^{-1})\\
&= -\frac{S^{*}_{(n)}}{\sqrt{n}}+O_{\mathbb{P}}(n^{-1/2}).\ \square
\end{aligned}$$

Burr VI. We proceed exactly as in the Proof related to Burr V with $s(u)=\left(\log(1/u)\right)^{-1}$, $0<u\leq u_0<1$, $c_0=\log\log(1/u_0)$,

$$\log\left(\log(1/u)\right)=c_0+\int_u^{u_0}\frac{s(t)}{t}\,dt.$$

We get that $F\in D(G_0)$. Let us use a direct method to find the asymptotic law of the record values. We apply the quantile function $V_{(n)}$ to get

$$\exp(X^{(n)}) = 2\log(kr)S_{(n)}(1+O_{\mathbb{P}}((\log n)^{-3}))$$

which leads to

$$\begin{aligned}
\frac{1}{2n\log(kr)}\exp(X^{(n)}) &= \frac{S_{(n)}}{n}(1+O_{\mathbb{P}}((\log n)^{-3}))\\
&= 1+\left(\frac{S_{(n)}}{n}-1\right)(1+O_{\mathbb{P}}((\log n)^{-3}))\\
&= 1+\left(\frac{S_{(n)}-n}{n}\right)(1+O_{\mathbb{P}}((\log n)^{-3})),
\end{aligned}$$

and next

$$\begin{aligned}
\sqrt{n}\left(\frac{1}{(2n)\log(kr)}\exp(X^{(n)})-1\right) &= \frac{S_{(n)}-n}{\sqrt{n}}(1+O_{\mathbb{P}}((\log n)^{-3}))\\
&= S^{*}_{n}(1+O_{\mathbb{P}}((\log n)^{-3})).
\end{aligned}$$

Finally, we have

$$\begin{aligned}\sqrt{n}\left(\frac{1}{(2n)\log(kr)}\exp(X^{(n)})-1\right) &= S_n^* + O_{\mathbb{P}}((\log n)^{-3}))\\ &= W_n^* + O_{\mathbb{P}}((\log n)^{-3}))\\ &\to \mathcal{N}(0,1).\end{aligned}$$

Burr X. We have

$$\log F^{-1}(1-u) = \frac{1}{2}\log\left(\log(1/u)\right)^{1/2} - \frac{r+1}{4r}\frac{u}{\log(1/u)} + O\left(\frac{u^2}{(\log(1/u))^2}\right).$$

When applied to $V_{(n)}$, we get

$$\log X^{(n)} = \frac{1}{2}\log S_{(n)} + O_{\mathbb{P}}((\log n)^{-2}d_n(\eta))$$

and hence, by routine computations,

$$\begin{aligned}\log X^{(n)} - \frac{1}{2}\log n &= \left(\frac{1}{2}\log S_{(n)} + O_{\mathbb{P}}((\log n)^{-2}d_n(\eta))\right) - \frac{1}{2}\log n\\ &= \frac{1}{2}\log\frac{S_{(n)}}{n} + O_{\mathbb{P}}((\log n)^{-2}d_n(\eta))\\ &= \frac{1}{2}\left(\left(\frac{S_{(n)}}{n}-1\right) + O_{\mathbb{P}}\left(\frac{S_{(n)}}{n}-1\right)\right) + O_{\mathbb{P}}((\log n)^{-2}d_n(\eta))\\ &= \frac{1}{2}\frac{1}{\sqrt{n}}S_n^* + O_{\mathbb{P}}(n^{-1}).\end{aligned}$$

We conclude that

$$\begin{aligned}\sqrt{n}\log\left(\frac{X^{(n)}}{\sqrt{n}}\right) &= \frac{1}{2}S_n^* + O_{\mathbb{P}}(n^{-1/2}) = \frac{1}{2}W_n^* + O_{\mathbb{P}}(c_n)\\ &\to \mathcal{N}(0,1/4).\end{aligned}$$

Burr Xa. We treat that case exactly as the case Burr X with the same final result,

$$F^{-1}(1-u) = (\log(1/u))^{1/2} + \frac{1-r}{4r}\frac{u}{(\log(1/u))^{1/2}} + O\left(\frac{u^2}{(\log(1/u))^{1/2}}\right). \tag{42}$$

$$\begin{aligned}\sqrt{n}\log\left(\frac{X^{(n)}}{\sqrt{n}}\right) &= \frac{1}{2}S_n^* + O_{\mathbb{P}}(n^{-1/2}) = \frac{1}{2}W_n^* + O_{\mathbb{P}}(c_n)\\ &\to \mathcal{N}(0,1/4).\end{aligned}$$

■

Remark. By the way, once we have a second order extension of the quantile function, we may directly find the asymptotic law of the record value without applying arguments in Theorems 1 and 3. However, those arguments, when put together, offer a unified approach to determine such asymptotic laws.

3. Simulations

Here we proceed to simulation studies of the asymptotic laws obtained. Since, we only want to illustrate how the results are for medium sizes, we will restric ourselves to two or three case in each extremal domain. The first issue we have to deal with concerns the sample sizes. In general, the sample size is fixed and the statistics using the generated sample are computed alongside related parameters. The situation is not the same with records theory.

Indeed, for a sample $X_1, \cdots, X_n$ of size $n \geq 1$, we study the nr-records. But, it is possible the sample does not have nr records up to n obervations. From [5], we have the following results. Let $N(n)$ be the number records in the sumple. The law of $N(n)$ is the sample is free-distribution. We have:

$$\mathbb{E}(N(n)) = (\log n)(1+o(1)) \text{ } and \text{ } \mathbb{V}ar(N(n)) = (\log n)(1+o(1)), \tag{43}$$

$$\frac{N(n)-\log n}{\sqrt{\log n}} \to \mathcal{N}(0,1), \tag{44}$$

and

$$\liminf_{n\to+\infty} \frac{N(n)-\log n}{\sqrt{2\log n \log\log\log n}} = -11 \text{ } and \text{ } \limsup_{n\to+\infty} \frac{N(n)-\log n}{\sqrt{2\log n \log\log\log n}} = 1. \tag{45}$$

So, for n fixed, we are not sure to have a fixed number of nr records values. For example, by using the gaussian approximation, we have the following probability $p(3)$ of having less that nr records values in a sample of size n in Table 2 (see page 654).

