EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 1, 2023, 404-417
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Isogeometric Analysis approximation of linear elliptic equations with L^1 data

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Abstract. Isogeometric Analysis (IgA) is a recent technique for the discretization of Partial Differential Equations (PDEs). The main feature of the method is the ability to maintain the same exact description of the computational geometry domain throughout the analysis process, including refinement.
In the present paper, we consider, in dimension $d \geq 2$ the Isogeometric Analysis approximation of second order elliptic equations in divergence form with right-hand side in L^1. We assume that the family of meshes is *shape regular* and satisfies the discrete maximum principle. When the right-hand side belongs to $L^1(\Omega)$, we prove that the unique solution of the discrete problem converges to the unique renormalized solution in $W_0^{1,q}(\Omega),\ 1 \leq q < \dfrac{d}{d-1}$. We also prove some error estimates and include numerical tests for data with low smoothness.
2020 Mathematics Subject Classifications: 65N30, 35J25

Key Words and Phrases: Isogeometric analysis, NURBS approximation, L^1 data, renormalized solution

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1. Introduction

This paper is devoted to the Isogeometric Analysis approximation of second order linear elliptic equations in divergence form with L^1-data. We study the following problem

$$\begin{cases} -\text{div}\ (A\nabla u) & = f \quad \text{ in } \Omega, \\ \qquad u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open, bounded and Lipschitz set of $\mathbb{R}^d$, with $d = 2$ or $d = 3$, A is a coercive matrix with coefficients in $L^\infty(\Omega)$ and f belongs to $L^1(\Omega)$.
This problem frequently appears in applied sciences, being one of the basic problems

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DOI: https://doi.org/10.29020/nybg.ejpam.v16i1.4674

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in mathematical fluid mechanics (see [7, 9]). For this class of problems, the maximum principle is required to obtain physically admissible solutions.
The problem (1) has been studied in [3] by the standard Finite Element Method (FEM). The authors proved that the discrete solution converges in $W_0^{1,q}(\Omega)$, $1 \leq q < \dfrac{d}{d-1}$ to the unique renormalized solution (see [6] for existence and uniqueness of renormalized solution).
The Isogeometric Analysis based on NURBS (Non-Uniform Rational B-Splines), which possesses improved properties, is a generalization of classical Finite Element Method. NURBS are capable of more precise geometric representation of complex objects and can exactly represent many engineered shapes. IgA also simplifies mesh refinement because the geometry is fixed at the coarsest level of refinement and is unchanged throughout the refinement process.
The rest of the paper is organized as follows: in Section 2, we gives setting of the problem and main result. We recall brievly the Isogeometric Analysis method. In Section 3, we study the convergence analysis and we obtain the error estimates for data in $L^{r,\infty}(\Omega)$ for $1 < r < 2$. To finish, we give numerical result. The novelty in our work is the convergence result in $W_0^{1,q}(\Omega)$, obtained for approximate solutions of (1) in NURBS space.

2. Preliminary

2.1. Renormalized solution

We investigate the Poisson's problem with homogeneous boundary conditions under the following conditions : the matrix A is such that

$$A \in L^\infty(\Omega)^{d\times d}, \tag{2}$$

$$\text{a.e. } x \in \Omega,\ \forall \phi \in \mathbb{R}^d,\ A(x)\phi \cdot \phi \geq \alpha |\phi|^2, \tag{3}$$

for some $\alpha > 0$, and the right-hand side f is such that

$$f \in L^1(\Omega). \tag{4}$$

We give the definition of the renormalized solution of problem(1).

Definition 2.1. *A function u is a renormalized solution of* (1) *if u satisfies*

$$u \in L^1(\Omega), \tag{5}$$

$$\forall k > 0,\ T_k(u) \in H_0^1(\Omega), \tag{6}$$

$$\lim_{k\longrightarrow\infty} \frac{1}{k}\int_\Omega |\nabla T_k(u)|^2 dx = 0, \tag{7}$$

$$\left\{\begin{array}{c} \forall k > 0,\ \forall S \in C_c^1(\mathbb{R}) \text{ with } suppS \subset [-k, +k], \\ \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega A\nabla T_k(u)\nabla v S(u)dx + \int_\Omega A\nabla T_k(u)\nabla T_k(u)S'(u)v dx = \int_\Omega f S(u) v dx. \end{array}\right. \tag{8}$$

As $T_k(u) \in H_0^1(\Omega)$, every term makes sense in (8).
When f belongs to $L^1(\Omega) \cap H^{-1}(\Omega)$, the usual weak solution of (1), namely

$$\begin{cases} \forall u \in H_0^1(\Omega), \\ \forall v \in H_0^1(\Omega), \displaystyle\int_\Omega A\nabla u \nabla v dx = \int_\Omega f v dx, \end{cases} \tag{9}$$

is also a renormalized solution of (1) and conversly.

