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Existence of nonoscillatory solutions of higher order nonlinear neutral differential equations

B. Çına[1], T. Candan[2,*] and M. Tamer Şenel[3]

[1] *Zara Veysel Dursun School of Applied Science, Cumhuriyet University, Sivas, Turkey*
[2] *College of Engineering and Technology, American University of the Middle East, Egaila 54200, Kuwait*
[3] *Department of Mathematics, Faculty of Sciences, Erciyes University, Kayseri, Turkey*

Abstract. In this paper, an n-th order neutral nonlinear differential equation is studied. By using the Banach contraction principle, some sufficient conditions are established for the existence of nonoscillatory solutions of nonlinear n-th order neutral differential equation. An example is included to illustrate the results obtained.
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1. Introduction

This paper is concerned with nonoscillatory solutions of nonlinear n-th order neutral differential equation of the form

$$[r(t)[x(t) - p(t)x(t-\tau)]^{(n-1)}]' + (-1)^n[f_1(t, x(\sigma_1(t))) - f_2(t, x(\sigma_2(t))) - g(t)] = 0, \quad (1)$$

where $n \geq 2$ is an integer, $\tau > 0$, $p, \sigma_i, g \in C([t_0, \infty), \mathbb{R})$, $r \in C([t_0, \infty), (0, \infty))$ and $\lim_{t\to\infty} \sigma_i(t) = \infty$, $i = 1, 2$.

Throughout this article, we assume that $f_i(t, x) \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$ is a nondecreasing in x for $i = 1, 2$, $xf_i(t, x) > 0$ for $x \neq 0$, $i = 1, 2$, and satisfies

$$|f_i(t, x) - f_i(t, y)| \leq q_i(t)|x - y| \quad \text{for} \quad t \in [t_0, \infty) \quad \text{and} \quad x, y \in [a, b], \quad (2)$$

where $q_i \in C([t_0, \infty), (0, \infty))$, $i = 1, 2$, and $[a, b]$ $(0 < a < b$ or $a < b < 0)$ is any closed interval. Furthermore, suppose that

$$\int_{t_0}^{\infty} \int_{t_0}^{s} \frac{s^{n-2}}{r(s)} q_i(u)duds < \infty, \quad i = 1, 2, \quad (3)$$

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Email addresses: bengu58cina@gmail.com (B. Çına),
Tuncay.Candan@aum.edu.kw (T. Candan) senel@erciyes.edu.tr (M. Tamer Şenel)

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$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}|f_i(u,d)|duds<\infty \quad \text{for some } d\neq 0,\ i=1,2, \tag{4}$$

and

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}|g(u)|duds<\infty \tag{5}$$

hold.

Oscillation and nonoscillation phenomena appear in different models from real world applications; see, for instance, oscillatory and nonoscillatory solutions may appear in impulsive partial neutral differential equations from mathematical biology, we refer to the papers [11, 12, 16] where impulsive effects are modelled by external sources complementing partial differential equations involving taxis mechanisms, and arising in biomathematics. We also refer the reader to the papers [9, 14, 15] for the oscillation and asymptotic behavior of solutions to various classes of neutral differential equations. In particular, Zhou and Zhang [21] and Candan [4] studied existence of nonoscillatory solutions of higher order neutral differential equations of the form

$$\frac{d^n}{dt^n}\left[x(t)+cx(t-\tau)\right]+(-1)^{n+1}\left[P(t)x(t-\sigma)-Q(t)x(t-\delta)\right]=0 \tag{6}$$

and

$$\begin{aligned}[r(t)[x(t) &+ P(t)x(t-\tau)]^{(n-1)}]' \\ &+ (-1)^n[Q_1(t)g_1(x(t-\sigma_1))-Q_2(t)g_2(x(t-\sigma_2))-f(t)]=0, \end{aligned}\tag{7}$$

respectively. Later, Çına et al.[8] studied the existence of nonoscillatory solutions of nonlinear second order neutral differential equation with forcing term of the form

$$\big(r(t)\left(x(t)-p(t)x(t-\tau)\right)'\big)'+f_1(t,x(\sigma_1(t)))-f_2(t,x(\sigma_2(t)))=g(t).$$

Motivated by the idea of [4, 8, 21], the goal of this paper is to present some sufficient conditions for the existence of nonoscillatory solutions of (1). For related studies on the existence of nonoscillatory solutions of second or higher order neutral differential and difference equations the reader is referred to the papers [3, 5–7, 17–20] and books [1, 2, 10, 13].