So, while we want to have a powerful test for small samples, we should ensure that we have enough data to use a meaningful number of records. From Table 2, we recommend to use the results for n al least equal to $n = 50$.

Now we are going to simulate the results on two for the cdf's of each extremal types: Burr II and Burr III for $\gamma > 0$, Burr I and Burr IV for $\gamma < 0$, Burr VI and XX for $\gamma = 0$. For each of them we will compute the p-values of the asymptotic normality tests, and we display the qq-plots and the Parzen estimators of the pdf's of the centered and normalized records values. In Figures 1 (for two Burr distributions in Type I), 2 (for two Burr distributions in Type III), 3 (for two Burr distributions in Type II), the *qq-plots* and the Parzen graphs support our findings. Table 3 (in page 655) provides the p-values that validate our statistical tests.

Description of the simulation works.

For each case, we proceed as follow

Step 1:

Generate a $N-$sample of standard uniform law

list the records obtained

if the number of record values is enough (compared to the number of records fixed (nr) at beginnig)

- Compute the statistic test associated to stantard normal law, $Z[i]$
- Compute the proportion of absolute value of Z greater than 1.96 (0.975-quantile of standard normal law)

Step 2:

Repeat **step 1**, $B = 1000$ times

report P_0 le mean of proportion obtained in the tries of **Step 1**

Step 3: Decision

If the value P_0 is less than 5%, we accept the normality

The analysis on the tables

4. Second order expansions of Burr's quantile functions

Quantile of Burr II distribution or parameter $r > 0$.

Its support is $\mathcal{V} = \mathbb{R}$ and its *cdf* is

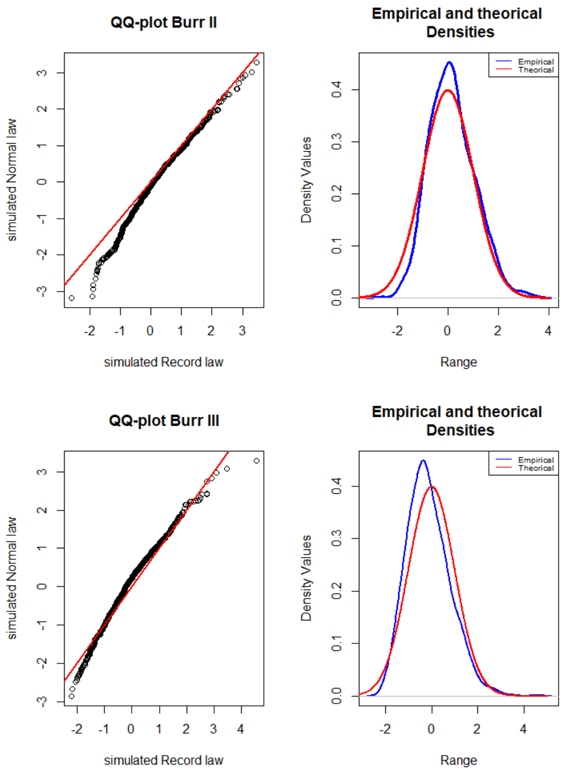

Figure 1: $\gamma > 0$: Burr II and Burr III

$$1 - u = \left(1 + e^{-x}\right)^{-r}, \; x \in \mathcal{V}, \; u \in]0, 1[. \quad (FII)$$

First, we will repeatedly need the following expansions, as $u \to 0$,

$$(1-u)^{-1/r} = 1 + \frac{u}{r} + \frac{r+1}{2r^2}u^2 + O(u^3) \tag{46}$$

$$(1-u)^{1/r} = 1 - \frac{u}{r} + \frac{1-r}{2r^2}u^2 + O(u^3). \tag{47}$$

By applying Expansion (46) on (FII), we get

$$\begin{aligned} -x &= \log\left(\frac{u}{r} + \frac{r+1}{2r^2}u^2 + O(u^3)\right) \\ &= \log\left(\frac{u}{r}\left(1 + \frac{r+1}{2r}u + O(u^2)\right)\right) \end{aligned}$$

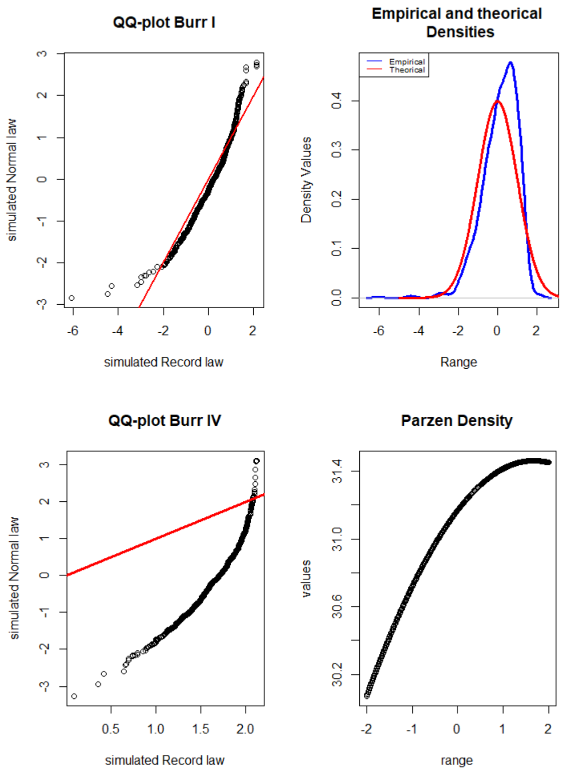

Figure 2: $\gamma < 0$: Burr I and Burr IV

$$= \quad -\log r + \log u + \log\left(1 + \frac{r+1}{2r}u + O(u^2)\right).$$

Now we develop the logarithm in $v = \frac{r+1}{2r}u + O(u^2) = O(u) \to 0$ at the first order, we get

$$-x = -\log r + \log u + \frac{r+1}{2r}u + O(u^2),$$

and we conclude

$$F^{-1}(1-u) = \log r + \log(1/u) - \frac{r+1}{2r}u + O(u^2). \tag{48}$$

Quantile of Burr III distribution of parameters $k > 0$ and $r > 0$.