2.2. NURBS-based Isogeometric Analysis

Here, we recall the basic concepts of the B-Splines and NURBS basis functions and geometrical representation.
NURBS are built from B-Splines. A *knot vector* in one dimenion is a set of coordinates in the parametric space, written $\Xi = \left\{\xi_1, \xi_2, ..., \xi_{n+p+1}\right\}$, where ξ_i is the i-knot index $i \in \{1, ..., n+p+1\}$ characterized by the polynomial degree p and the number of basis functions n defining the B-Splines basis, respectively. By convention, we assume that $\xi_1 = 0$ and $\xi_{n+p+1} = 1$. The consequence is that parametric domain is defined as $\widehat{\Omega} := (\xi_1, \xi_{n+p+1}) = (0,1) \subset \mathbb{R}$. Knots may be repeated with the number of repetitions indicating its *multiplicity.* To investigate the concept of *mesh* elements in the parametric domain, we collect all the r distinct and ordered knots of Ξ, say ζ_j for $j = 1, ..., r$ into a vector $Z = \{\zeta_1, ..., \zeta_r\}$ with $\zeta_1 \equiv \xi_1 = 0$ and $\zeta_r \equiv \xi_{n+p+1} = 1$. In particular, the one dimensional *mesh* over $\widehat{\Omega}$, say $\mathcal{Q}_h$, is given by

$$\mathcal{Q}_h := \{Q = (\zeta_j, \zeta_{j+1}) : j = 1, ..., r-1\};$$

We denote by

$$h := \max\{h_Q : Q \in \mathcal{Q}_h\}, \text{ where } h_Q := diam(Q) \; \forall Q \in \mathcal{Q}_h \tag{10}$$

the *global mesh size* in the parametric domain $\widehat{\Omega}$.
By means of the *Cox-de Boor recursion formula*, (see [5], [10]), univariate B-Splines basis functions $N_i : \widehat{\Omega} \longrightarrow \mathbb{R}$ for $i = 1, ..., n$, are built as piecewise polynomials of degree p with compact support over the interval (ξ_i, ξ_{i+p+1}). The basis functions are everywhere pointwise nonnegative and $C^\infty-$continuous, except in the knot values ζ_j, where they are only $C^{p-m_j}-$ continuous. In particular, we define for all $j = 1, ..., r$, the smoothness integer parameters $k_j = p - m_j + 1$ such that $0 \le k_j \le p$, we collect them in a vector $\mathcal{K} = \{k_1, ..., k_r\}$, and we introduce the minimum integer parameter $k_{min} := \min\limits_{j=2,...,r-1}\{k_j\}$.
The B-Splines space built from the basis function in the parametric domain $\widehat{\Omega}$ reads:

$$\mathcal{S}_h := span\{N_i\}_{i=1}^n. \tag{11}$$

By definition, the B-Splines in $\mathcal{S}_h$ are globally $C^{k_{min}}-$continuous.

For each multi-index $\mathbf{i} := (i_1, ..., i_\kappa)$ in the set $I = \{\mathbf{i} = (i_1, ..., i_\kappa) : 0 \le i_\alpha \le n_\alpha,\ for\ 1 \le \alpha \le \kappa\}$, we define the *multivariate* B-Splines basis functions as:

$$N_i : \widehat{\Omega} \to \mathbb{R}, \qquad N_i(\eta) := \prod_{\alpha=1}^{\kappa} N_{i_\alpha}^{\alpha}(\eta_\alpha), \tag{12}$$

and we denote the tensor product B-Splines space, as:

$$\mathcal{S}_h := span\{N_i\}_{i\in I}. \tag{13}$$

Uni- and multivariate NURBS basis functions are defined on the parametric domain $\widehat{\Omega} = (0,1)^\kappa$ once provided κ knot vectors Ξ_α for $\alpha = 1, ..., \kappa$ and the corresponding B-Splines basis $\{N_i\}_{i\in I}$, by introducing a set of real numbers $\omega = \{\omega_i\}_{i\in I}$, called the *weights*. We assume that the weights are positive and we define a positive scalar piecewise polynomial function, called *weighting function*, as:

$$W : \widehat{\Omega} \to \mathbb{R}, \qquad W(\eta) := \sum_{i\in I} \omega_i N_i(\eta). \tag{14}$$