Let $T_0=\min\{t_1-\tau,\ \inf_{t\geq t_1}\sigma_1(t),\ \inf_{t\geq t_1}\sigma_2(t)\}$ for $t_1\geq t_0$. By a solution of equation (1), we mean a function $x\in C([T_0,\infty),\mathbb{R})$ in the sense that $x(t)-p(t)x(t-\tau)$ is $n-1$ times continuously differentiable and $r(t)(x(t)-p(t)x(t-\tau))^{(n-1)}$ is continuously differentiable on $[t_1,\infty)$ and such that equation (1) is satisfied for $t\geq t_1$.

As usual, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

2. Main Results

Theorem 1. *Assume that (3)-(5) hold and $0\leq p(t)\leq p<1$. Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$, the case $d < 0$ can be treated similarly. Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the $\|x\| = \sup_{t \geq t_0} |x(t)| < \infty$ norm. Set

$$A = \{x \in X : N_1 \leq x(t) \leq d, \quad t \geq t_0\},$$

where N_1 is a positive constant such that $N_1 < (1-p)d$. Clearly, A is a closed, bounded and convex subset of X. By (3)-(5) there exists a $t_1 > t_0$ sufficiently large such that $t - \tau \geq t_0$, $\sigma_1(t) \geq t_0$, $\sigma_2(t) \geq t_0$ for $t \geq t_1$ and

$$p + \frac{2}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} q_i(u) du ds \leq \theta_1 < 1, \quad i = 1, 2, \tag{8}$$

where θ_1 is a constant,

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} [f_1(u, d) + |g(u)|] du ds \leq (1-p)d - \alpha, \tag{9}$$

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} [f_2(u, d) + |g(u)|] du ds \leq \alpha - N_1, \tag{10}$$

where α is a positive constant such that $N_1 < \alpha < (1-p)d$. Define the operator S on A by

$$(Sx)(t) = \begin{cases} \alpha + p(t)x(t-\tau) + \frac{1}{(n-2)!} \int_t^{\infty} \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^{s} [f_1(u, x(\sigma_1(u))) \\ -f_2(u, x(\sigma_2(u))) - g(u)] du ds, \quad t \geq t_1 \\ (Sx)(t_1), \quad t_0 \leq t \leq t_1. \end{cases}$$

We can easily see that Sx is continuous. We shall show that $SA \subset A$. In fact, for every $x \in A$ and $t \geq t_1$, due to (9), we have

$$\begin{aligned} (Sx)(t) &= \alpha + p(t)x(t-\tau) + \frac{1}{(n-2)!} \int_t^{\infty} \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^{s} [f_1(u, x(\sigma_1(u))) \\ &- f_2(u, x(\sigma_2(u))) - g(u)] du ds \\ &\leq \alpha + pd + \frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} [f_1(u, d) + |g(u)|] du ds \\ &\leq d. \end{aligned}$$

Furthermore, by using (10), we obtain

$$\begin{aligned} (Sx)(t) &= \alpha + p(t)x(t-\tau) + \frac{1}{(n-2)!} \int_t^{\infty} \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^{s} [f_1(u, x(\sigma_1(u))) \\ &- f_2(u, x(\sigma_2(u))) - g(u)] du ds \\ &\geq \alpha - \frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} [f_2(u, d) + |g(u)|] du ds \end{aligned}$$

$$\geq \quad N_1.$$

Thus, we proved that $SA \subset A$. Now we shall show that operator S is a contraction operator on A. In fact, for $x, y \in A$ and $t \geq t_1$, in view of (2) and (8), we have

$$\begin{aligned}
|(Sx)(t) - (Sy)(t)| &\leq p|x(t-\tau) - y(t-\tau)| \\
&+ \frac{1}{(n-2)!}\sum_{i=1}^{2}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s |f_i(u, x(\sigma_i(u))) - f_i(u, y(\sigma_i(u)))|duds \\
&\leq p|x(t-\tau) - y(t-\tau)| \\
&+ \frac{1}{(n-2)!}\sum_{i=1}^{2}\int_{t_1}^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s q_i(u)|x(\sigma_i(u)) - y(\sigma_i(u)))|duds \\
&\leq \|x-y\|\Big[p + \frac{1}{(n-2)!}\sum_{i=1}^{2}\int_{t_1}^\infty\int_{t_1}^s \frac{(s-t)^{n-2}}{r(s)}q_i(u)duds\Big] \\
&\leq \theta_1\|x-y\|.
\end{aligned}$$

This implies that

$$\|Sx - Sy\| \leq \theta_1\|x-y\|.$$

Since $\theta_1 < 1$ by (8), it follow that S is a contraction mapping on A. By the Banach contraction mapping principle, S has a fixed point $x \in A$, which is obviously a positive solution of (1). This completes the proof.