Its support is $\mathcal{V} = \mathbb{R}_+$ and its *cdf* is

$$1 - u = \left(1 + x^{-k}\right)^{-r}, \ x \in \mathcal{V}, \ u \in]0,1[. \quad (FIII)$$

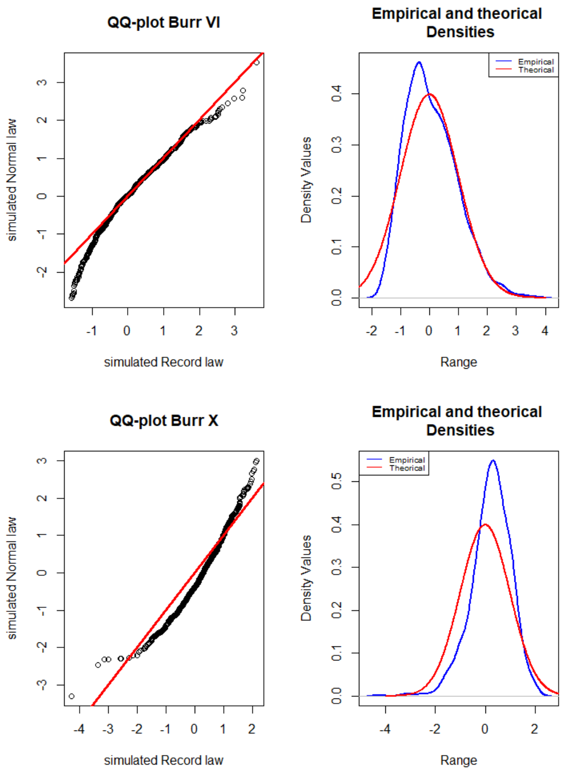

Figure 3: $\gamma = 0$: Burr VI and Burr X

By applying Expansion (46) on $(FIII)$, we get

$$
\begin{aligned}
x &= \left(\frac{u}{r}\left(1+\frac{r+1}{2r}u+O(u^2)\right)\right)^{-1/k} \\
&= r^{1/k}u^{-1/k}\left(1+\frac{r+1}{2r}u+O(u^2)\right)^{-1/k} \\
&= r^{1/k}u^{-1/k}\left(1-\frac{r+1}{2kr}u+O(u^2)\right).
\end{aligned}
$$

We conclude

$$
F^{-1}(1-u) = r^{1/k}u^{-1/k}\left(1-\frac{r+1}{2kr}u+O(u^2)\right). \tag{49}
$$

Quantile of Burr IV distribution of parameters $c > 0$ and $r > 0$. .

Its domain is $\mathcal{V} = [0, c]$ and its *cdf* is given by

$$F(x) = \left[1 + \left(\frac{c-x}{x}\right)^{1/c}\right]^{-r}, \; x \in]0, c].$$

We have for $F(x) = 1 - u$ with $u \in [0; 1[$,

$$1 - u = \left[1 + \left(\frac{c-x}{x}\right)^{1/c}\right]^{-r}$$

$$\begin{aligned}
(1-u)^{-1/r} &= 1 + \left(\frac{c-x}{x}\right)^{1/c} \\
1 + \frac{1}{r}u + \frac{r+1}{2r^2}u^2 + O\left(u^3\right) &= 1 + \left(\frac{c-x}{x}\right)^{1/c} \\
\frac{1}{r}u + \frac{r+1}{2r^2}u^2 + O\left(u^3\right) &= \left(\frac{c-x}{x}\right)^{1/c}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\left(\frac{c-x}{x}\right)^{1/c} &= \frac{1}{r}u + \frac{r+1}{2r^2}u^2 + O\left(u^3\right) \\
\left(\frac{c}{x} - 1\right)^{1/c} &= \frac{1}{r}u\left(1 + \frac{r+1}{2r}u + O\left(u^2\right)\right).
\end{aligned}$$

The last equation leads to,

$$\begin{aligned}
\frac{c}{x} - 1 &= \left[\frac{1}{r}u\left(1 + \frac{r+1}{2r}u + O\left(u^2\right)\right)\right]^c \\
&= r^{-c}u^c\left(1 + \frac{r+1}{2r}u + O\left(u^2\right)\right)^c \\
&= r^{-c}u^c\left(1 + \frac{c(r+1)}{2r}u + O\left(u^2\right)\right).
\end{aligned}$$

Then, we have

$$\frac{c}{x} = 1 + r^{-c}u^c\left(1 + \frac{c(r+1)}{2r}u + O\left(u^2\right)\right). \tag{50}$$

The Equation (50), leads to

$$\frac{x}{c} = \left[1 + r^{-c}u^c \left(1 + \frac{c(r+1)}{2r}u + O\left(u^2\right)\right)\right]^{-1}. \tag{51}$$

Since $c > 0$, we have $r^{-c}u^c \left(1 + \frac{c(r+1)}{2r}u + O\left(u^2\right)\right) \to 0$ as $u \to 0$.

Equation (51), leads to

$$\frac{x}{c} = 1 - r^{-c}u^c \left(1 + \frac{c(r+1)}{2r}u + O\left(u^2\right)\right).$$

That leads to,

$$x = c - cr^{-c}u^c \left(1 + \frac{c(r+1)}{2r}u + O\left(u^2\right)\right).$$

Finally, we have

$$uep(F) - F^{-1}(1-u) = cr^{-c}u^c \left(1 + \frac{c(r+1)}{2r}u + O\left(u^2\right)\right). \tag{52}$$

Quantile of Burr V distribution of parameters $k > 0$ and $r > 0$.