The i-th multivariate NURBS basis function is defined as

$$R_i : \widehat{\Omega} \to \mathbb{R}, \qquad R_i(\eta) = \frac{N_i(\eta)\omega_i}{W(\eta)}\ \forall i \in I, \tag{15}$$

and the corresponding NURBS space over the parametric domain Ω reads:

$$\mathcal{N}_h := span\{R_i\}_{i\in I}. \tag{16}$$

Therefore, we consider the NURBS space over the parametric domain $\widehat{\Omega}$ of (16) and a set of *control points* $\{\mathbf{P}_i\}_{i\in I} \subset \mathbb{R}^d$.Then a NURBS geometry Ω in $\mathbb{R}^d$ is defined from the parametric domain $\widehat{\Omega} = (0,1)^\kappa$ by means of the *geometrical mapping*:

$$\mathbf{x} : \widehat{\Omega} \to \Omega \subseteq \mathbb{R}^d \qquad \mathbf{x}(\eta) = \sum_{i\in I} R_i(\eta)\mathbf{P}_i. \tag{17}$$

By means of the geometrical mapping (17), we define the *physical mesh* $\mathcal{K}_h$ in the computational domain Ω, whose elements are obatained as the image of the elements in the parametric domain, i.e.:

$$\mathcal{K}_h := \{K = \mathbf{x}(Q) : Q \in \mathcal{Q}_h\}.$$

We denote the global mesh size of the mesh in the physical domain by

$$h := \max\{h_K : K \in \mathcal{K}_h\}, \text{ with } h_K := \|\nabla \mathbf{x}\|_{L^\infty(K)} \widehat{h}_{\widehat{K}} \text{ and } \widehat{h}_{\widehat{K}} = diam(\widehat{K}).$$

Further, we assume that the physical mesh is *quasi-uniform*, i.e. there exists a positive constant C_u, independent of h, such that

$$h_K \le h \le C_u h_K \qquad \forall K \in \mathcal{K}_h. \tag{18}$$

Moreover, we define the space of NURBS in the domain Ω as the push-forward of the space $\mathcal{N}_h$ of (16), i.e.:

$$\mathcal{V}^h := span\{R_i \circ \mathbf{x}^{-1}\}_{i\in I} = span\{\mathcal{R}_i\}_{i\in I}, \tag{19}$$

where $\{R_i\}_{i\in I}$ is the NURBS basis in the physical domain, with $\mathcal{R}_i := R_i \circ \mathbf{x}^{-1}$ for all $i \in I$. The geometrical mapping (17) is assumed to be invertible a.e. in Ω, with smooth inverse on each element K of the physical mesh $\mathcal{K}_h$.
In our analysis, we restricted ourselves to the case $d = \kappa$.
In standard FEM, the space V_h is a space of piecewise polynomials. In an IgA context, as introduced in [8], this space is formed by NURBS functions. For this, we introduce finite-dimensional spaces on the patch $(0,1)^d$. The approximate solution u_h of problem (9) is obtained by solving the following problem:

$$\begin{cases} \text{Find } u_h \in V^h, \\ \forall v_h \in V^h, \ \displaystyle\int_\Omega A\nabla u_h \nabla v_h dx = \int_\Omega f v_h dx, \end{cases} \tag{20}$$

where

$$V^h := \mathcal{V}^h \cap H_0^1(\Omega),$$

and $\mathcal{V}^h$ is a NURBS space described in (19). In our framework we prefer to define this space in the following general way, (see [1]):

$$V^h = \{v_h \in H_0^1(\Omega) : v_h = \widehat{v}_h \circ \mathbf{x}^{-1} \in \widehat{V}_h\}. \tag{21}$$

$\widehat{V}^h$ is a discrete space defined in the parametric domain $\widehat{\Omega}$ such that

$$\widehat{V}^h = \Big\{v_h : (0,1)^d \longrightarrow \mathbb{R}^d \mid \widehat{v}_h = v_h \circ \mathbf{x}, \ v_h \in V^h\}.$$

Note that the discrete problem (20) has a *unique solution.* Indeed, it is square system of linear equations in finite dimension, and the integral in the right-hand side is well-defined because the functions of V^h belong to $L^\infty(\Omega)$.
We denote $a_h(u_h, v_h) := \displaystyle\int_\Omega A\nabla u_h \nabla v_h dx$, the bilinear form of (20).
Since Splines are not in general interpolatory, a common way to define projection is by giving a dual basis. Given a function $\widehat{v} \in L^2(\widehat{\Omega})$ defined in the parametric domain $\widehat{\Omega}$, we use the projective operator over the B-Splines space $\mathcal{S}_h$, say $\Pi_{\mathcal{S}_h}$, introduced in [1] and defined as:

$$\Pi_{\mathcal{S}_h} : L^2(\widehat{\Omega}) \to \mathcal{S}_h, \qquad \Pi_{\mathcal{S}_h}\widehat{v} := \sum_{i\in I} \lambda_i(\widehat{v}) N_i, \tag{22}$$

where the linear functionals $\lambda_j \in L^2(\widehat{\Omega})'$ determine the dual basis for the set of B-Splines [11], i.e. they are such that $\lambda_j(N_i) := \delta_{j,i}$ for $i, j \in I$. The corresponding projective operator over the NURBS space $\mathcal{N}_h$ in the parametric domain (16), say $\Pi_{\mathcal{N}_h}$, is defined by means of $\Pi_{\mathcal{S}_h}$ and the definition of the NURBS basis functions of (15) through the weighting function W of (14). In particular, $\Pi_{\mathcal{N}_h}$ reads:

$$\Pi_{\mathcal{N}_h} : L^2(\widehat{\Omega}) \to \mathcal{N}_h, \qquad \Pi_{\mathcal{N}_h}\widehat{v} := \frac{\Pi_{\mathcal{S}_h}(W\widehat{v})}{W}, \tag{23}$$

for all $\widehat{v} \in L^2(\widehat{\Omega})$. In this manner, the projective operator over $\mathcal{V}^h$, the NURBS space in the physical domain Ω defined in (19) as the push-forward of the space $\mathcal{N}_h$, is given by:

$$\Pi_{\mathcal{V}^h} : L^2(\Omega) \to \mathcal{V}^h, \qquad \Pi_{\mathcal{V}^h} v := \left(\Pi_{\mathcal{N}_h}(\widehat{v})\right) \circ \mathbf{x}^{-1}. \tag{24}$$

The following result is proved in [2] and shows that $\Pi_{\mathcal{V}^h}$ is actually a projector on $\mathcal{V}^h$.

Proposition 2.1. *Its holds that $\Pi_{\mathcal{V}^h} v_h = v_h$ for all $v_h \in \mathcal{V}^h$. That is, $\Pi_{\mathcal{V}^h}$ is a projector.*

Now, we define the real number

$$D_{ij} = \int_{\Omega} A\nabla\lambda_i \nabla\lambda_j dx; \tag{25}$$

this defines an $I \times I$ matrix D.
Here, D satisfies

$$\forall i \in I, \qquad D_{ii} - \sum_{j\in I, j\neq i} |D_{ij}| \geq 0. \tag{26}$$

D is assumed to be diagonally dominant matrix.
This assumption is close to the usual assumption which ensures the discrete maximum principle.

Lemma 1. *The matrix $D(s_h)$ is an M-matrix and satisfies property (26), $\forall\ s_h \in V^h$.*

The coerciveness of the bilinear form a_h is the a consequence of this Lemma:

$$a_h(v_h, v_h) \geq \alpha \|\nabla v_h\|_2^2, \qquad \forall v_h \in V^h. \tag{27}$$

3. Convergence Analysis and Error estimates

In this Section, we give a priori estimates on the solution u_h of (20). These results allow to prove our main result.

Theorem 2. *(see [4]) Assume that A satisfies (3) and (27). Then, for every $h > 0$, let u_h the unique solution of problem (20), then $\{u_h\}_{h>0}$ is bounded in $W_0^{1,q}(\Omega)$ $(1 \leq q < \dfrac{d}{d-1})$ and there exists a constant $C > 0$ independent of h, such that*

$$\|u_h\|_{W_0^{1,q}(\Omega)} \leq C\|f\|_{L^1(\Omega)}. \tag{28}$$

Proposition 3.1. *Under assumption* (26), *on has for every* $v_h \in V^h$ *and every* $k > 0$

$$\int_\Omega A\nabla\Big(v_h - \Pi_{\mathcal{V}^h}(T_k(v_h))\Big)\nabla\Pi_{\mathcal{V}^h}(T_k(v_h))dx \geq 0. \tag{29}$$

Proof. We use the technique applied in [3].
Since

$$v_h = \sum_{i\in I} v_h(x_i)\lambda_i \text{ and } \Pi_{\mathcal{V}^h}(T_k(v_h)) = \sum_{i\in I} T_k(v_h)(x_i)\lambda_i,$$

using the definition (26) of D_{ij}, we have

$$\int_\Omega A\nabla\Big(v_h - \Pi_{\mathcal{V}^h}(T_k(v_h))\Big)\nabla\Pi_{\mathcal{V}^h}(T_k(v_h))dx =$$

$$= \sum_{i,j\in I} D_{ij}\Big(v_h(x_i) - T_k(v_h(x_i))\Big)T_k(v_h(x_j)) = \sum_{i\in I} S_i,$$

where

$$x_i = (\xi_{i+1} + ... + \xi_{i+p})/p$$

and

$$S_i = D_{ii}\Big(v_h(x_i) - T_k(v_h(x_i))\Big)T_k(v_h(x_i)) +$$

$$+ \sum_{j\in I, j\neq i} D_{ij}\Big(v_h(x_i) - T_k(v_h(x_i))\Big)T_k(v_h(x_j)).$$