Its support is $\mathcal{V} = [-\pi/2,\ \pi/2]$ and its *cdf* is

$$1 - u = \left(1 + ke^{-\tan x}\right)^{-r},\ x \in \mathcal{V},\ u \in]0, 1[.\ \ (FV)$$

By applying Expansion (46) on (FV), we get $u = 1 - F\left(x\right)$,

$$(1-u)^{-1/r} = 1 + ke^{-\tan(x)}.$$

By (46) , we get

$$1 + ke^{-\tan(x)} = 1 + \frac{u}{r} + \frac{r-1}{2r^2}u^2 + O\left(u^3\right).$$

And so, we get

$$e^{-\tan(x)} = \frac{1}{k}\left(\frac{u}{r} + \frac{(r-1)}{2r^2}u^2 + O\left(u^3\right)\right). \tag{53}$$

Let us set

$$h(x) = e^{-\tan(x)}, \quad x \in]-\frac{\pi}{2}, \frac{\pi}{2}[.$$

So, we get

$$x = h^{-1}\left(\frac{1}{k}\left(\frac{u}{r} + \frac{(r-1)}{2r^2}u^2 + O\left(u^3\right)\right)\right).$$

Let us find h^{-1}. We begin by remarking that

$$\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[,\ \tan(x) = \frac{1}{\tan\left(\frac{\pi}{2} - x\right)}.$$

We set

$$X = \frac{\pi}{2} - x.$$

and remark that $X \to 0+$ as $x \to (\pi/2)^{-}$.

We expand $\tan(X)$ as follows

$$\tan(X) = X + \frac{X^3}{3} + \frac{2}{15}X^5 + O\left(X^7\right).$$

Hence,

$$\begin{aligned} \tan(x) &= \frac{1}{\tan(X)} = X^{-1}\left(1 + \frac{X^2}{3} + \frac{2}{15}X^4 + O\left(X^6\right)\right)^{-1} \\ &= X^{-1}\left(1 - \frac{X^2}{3} + \frac{7}{90}X^4 + O\left(X^6\right)\right). \end{aligned}$$

We had already set

$$\tan(x) = Y = -\log y, as \quad y \downarrow 0. \tag{54}$$

So, we have

$$Y = X^{-1}\left(1 - \frac{X^2}{3} + \frac{7}{90}X^4 + O\left(X^6\right)\right).$$

By the routine methods developed earlier, we have

$$X = Y^{-1}\left(1 - \frac{1}{3}Y^{-2} + O\left(Y^{-4}\right)\right).$$

So

$$\frac{\pi}{2} - x = (\log(1/y))^{-1}\left(1 - \frac{(\log(1/y))^{-2}}{3} + O\left(\log(1/y)^{-4}\right)\right). \tag{55}$$

By formula (54) , we have

$$\tan x = -\log y \Longleftrightarrow e^{-\tan x} = y \Longleftrightarrow h(x) = y.$$

But from Formulas (53) and (54), we may take

$$y =: y(u) = \frac{u}{kr}\left(1 + \frac{r+1}{2r}u + O\left(u^2\right)\right).$$

Hence, Formula (55) becomes

$$\begin{aligned}\frac{\pi}{2} - x &= \log(1/y(u))^{-1}\left(1 - \frac{\log(1/y(u))^{-2}}{3} + O\left(\log(1/y(u))^{-4}\right)\right)\\ &= \log(1/y(u))^{-1} - \frac{\log(1/y(u))^{-3}}{2} + O\left(\log(1/y(u))\right)^{-5}.\end{aligned}$$

But

$$\begin{aligned}\log\left(\frac{1}{y(u)}\right) &= -\log(y(u))\\ &= \log\left(\frac{kr}{u}\right) - \log\left[1 + \frac{r+1}{2r}u + O\left(u^2\right)\right]^{-1}\\ &= \log\left(\frac{kr}{u}\right)\left(1 - \frac{r+1}{2r}\frac{u}{\log\frac{kr}{u}} + O\left(\frac{u^2}{\log\frac{kr}{u}}\right)\right).\end{aligned}$$

Then

$$\left(\log\frac{1}{y(u)}\right)^{-1} = \left(\log\frac{kr}{u}\right)^{-1}\left(1 + \frac{r+1}{2r}\frac{u}{\log\frac{kr}{u}} + O\left(\frac{u^2}{\log\frac{kr}{u}}\right)\right)$$

$$= \left(\log \frac{kr}{u}\right)^{-1} + \frac{r+1}{2r}\frac{u}{\left(\log \frac{kr}{u}\right)^{2}} + O\left(\left(\frac{u}{\log \frac{kr}{u}}\right)^{2}\right).$$

Finally, we have

$$\frac{\pi}{2} - F^{-1}(1-u) = \left(\log\left(\frac{kr}{u}\right)\right)^{-1} - \frac{1}{2}\left(\log\left(\frac{kr}{u}\right)\right)^{-3} + O\left((\log(1/u))^{-5}\right). \qquad (56)$$

Quantile of Burr VI distribution of parameters $k > 0$ and $r > 0$.

Its support is $\mathcal{V} = \mathbb{R}$ and its *cdf* is

$$1 - u = \left(1 + ke^{-\sinh x}\right)^{-r}, \ x \in \mathcal{V}, \ u \in]0, 1[. \quad (FVI)$$

By applying Expansion (46) to Formula (FVI), we have

$$e^{-\sinh(x)} = \frac{1}{k}\left(\frac{u}{r} + \frac{r+1}{2r}u^2 + O\left(u^3\right)\right).$$

Thus

$$y = e^{-\sinh(x)} \Longleftrightarrow \sinh(x) = -\log y =: Y.$$

Then

$$\begin{aligned} \sinh(x) &= \frac{e^x - e^{-x}}{2} = Y \\ &\Longleftrightarrow e^{2x} - 2Ye^x - 1 = 0. \end{aligned}$$

This equation has two solutions:

$$e^x = Y - \sqrt{Y^2+1}(i), \text{ or } e^x = Y + \sqrt{Y^2+1} \ (ii).$$

The solution (i) is impossible since for $Y \geq 0, Y - \sqrt{Y^2+1} \leq 0$ and so, $e^x \neq Y - \sqrt{Y^2+1}$ for any $x \in \mathbb{R}$. We keep Solution (ii). Hence

$$\begin{aligned} x &= \log Y + \log\left(1 + \left(1 + y^{-2}\right)^{\frac{1}{2}}\right) \\ &= \log Y + \log\left(2 + \frac{1}{2}Y^{-2} - \frac{1}{8}Y^{-4} + O\left(Y^{-6}\right)\right) \\ &= \log Y + \log 2 + \frac{1}{4}Y^{-2} - \frac{3}{32}Y^{-4} + O\left(Y^{-6}\right). \qquad (57) \end{aligned}$$