Fix $i \in I$. If $|v_h(x_i)| \leq k$, then $v_h(x_i) - T_k(v_h(x_i)) = 0$ and $S_i = 0$.
If $|v_h(x_i)| > k$, then

$$\Big(v_h(x_i) - T_k(v_h(x_i))\Big)T_k(v_h(x_i)) = |v_h(x_i) - T_k(v_h(x_i))|k.$$

Since $|T_k(v_h(x_j))| \leq k$ for every j, one has

$$S_i \geq D_{ii}|v_h(x_i) - T_k(v_h(x_i))|k - \sum_{j\in I, j\neq i} |D_{ij}||v_h(x_i) - T_k(v_h(x_i))|k$$

$$= |v_h(x_i) - T_k(v_h(x_i))|k\Big(D_{ii} - \sum_{j\in I, j\neq i}|D_{ij}|\Big) \geq 0,$$

owing the hypothesis (26). This proves that

$$\forall i \in I, \qquad S_i \geq 0,$$

and therefore we obtain (29).

Now, we establish a priori estimate on the solution u_h of (20).

Proposition 3.2. *Under the assumptions* (2.1), (3), (4), (10), (18) *and* (26). *Then the unique solution* u_h *of* (20) *satisfies for every* $h > 0$ *and every* $k > 0$

$$\int_\Omega A\nabla\Pi_{\mathcal{V}^h}(T_k(u_h))\nabla\Pi_{\mathcal{V}^h}(T_k(u_h))dx \leq \int_\Omega f\Pi_{\mathcal{V}^h}(T_k(u_h))dx. \tag{30}$$

In particular, u_h *satisfies*

$$\alpha \int_\Omega |\nabla\Pi_{\mathcal{V}^h}(T_k(u_h))|^2 dx \leq k\|f\|_{L^1(\Omega)}. \tag{31}$$

Proof. Since $T_k(u_h)$ is continuous, the function $\Pi_{\mathcal{V}^h}(T_k(u_h))$ belongs to $\mathcal{V}^h$. Using this function as test function in (20) we have

$$\int_\Omega A\nabla u_h\nabla\Pi_{\mathcal{V}^h}(T_k(u_h))dx = \int_\Omega f\Pi_{\mathcal{V}^h}(T_k(u_h))dx. \tag{32}$$

Proposition (3.1) shows that

$$\int_\Omega A\nabla\Big(v_h - \Pi_{\mathcal{V}^h}(T_k(v_h))\Big)\nabla\Pi_{\mathcal{V}^h}(T_k(v_h))dx \geq 0,$$

which immediately implies (30).
As a consequence, we consider (30) and the coercivity (3) of A to obtain (31).

Our main result is the following.

Theorem 3. *Under the assumptions of Proposition* (3.2), *the unique solution* u_h *of* (20) *satisfies for every* $k > 0$ *and for every* q *with* $1 \leq q < \dfrac{d}{d-1}$

$$u_h \longrightarrow u \text{ strongly in } W_0^{1,q}(\Omega), \tag{33}$$

when the mesh size h *tends to zero, where* u *is the unique renormalized solution of* (1).

Proof. Let us consider $\big(f^\varepsilon\big)_\varepsilon$, a sequence of functions such that

$$f^\varepsilon \in L^2(\Omega), \qquad f^\varepsilon \longrightarrow f \text{ strongly in } L^1(\Omega).$$

We can take for example $f^\varepsilon = T_{\frac{1}{\varepsilon}}(f)$.
Let u_h^ε be the unique solution of problem (20) with regularized data $f^\varepsilon \in L^2(\Omega)$. Then $u_h - u_h^\varepsilon$ satisfies

$$\begin{cases} u_h - u_h^\varepsilon \in V^h, \\ \forall v_h \in V^h, \displaystyle\int_\Omega A\nabla(u_h - u_h^\varepsilon)\nabla v_h dx = \int_\Omega (f - f^\varepsilon)v_h dx. \end{cases}$$

We consider this problem and we apply estimate (31). We have for every $k > 0$, every $h > 0$ and every $\varepsilon > 0$

$$\alpha \int_{\Omega} \left| \nabla \Pi_{\mathcal{V}^h} \Big(T_k(u_h - u_h^{\varepsilon}) \Big) \right|^2 dx \leq k \|f - f^{\varepsilon}\|_{L^1(\Omega)},$$