By (57), we have

$$\begin{aligned} y &= y(u) = \frac{1}{k}\left(\frac{u}{r} + \frac{r+1}{2r^2}u^2 + O\left(u^3\right)\right) \\ &= \frac{u}{kr}\left(1 + \frac{r+1}{2r}u + O\left(u^2\right)\right). \end{aligned}$$

So

$$\begin{aligned} -\log y(u) &= \log\frac{kr}{u} - \frac{r+1}{2r}u + O\left(u^2\right) \\ &= \log\frac{kr}{u}\left(1 - \frac{r+1}{2r}\frac{u}{\log\frac{kr}{u}} + O\left(\frac{u^2}{\log\frac{kr}{u}}\right)\right). \end{aligned}$$

Hence

$$\log Y = \log\log\frac{kr}{u} - \frac{r+1}{2r}\left(\frac{u}{\log\frac{kr}{u}}\right) + O\left(\frac{u}{\log 1/u}\right). \qquad (58)$$

By (58), we have

$$Y^{-\alpha} = \left(\log\frac{kr}{u}\right)^{-\alpha}\left(1 + \frac{\alpha(r+1)}{2r}\frac{u}{\log\frac{kr}{u}} + O\left(\frac{u}{\log\frac{kr}{u}}\right)^2\right).$$

Finally, by (57),

$$F^{-1}(1-u) = \log 2 + \log\log kr + \log\log(1/u) + \frac{1}{4}\left(\log\log\frac{kr}{u}\right)^{-2} + O\left(\log\log(1/u)^{-3}\right),\ u < \frac{1}{e}. \qquad (59)$$

Quantile of Burr VII distribution of parameter $r > 0$.

Its support is $\mathcal{V} = \mathbb{R}$ and its *cdf* is

$$1 - u = 2^r \left(1 + \tanh x\right)^r, \; x \in \mathcal{V}, \; u \in]0, 1[. \quad (FVII)$$

By applying Expansion (47) on $(FVII)$, we get

$$\tanh(x) = 2(1-u)^{1/r} - 1 = 1 - \frac{2}{r}u + \frac{1-r}{r^2}u^2 + O\left(u^3\right) =: y.$$

So, we have

$$\frac{e^x - e^{-x}}{e^x + e^x} = \frac{e^{2x} - 1}{e^{2x} + 1} = y \in]-1, 1[.$$

Then,

$$e^{2x} = \frac{y+1}{y-1} \;, \quad y \in]-1, 1[, \; x \in \mathbb{R}.$$

In the formula above, $x \uparrow +\infty$ as $y \uparrow 1$. So

$$\begin{aligned}
x &= \frac{1}{2} \log \frac{y+1}{y-1} \\
&= \frac{1}{2} \left[\log(1+y) - \log(1-y)\right] \\
&= \frac{1}{2} \left[\log\left(2 - \frac{2}{r}u + \frac{1-r}{r^2}u^2 + O\left(u^3\right)\right) - \log\left(\frac{2}{r}u\left(1 - \frac{1-r}{2r}u + O\left(u^2\right)\right)\right)\right] \\
&= \frac{1}{2} \left[\left(\log 2 - \frac{1}{r}u + \frac{1-r}{2r^2}u^2 - \frac{1}{2}\frac{1}{r^2}u^2 + O\left(u^3\right)\right) - \left(\log \frac{2}{r}u - \frac{1-r}{2r}u + O\left(u^2\right)\right)\right].
\end{aligned}$$

So we have,

$$2x = \log r + \log(1/u) - \frac{1+r}{2r}u + O\left(u^2\right).$$

Thus,

$$x = \log \sqrt{r} + \frac{1}{2} \log(1/u) - \frac{1+r}{4r}u + O\left(u^2\right).$$

Hence,

$$F^{-1}(1-u) = \log\sqrt{r} + \frac{1}{2}\log(1/u) - \frac{1+r}{4r}u + O\left(u^2\right). \quad (60)$$

Quantile of Burr VIII distribution of parameter $r > 0$.

Its support is $\mathcal{V} = \mathbb{R}$ and its *cdf* is

$$1-u = \left(\frac{2}{\pi}\arctan(e^x)\right)^r, \ x \in \mathcal{V}, \ u \in]0,1[. \quad (FVIII)$$

Case $r \neq 1$.

By applying Expansion (47) on $(FVIII)$, we get

$$x = \log\left[\tan\left(\frac{\pi}{2}\left\{1 - \frac{u}{r} + \frac{1-r}{2r^2}u^2 + O(u^3)\right\}\right)\right].$$

From there, we use the property that $\tan(\pi/2-u) = 1/\tan(u)$ for u positive and small, to have

$$x = -\log\left[\tan\left(\left\{\frac{u}{r} + \frac{1-r}{2r^2}u^2 + O(u^3)\right\}\right)\right].$$

Next, using expansion $\tan(v) = v + v^3/3 + 2v^5/15 + O(v^7)$ as $v \to 0$ but restricting to the first order, we have

$$x = -\log\left[\tfrac{\pi}{2r}u\left(1 - \tfrac{1-r}{2r}u + O(u^2)\right)\right].$$

Finally, we have

$$F^{-1}(1-u) = \log(2r/\pi) + \log(1/u) - \frac{1-r}{2r}u + O(u^2). \quad (61)$$

Quantile of Burr IX distribution of parameters $k > 0$ **and** $r > 0$.