Next, applying Theorem 2.1 of [3] and using Theorem 2, we deduce that, for every q with $1 \leq q < \dfrac{d}{d-1}$, every $h > 0$ and every $\varepsilon > 0$

$$\|u_h - u_h^{\varepsilon}\|_{W_0^{1,q}(\Omega)} \leq C_2 \frac{1}{\alpha} \|f - f^{\varepsilon}\|_{L^1(\Omega)}, \tag{34}$$

where C_2 is a constant which depends of d and q.
On the other hand, since $f^{\varepsilon} \in L^2(\Omega)$ and the fact that the physical mesh is quasi-uniform under hypothesis (10) and (18), we have that, for any $\varepsilon > 0$

$$\lim_{h \longrightarrow 0} \|u_h^{\varepsilon} - u^{\varepsilon}\|_{H_0^1(\Omega)} = 0, \tag{35}$$

where u^{ε} is the unique solution of

$$\begin{cases} u^{\varepsilon} \in H_0^1(\Omega), \\ -div(A \nabla u^{\varepsilon}) = f^{\varepsilon} \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{36}$$

Finally, the function u^{ε}, which is the unique weak solution of (36), is also the unique renormalized solution in the sense of Definition 1.1 of the problem

$$\begin{cases} -div(A \nabla u^{\varepsilon}) = f^{\varepsilon} \text{ in } \Omega, \\ u^{\varepsilon} = 0 \text{ on } \partial\Omega. \end{cases} \tag{37}$$

We consider u and u^{ε} the unique renormalized solutions of (1) and (37) respectively. We have, indeed the continuous dependence of the renormalized solution with respect to the data implies that

$$\|u^{\varepsilon} - u\|_{W_0^{1,q}(\Omega)} \leq C_3 \frac{1}{\alpha} \|f^{\varepsilon} - f\|_{L^1(\Omega)}. \tag{38}$$

for every q with $1 \leq q < \dfrac{d}{d-1}$.
Inequality (38) is given by Theorem 1.2 in [3].
Writing now

$$\|u_h - u\|_{W_0^{1,q}(\Omega)} \leq \|u_h - u_h^{\varepsilon}\|_{W_0^{1,q}(\Omega)} + \|u_h^{\varepsilon} - u^{\varepsilon}\|_{W_0^{1,q}(\Omega)} + \|u^{\varepsilon} - u\|_{W_0^{1,q}(\Omega)}$$

and using (34), (35) and (38), we have proved that for every $\varepsilon > 0$ and every q with $1 \leq q < \dfrac{d}{d-1}$

$$\limsup_{h \longrightarrow 0} \|u_h - u\|_{W_0^{1,q}(\Omega)} \leq \Big(C_1, C_2, C_3 \Big) \frac{1}{\alpha} \|f^{\varepsilon} - f\|_{L^1(\Omega)}.$$

Taking the limit when ε tends to zero proves (33).

For every r with $1 < r < +\infty$, we denote by $L^{r,\infty}(\Omega)$ the Marcinkiewicz space whose norm is defined by

$$\|f\|_{L^{r,\infty}(\Omega)} = \sup_{\nu>0} \left(\nu \Big|\{x \in \Omega : |f(x)| \geq \nu\}\Big|^{1/r}\right). \tag{39}$$

Next, error estimates for data $f \in L^{r,\infty}(\Omega)$ may be derived using the techniques introduced in [3].

Theorem 4. *Under the assumptions of Theorem 2.2 and $f \in L^{r,\infty}(\Omega)$ for some r with $1 < r < 2$, there exists a constant C independent of the mesh size h such that we have the error estimate*

$$\|u_h - u\|_{W_0^{1,q}(\Omega)} \leq Ch^{2(1-\frac{1}{r})}\|f\|_{L^{r,\infty}(\Omega)}. \tag{40}$$

Proof. We assume that f belongs to the Marcinkiewicz space $L^{r,\infty}(\Omega)$ for some r with $1 < r < 2$ (this holds in particular if f belongs to $L^r(\Omega)$). For every $\varepsilon > 0$, we set

$$f^\varepsilon = T_{\frac{1}{\varepsilon}}(f),$$

which belongs to $L^\infty(\Omega) \subset L^2(\Omega)$, and we denote by u_h^ε the solution of (20) with right-hand side f^ε. Defining also u^ε at the solution of (36), we write for every q with $1 \leq q < \dfrac{d}{d-1}$

$$\|u_h - u\|_{W_0^{1,q}(\Omega)} \leq \|u_h - u_h^\varepsilon\|_{W_0^{1,q}(\Omega)} + \|u_h^\varepsilon - u^\varepsilon\|_{W_0^{1,q}(\Omega)} + \|u^\varepsilon - u\|_{W_0^{1,q}(\Omega)}. \tag{41}$$

We have for a new constant C (which depends on q, Ω,)

$$\|u_h^\varepsilon - u^\varepsilon\|_{W_0^{1,q}(\Omega)} \leq Ch\|f^\varepsilon\|_{L^2(\Omega)}.$$