Its support is $\mathcal{V} = \mathbb{R}$ and its *cdf* is

$$1-u = 1-\left(\frac{2}{2+k\left((1+e^x)^r-1\right)}\right),\ x\in\mathcal{V},\ u\in]0,1[.\quad (FIX)$$

We get

$$\frac{2}{u} = 2+k\left((1+e^x)^r-1\right),$$

which leads to

$$(1+e^x)^r-1 = \frac{1}{k}\left(\frac{2}{u}-2\right),$$

that is

$$\begin{aligned}(1+e^x)^r &= \frac{2u^{-1}}{k}(1-u)+1\\ &= \frac{2u^{-1}}{k}\left(1-u+\frac{ku}{2}\right)\\ &= \frac{2u^{-1}}{k}\left(1-\frac{2-k}{2}u\right).\end{aligned}$$

So, we have by expanding

$$1+e^x = \left(\frac{2u^{-1}}{k}\right)^{1/r}\left(1-\frac{2-k}{2r}u+O(u^2)\right).$$

Then we have

$$e^x = \left(\frac{2u^{-1}}{k}\right)^{1/r}\left(1-\left(\frac{2-k}{2r}\right)u-\left(\frac{ku}{2}\right)^{1/r}+O(u^2)\right)$$

and

$$x = \frac{1}{r}\log\left(\frac{2u^{-1}}{k}\right) - \left(\frac{2-k}{2r}\right)u - \left(\frac{ku}{2}\right)^{1/r} + O(u^2).$$

Now, we conclude, that the quatile depends on the value of $r > 0$, as follows.

$$F^{-1}(1-u) = \begin{cases} \frac{1}{r}\log\left(\frac{2}{uk}\right) - \left(\frac{2-k}{2r}\right)u + O(u^2) & \text{if } 0 < r \leq 1/2 \\ \frac{1}{r}\log\left(\frac{2}{uk}\right) - \left(\frac{2-k}{2r}\right)u + O(u^{1/r}) & \text{if } r > 1/2. \end{cases}$$

Quantile of Burr X distribution of parameter $r > 0$.

Its support is $\mathcal{V} = \mathbb{R}_+$ and its *cdf* is

$$1-u = \left(1 + e^{-x^2}\right)^{-r}, \ x \in \mathcal{V}, \ u \in]0,1[. \quad (FXa)$$

By applying Expansion (46) on (FXa), we get

$$e^{-x^2} = \frac{u}{r}\left(1 + \frac{r+1}{2r}u + O(u^2))\right)$$

and

$$\begin{aligned} -x^2 &= -\log r + \log u + \frac{r+1}{2r}u + O(u^2)) \\ &= -\left(\log(1/u)\left\{1 - \frac{r+1}{2r}\frac{u}{\log(1/u)} + \frac{\log r}{\log(1/u)} + O\left(\frac{u^2}{\log(1/u)}\right)\right\}\right). \end{aligned}$$

So by expanding the latter line at the power $1/2$, we get

$$\begin{aligned} x &= \left(\log(1/u)\left\{1 - \frac{r+1}{2r}\frac{u}{\log(1/u)} + \frac{\log r}{\log(1/u)} + O\left(\frac{u^2}{\log(1/u)}\right)\right\}\right)^{1/2} \\ &= (\log(1/u))^{1/2}\left\{1 - \frac{r+1}{4r}\frac{u}{\log(1/u)} + O\left(\frac{u^2}{\log(1/u)}\right)\right\}. \end{aligned}$$

Finally, we have

$$F^{-1}(1-u) = (\log(1/u))^{1/2}\left\{1 - \frac{r+1}{4r}\frac{u}{\log(1/u)} + O\left(\frac{u^2}{\log(1/u)}\right)\right\}.$$

Quantile of Burr XI distribution of parameter $r > 0$.

Its support is $\mathcal{V} = [0,1]$ and its *cdf* is

$$F(x) = \left(x - \frac{1}{2\pi}\sin(2\pi x)\right)^r, \ x \in \mathcal{V}, \ u \in]0,1[. \ \ (FXI)$$

Let us set $g(x) = \sin(2\pi x)$ of $x \in [0,1]$. The first seven derivatives (at left of $x = 1$) are

$$g'(x) = 2\pi\cos(2\pi x), \ \ g''(x) = -(2\pi)^2\sin(2\pi x) \ , \ g^{(3)}(x) = -(2\pi)^3\cos(2\pi x),$$

$$g^{(4)}(x) = +(2\pi)^4\sin(2\pi x), \ \ g^{(5)}(x) = (2\pi)^5\cos(2\pi x),$$

$$g^{(6)}(x) = -(2\pi)^6\sin(2\pi x) \ \text{ and } \ g^{(7)}(x) = -(2\pi)^6\cos(2\pi x).$$

The even derivatives vanish at $x = 1$ and the odd derivatives take values $(-1)^k(2\pi)^{2k+1}$, $k \geq 0$. Thus, g is expanded at $x = 1$ as follows.

$$\sin(2\pi x) = (2\pi)(x-1) - \frac{(2\pi)^3}{6}(x-1)^3 + \frac{(2\pi)^5}{5!}(x-1)^5 - \frac{(2\pi)^7}{7!}(x-1)^7 + O\left((x-1)^9)\right).$$

So, we have

$$\begin{aligned} F(x) &= \left(x - (x-1) + \frac{(2\pi)^2}{6}(x-1)^3 - \frac{(2\pi)^4}{5!}(x-1)^5 + \frac{(2\pi)^6}{7!}(x-1)^7 + O\left((x-1)^9)\right)\right)^r \\ &= \left(1 + \frac{(2\pi)^2}{6}(x-1)^3 - \frac{(2\pi)^4}{5!}(x-1)^5 + \frac{(2\pi)^6}{7!}(x-1)^7 + O\left((x-1)^9)\right)\right)^r. \end{aligned}$$

We set $v = \frac{(2\pi)^2}{6}(x-1)^3 - \frac{(2\pi)^4}{5!}(x-1)^5 + \frac{(2\pi)^6}{7!}(x-1)^7 + O\left((x-1)^9)\right)$ and the *cdf* becomes

$$\begin{aligned} F(x) &= (1+v)^r \\ &= 1 + rv + \frac{r(r-1)}{2}v^2 + O(v^3) \end{aligned}$$

$$= \quad 1+\frac{(2\pi)^2}{6r}(x-1)^3-\frac{(2\pi)^4}{120r}(x-1)^5+\frac{r(r-1)}{2}\frac{(2\pi)^4}{36}(x-1)^6+O\left((x-1)^7\right).$$

Hence

$$\begin{aligned} 1-F(x) &= \frac{(2\pi)^2}{6r}(1-x)^3-\frac{(2\pi)^4}{120r}(1-x)^5-\frac{r(r-1)}{2}\frac{(2\pi)^4}{36}(x-1)^6+O\left((x-1)^7\right) \\ &= \frac{(2\pi)^2}{6r}(1-x)^3-\frac{(2\pi)^4}{120r}(1-x)^5+O\left((x-1)^6\right) \\ &= \alpha X^3+\beta X^5+O\left(X^6\right). \end{aligned}$$

where $\alpha = \frac{(2\pi)^2}{6r}$, $\beta = -\frac{(2\pi)^4}{120r}$ and $X = 1-x$. We set $u = 1-F(x)$ and we have the following

$$\begin{aligned} u &= \alpha X^3+\beta X^5+O\left(X^6\right) \\ &= \alpha X^3\left(1+\frac{\beta}{\alpha}X^5+O\left(X^3\right)\right). \end{aligned}$$

By the same method used previously,

$$X=\alpha^{-1/3}u^{1/3}\left(1-\frac{\beta}{3\alpha}\alpha^{-1/3}u^{2/3}+O\left(u^{4/9}\right)\right).$$

That leads to,

$$1-x=\alpha^{-1/3}u^{1/3}\left(1-\frac{\beta}{3\alpha}\alpha^{-1/3}u^{2/3}+O\left(u^{4/9}\right)\right).$$

Hence

$$uep\,(F)-F^{-1}\,(1-u)=\alpha^{-1/3}u^{1/3}\left(1-\frac{\beta}{3\alpha}\alpha^{-1/3}u^{2/3}+O\left(u^{4/9}\right)\right). \tag{62}$$

Quantile of Burr XII distribution of parameters $c>0$ and $r>0$.

Its support is $\mathcal{V}=\mathbb{R}_+$ and its *cdf* is

$$1-u=1-\left(1+x^c\right)^{-r},\ x\in\mathcal{V},\ u\in]0,1[.\ \ (FXII)$$

We have

$$\begin{aligned} x &= u^{-1/(rc)}\biggl(1-u^{1/r}\biggr)^{1/c} \\ &= u^{-1/(rc)}\biggl(1-\frac{1}{c}u^{1/r}+\frac{1-c}{2c^2}u^{2/r}+O(u^{3/r})\biggr). \end{aligned}$$

Finally, we have

$$F^{-1}(1-u)=u^{-1/(rc)}\biggl(1-\frac{1}{c}u^{1/r}+\frac{1-c}{2c^2}u^{2/r}+O(u^{3/r})\biggr). \tag{63}$$

Quantile of distribution (Xa) of parameter $r>0$.

Its support is $\mathcal{V}=\mathbb{R}_+$ and its *cdf* is

$$1-u=\biggl(1-e^{-x^2}\biggr)^r,\ x\in\mathcal{V},\ u\in]0,1[.\ \ (Fx)$$

We have

$$x^2=-\log\biggl(1-(1-u)^{-1/r}\biggr)$$

By using the computations as defined in Burr II's case, we get

$$F^{-1}(1-u)=(\log(1/u))^{1/2}\biggl(1+\frac{1-r}{4r}\frac{u}{\log(1/u)}+O\left(\frac{u^2}{\log(1/u)}\right)\biggr). \tag{64}$$

5. Conclusion

Appendix (A1) : Applying Theorem 4 for Burr V, VI, X and Xa.

We compute $s(u)=-u(F^{-1}(1-u))'$ for each of these four cases and remark that $s(u)\to 0$ as $u\to 0$ and $s(\circ)$ is slowly varying at zero. We consider the expression of $F(\circ)$ and $F^{-1}(\circ)$ for Burr V, VI, X and Xa in Table 1 and find the following expressions of $s(\circ)$:

$$(Burr\ V)\quad:\quad F^{-1}(1-u)=arctan\biggl(-\log\left\{\frac{(1-u)^{-1/r}-1}{k}\right\}\biggr)$$

$$s(u) = (\log r/u)^{-2} \frac{\varepsilon(u)d(u)}{\left(1 + \frac{\log k + \log d(u)}{\log r/u}\right)^2 + (\log r/u)^{-2}},$$

with $\varepsilon(u) = (1-u)^{-(r+1)/r}$, $d(u) = (u/r)/\{(1-u)^{-1/r} - 1\}$, $arctan(\circ)$ is the inverse function of the tangent function $\tan(\circ)$

$$(Burr\ VI) \quad : \quad F^{-1}(1-u) = arcsinh\Bigg(-\log\left(\frac{(1-u)^{-1/r} - 1}{k}\right)\Bigg)$$

$$s(u) = (\log r/u)^{-1} \frac{\varepsilon(u)d(u)}{\left(1 + \left(\frac{\log k + \log d(u)}{\log r/u}\right)^2 + (\log r/u)^{-2}\right)^{1/2}},$$

with $\varepsilon(u) = (1-u)^{-(r+1)/r}$, $d(u) = (u/r)/\{(1-u)^{-1/r} - 1\}$, $arcsinh(\circ)$ is the inverse function of the hyperbolic sine function $\sinh(\circ)$

$$(Burr\ X) \quad : \quad F^{-1}(1-u) = \Big(-\log\left((1-u)^{-1/r} - 1\right)\Big)^{1/2}$$

$$s(u) = \frac{1}{2}(\log r/u)^{-1/2} \frac{\varepsilon(u)d(u)}{\left(1 + \frac{\log d(u)}{\log r/u}\right)},$$

with $\varepsilon(u) = (1-u)^{-(r+1)/r}$, $d(u) = (u/r)/\{(1-u)^{-1/r} - 1\}$.

$$(Burr\ Xa) \quad : \quad F^{-1}(1-u) = \Big(-\log\left(1 - (1-u)^{1/r}\right)\Big)^{1/2}$$

$$s(u) = \frac{1}{2}(\log r/u)^{-1/2} \frac{\varepsilon(u)d(u)}{\left(1 + \frac{\log d(u)}{\log r/u}\right)},$$

with $\varepsilon(u) = (1-u)^{(1-r)/r}$, $d(u) = (u/r)/\{1 - (1-u)^{1/r}\}$.