Using then (34) and (38), we deduce that for a new constant C, which is independent of ε, h and f (but depends on d, q, Ω), one has

$$\|u_h - u\|_{W_0^{1,q}(\Omega)} \leq C\Big(\|f - f^\varepsilon\|_{L^1(\Omega)} + h\|f^\varepsilon\|_{L^2(\Omega)}\Big). \tag{42}$$

We now estimate the right-hand side of this inequality by considering

$$\|g\|^p_{L^p(\Omega)} = p\int_0^{+\infty} t^{p-1}\Big|\Big\{x \in \Omega : |g(x)| \geq t\Big\}\Big|dt,$$

which gives

$$\begin{cases} \|f - f^\varepsilon\|_{L^1(\Omega)} = & \displaystyle\int_0^{+\infty} \Big|\Big\{x \in \Omega : |f(x) - T_{\frac{1}{\varepsilon}}(f)(x)| \geq t\Big\}\Big|dt \\ \qquad\qquad = & \displaystyle\int_0^{+\infty} \Big|\Big\{x \in \Omega : |f(x) - \frac{1}{\varepsilon}| \geq t\Big\}\Big|dt \\ \qquad\qquad = & \displaystyle\int_{\frac{1}{\varepsilon}}^{+\infty} \Big|\Big\{x \in \Omega : |f(x)| \geq t\Big\}\Big|dt, \end{cases} \tag{43}$$

and

$$\begin{cases} \|f^\varepsilon\|^2_{L^2(\Omega)} = & 2\int_0^{+\infty} t\Big|\Big\{x \in \Omega : |T_{\frac{1}{\varepsilon}}(f)(x)| \geq t\Big\}\Big| dt \\ = & 2\int_0^{\frac{1}{\varepsilon}} t\Big|\Big\{x \in \Omega : |f(x)| \geq t\Big\}\Big| dt. \end{cases} \tag{44}$$

By the norm in the Marcinkiewicz space $L^{r,\infty}(\Omega)$ define in (39) , we have

$$\Big|\Big\{x \in \Omega : |f(x)| \geq t\Big\}\Big| \leq \min\Big\{|\Omega|, \frac{\|f\|^r_{L^{r,\infty}(\Omega)}}{t^r}\Big\},$$

and thus

$$\begin{cases} \|f - f^\varepsilon\|_{L^1(\Omega)} \leq & \frac{1}{r-1}\varepsilon^{r-1}\|f\|^r_{L^{r,\infty}(\Omega)}, \\ \|f^\varepsilon\|_{L^2(\Omega)} \leq & \sqrt{\frac{2}{2-r}}\frac{1}{\varepsilon^{1-\frac{r}{2}}}\|f\|^{\frac{r}{2}}_{L^{r,\infty}(\Omega)}. \end{cases} \tag{45}$$

Then, (42) gives

$$\|u_h - u\|_{W_0^{1,q}(\Omega))} \leq C\Big(\frac{1}{r-1}\varepsilon^{r-1}\|f\|^r_{L^{r,\infty}(\Omega)} + \sqrt{\frac{2}{2-r}}\frac{h}{\varepsilon^{1-\frac{r}{2}}}\|f\|^{\frac{r}{2}}_{L^{r,\infty}(\Omega)}\Big).$$

Taking in this inequality $\varepsilon = \dfrac{h^{\frac{2}{r}}}{\|f\|^{\frac{r}{2}}_{L^{r,\infty}(\Omega)}}$ yields, for every q with $1 \leq q < \dfrac{d}{d-1}$ and for every $h > 0$, we obtain

$$\|u_h - u\|_{W_0^{1,q}(\Omega)} \leq C\Big(d, q, r, |\Omega|, \Big)h^{2(1-\frac{1}{r})}\|f\|^r_{L^{r,\infty}(\Omega)}.$$

4. Numerical implementation

In this Section, we give the numerical tests to attest our main error estimate result, namely Theorem (4). We consider in this paper for the numerical test a simple geometry : a quarter of a ring with inner and outer radius equal to 1 or 2, respectively, and described through a quadratic NURBS parametrization, as the one in Figure 1.

We solve the initial problem (1) with $A(x)$ the identity matrix. This matrix satisfy the assumptions of Theorem (3), namely (2), (3), (4), (10), (18) and (26), are satisfied. The right-hand side f is imposed to obtain the renormalized solution $u = e^{x_1}\sin(x_2)$.