So, for all four cases, we have $s(\circ)$ is slowly varying at zero and hence by Representation (7) in Proposition 2, we have that $F \in D(G_0)$. Also, $s(u) \to 0$ as $u \to 0$.

Now, we are going to use Theorem 4 to all four cases. We give the details for one case, the first for example ($Burr\ V$). We remark that $\varepsilon(u) = 1 + O(u)$ and $d(u) = O(u)$. For

$min(v_n, V_n) \leq u \leq max(v_n, V_n)$, we have $A_n = \min(n, S_{(n)}) \leq \log(1/u) \leq \max(n, S_{(n)}) = B_n$. So uniformly in $(u, v) \in [min(v_n, V_n), max(v_n, V_n)]^2$, we have for any $\eta > 1$

$$\varepsilon(u) = 1 + O_{\mathbb{P}}(d_n(\eta)), \quad d(u) = O_{\mathbb{P}}(d_n(\eta)), \ \log u = O_{\mathbb{P}}(n),$$

which leads to

$$\frac{s(u)}{s(v)} = \left(\frac{\log(1/u) + \log r}{\log(1/v) + \log r}\right)^{-1/2} (1 + O_{\mathbb{P}}(f_n)),$$

for $f_n = n^{-2}$, since the rates $O_{\mathbb{P}}(n^{-1/2})$ are much lower that those of $O_{\mathbb{P}}(d_n(\eta))$. Since

$$A_n = min(n, S_{(n)}) \leq \log(1/u), \log(1/v) \leq max(n, S_{(n)}) = B_n,$$

we have

$$C_n = \left(\frac{\log A_n - \log r}{\log B_n - \log r}\right)^{-1/2} \leq \left(\frac{\log u - \log r}{\log v - \log r}\right)^{-1/2} \leq \left(\frac{\log B_n - \log r}{\log A_n - \log r}\right)^{-1/2} = D_n.$$

It straightforward to show that $(C_n - 1)$ and $(D_n - 1)$ are both $O_{\mathbb{P}}(n^{-1/2})$. So the condition of Theorem 4 holds with $d_n = n^{-1/2}$. By handling the rates appropriately, we get (we recall that $c_n = (\log n)/\sqrt{n}$, $n \geq 1$):

(V) Alternative form of the asymptotic law of record values from Burr V. (with $f_n = n^{-2}$, $d_n = n^{-2}$)

$$\frac{(\log r + n)^2\left(X^{(n)} - \arctan\left(- \log\left\{\frac{(1-e^{-n})^{-1/r} - 1}{k}\right\}\right)\right)}{\sqrt{n}} = S_n^* + O_{\mathbb{P}}(n^{-1/2}) = W_n^* + O_{\mathbb{P}}(c_n) \rightarrow \mathcal{N}(0,1). \quad \textit{(VAlt)}$$

By using again the same techniques, we get the following forms.

(VI) Alternative form of the asymptotic law of record values from Burr VI. (with $f_n = n^{-1/2}$, $d_n = n^{-1/2}$)

$$\frac{(\log r + n)^2\left(X^{(n)} - arcsinh\left(- \log\left\{\frac{(1-e^{-n})^{-1/r} - 1}{k}\right\}\right)\right)}{\sqrt{n}} = S_n^* + O_{\mathbb{P}}(n^{-1/2}) = W_n^* + O_{\mathbb{P}}(c_n) \rightarrow \mathcal{N}(0,1). \quad \textit{(VIAlt)}$$

(X) Alternative form of the asymptotic law of record values from Burr X. (with $f_n = n^{-1}$, $d_n = n^{-1/2}$)

$$\frac{(\log r + n)^2\left(X^{(n)} - \left(-\log\left((1-e^{-n})^{-1/r} - 1\right)\right)^{1/2}\right)}{\sqrt{n}} = S_n^* + O_{\mathbb{P}}(n^{-1/2}) = W_n^* + O_{\mathbb{P}}(c_n) \rightarrow \mathcal{N}(0,1). \quad (XAlt)$$

(Xa) Alternative form of the asymptotic law of record values from Burr Xa. (with $f_n = n^{-1/2}$, $d_n = n^{-1/2}$)

$$\frac{(\log r + n)^2\left(X^{(n)} - \left(-\log\left(1-(1-e^{n})^{-1/r}\right)\right)^{1/2}\right)}{\sqrt{n}} = S_n^* + O_{\mathbb{P}}(n^{-1/2}) = W_n^* + O_{\mathbb{P}}(c_n) \rightarrow \mathcal{N}(0,1). \quad (XaAlt)$$

Conclusions and Perspective

This paper has its own interests by giving all the asymptotic laws of the upper records values that can be adapted to give the corresponding results for the lower records values. Actually, the results are at intersection of three major sub-disciplines of Statistics (Extreme value theory, records theory and asymptotic expansions) and the opportunity to summarize them has been seized. Most importantly, the study of the distributions of the so important Burr law paves the way of a handbook of similar results for as much as possible continuous distributions. This handbook already exists as a draft. The current paper will be main source of citations of it.

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n	25	50	75	100	150	200	250	300	350	500	1000
$p(3)$	0.45	0.322	0.26	0.22	0.18	0.15	0.14	0.13	0.11	0.098	0.069

Table 2: Probabilities table of having less three records in a sample of size n

	$\gamma > 0$		$\gamma < 0$		$\gamma = 0$	
Burr ($parameters$)	II ($r = 2$)	III ($r = 2, k = 3$)	I	IV ($r = 2, c = 3$)	VI ($k = 2, r = 3$)	X ($k = 2, r = 3$)
$P_0(\%)$	3.78	3.05	2.28	14.65	3.54	1.77

Table 3: Empirical $p-$value of test record values normality ($nr = 3$ records)

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