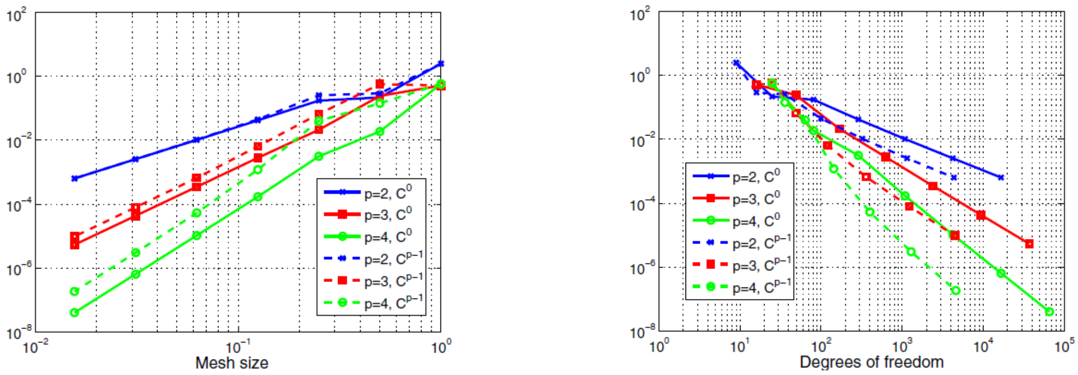
We solve the problem ina set of successively refined meshes, the coarest three meshes are plotted in Figure 1, for degree p varying from 2 to 4, and in NURBS spaces of maximum (C^{p-1}) and minimum (C^0) continuity.

In Figure 2, we present the error in the $W_0^{1,q}$-norm with respect to the mesh size h, and with respect to the number of degree of freedom. The result in terms of the mesh size confirm the estimate of Theorem (4) when we take for example

Figure 1: Mesh parametrization.

$f(x) = \dfrac{1}{|x|^{2/r}} \in L^{r,\infty}(\Omega)$.

In terms of the degrees of freedom, the results always converges like $O(N_{dof}^{-p/2})$ where N_{dof} is the number of degrees of freedom



(a) Error in the terms of the mesh size.

(b) Error in terms of the degrees of freedom.

Figure 2: Error estimates in the $W_0^{1,1}$- norm in the quarter-ring: error in terms of (A) the mesh size, and (B) the degrees of freedom.

5. Conclusion

In this paper, we discussed in dimension $d \geq 2$, the Isogeometric Analysis approximation of second order elliptic equations in divergence form with right-hand side in L^1. We have proven that the unique solution of the discrete problem converges, in NURBS space, to the unique renormalized solution in $W_0^{1,q}(\Omega)$, $1 \leq q < \dfrac{d}{d-1}$. We have also studied the convergence analysis and we have obtained the error estimates for data in $L^{r;\infty}(\Omega)$ for $1 < r < 2$. To finish, we gave numerical results using Python.

References

[1] Y. Bazilevs, L. Beirão da Veiga, J. A. Cottrell, T. J. R. Hughes, and G. Sangalli. Isogeometric analysis: Approximation, stability and error estimates for h-refined meshes. *Math. Models Methods Appl. Sci.*, 16(7):1031–1090, 2006.

[2] A. Buffa, G. Sangalli, and R. Vázquez. Isogeometric analysis in electromagnetics: B-splines approximation. *Comput. Methods Appl. Mech. Eng.*, 199(17-20):1143–1152, 2010.

[3] J. Casado-Díaz, T. Chacón Rebollo, V. Girault, M. Gómez Mármol, and F. Murat. Finite elements approximation of second order linear elliptic equations in divergence form with right-hand side in L^1. *Numer. Math.*, 105(3):337–374, 2007.

[4] T. Chacón Rebollo, E. D. Fernández Nieto, and M. Gómez Mármol. Some remarks on a model for the atmospheric pressure in ocean dynamics. In *Numerical mathematics and advanced applications. Proceedings of ENUMATH 2005, the 6th European conference on numerical mathematics and advanced applications, Santiago de Compostela, Spain, July 18–22, 2005.*, pages 279–287. Berlin: Springer, 2006.

[5] J. Austin Cottrell, Thomas J. R. Hughes, and Yuri Bazilevs. *Isogeometric analysis. Toward integration of CAD and FEA.* Hoboken, NJ: John Wiley & Sons, 2009.

[6] Gianni Dal Maso, François Murat, Luigi Orsina, and Alain Prignet. Renormalization solutions of elliptic equations with general measure data. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, 28(4):741–808, 1999.

[7] B. Faraj and M. Modanli. Using difference scheme method for the numerical solution of telegraph partial differential equation. *Journal of Garmian University*, 4(ICBS Conference):157–163, 2017.

[8] T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Eng.*, 194(39-41):4135–4195, 2005.

[9] M. Modanli, B. M. Faraj, and F.W. Ahmed. Using matrix stability for variable telegraph partial differential equation. *Int. J. Optim. Control, Theor. Appl.*, 10(2):237–243, 2020.

[10] Les Piegl and Wayne Tiller. *The NURBS books.* Berlin: Springer, 2nd ed. edition, 1997.

[11] Larry L. Schumaker. *Spline functions: basic theory.* Camb. Math. Libr. Cambridge: Cambridge University Press, 2007.