Theorem 2. *Assume that (3)-(5) hold and* $1 < p_1 \leq p(t) \leq p_2 < \infty$. *Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$, the case $d < 0$ can be treated similarly. Let X be the set as in the proof of Theorem 1. Set

$$A = \{x \in X : N_2 \leq x(t) \leq d, \quad t \geq t_0\},$$

where N_2 is a positive constant such that $p_2N_2 < (p_1 - 1)d$. It is clear that A is a closed, bounded and convex subset of X. By (3)-(5), we can choose a $t_1 > t_0$ sufficiently large such that $\sigma_1(t+\tau) \geq t_0$, $\sigma_2(t+\tau) \geq t_0$ for $t \geq t_1$ and

$$\frac{1}{p_1}\Big[1 + \frac{2}{(n-2)!}\int_{t_1}^\infty\int_{t_1}^s \frac{(s-t)^{n-2}}{r(s)}q_i(u)duds\Big] \leq \theta_2 < 1, \quad i = 1, 2, \tag{11}$$

where θ_2 is a constant,

$$\frac{1}{(n-2)!}\int_{t_1}^\infty\int_{t_1}^s \frac{(s-t)^{n-2}}{r(s)}[f_1(u,d) + |g(u)|]duds \leq \alpha - p_2N_2, \tag{12}$$

$$\frac{1}{(n-2)!}\int_{t_1}^\infty\int_{t_1}^s \frac{(s-t)^{n-2}}{r(s)}[f_2(u,d) + |g(u)|]duds \leq (p_1 - 1)d - \alpha, \tag{13}$$

where α is a positive constant such that $p_2N_2 < \alpha < (p_1-1)d$. Define the operator S on A by

$$(Sx)(t)=\begin{cases} \frac{1}{p(t+\tau)}\Big[\alpha+x(t+\tau)-\frac{1}{(n-2)!}\int_{t+\tau}^{\infty}\frac{(s-t-\tau)^{n-2}}{r(s)}\int_{t_1+\tau}^{s}[f_1(u,x(\sigma_1(u))) \\ -f_2(u,x(\sigma_2(u)))-g(u)]duds\Big], \quad t\geq t_1 \\ (Sx)(t_1), \quad t_0\leq t\leq t_1. \end{cases}$$

Clearly, Sx is continuous. First, we shall show that $SA\subset A$. In fact, for every $x\in A$ and $t\geq t_1$, using (13), we obtain

$$\begin{aligned} (Sx)(t) &= \frac{1}{p(t+\tau)}\Bigg[\alpha+x(t+\tau)-\frac{1}{(n-2)!}\int_{t+\tau}^{\infty}\frac{(s-t-\tau)^{n-2}}{r(s)}\int_{t_1+\tau}^{s}[f_1(u,x(\sigma_1(u))) \\ &- f_2(u,x(\sigma_2(u)))-g(u)]duds\Bigg] \\ &\leq \frac{1}{p_1}\Bigg[\alpha+d+\frac{1}{(n-2)!}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}[f_2(u,d)+|g(u)|]duds\Bigg] \\ &\leq d \end{aligned}$$

and taking (12) into account, we have

$$\begin{aligned} (Sx)(t) &= \frac{1}{p(t+\tau)}\Bigg[\alpha+x(t+\tau)-\frac{1}{(n-2)!}\int_{t+\tau}^{\infty}\frac{(s-t-\tau)^{n-2}}{r(s)}\int_{t_1+\tau}^{s}[f_1(u,x(\sigma_1(u))) \\ &- f_2(u,x(\sigma_2(u)))-g(u)]duds\Bigg] \\ &\geq \frac{1}{p(t+\tau)}\Bigg[\alpha-\frac{1}{(n-2)!}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}[f_1(u,d)+|g(u)|]duds\Bigg] \\ &\geq \frac{1}{p_2}\Bigg[\alpha-\frac{1}{(n-2)!}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}[f_1(u,d)+|g(u)|]duds\Bigg] \\ &\geq N_2. \end{aligned}$$

Thus, we proved that $SA\subset A$. Second, we shall show that S is a contraction operator on A. In fact, for $x,y\in A$ and $t\geq t_1$, in view of (2) and (11), we have

$$\begin{aligned} &|(Sx)(t)-(Sy)(t)|\leq\frac{1}{p(t+\tau)}\Bigg[|x(t+\tau)-y(t+\tau)| \\ &+\frac{1}{(n-2)!}\sum_{i=1}^{2}\int_{t}^{\infty}\frac{(s-t-\tau)^{n-2}}{r(s)}\int_{t_1}^{s}|f_i(u,x(\sigma_i(u)))-f_i(u,y(\sigma_i(u)))|duds\Bigg] \\ &\leq\frac{\|x-y\|}{p_1}\Bigg[1+\frac{1}{(n-2)!}\sum_{i=1}^{2}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}q_i(u)duds\Bigg] \\ &\leq\theta_2\|x-y\|. \end{aligned}$$

This immediately implies that

$$\|Sx - Sy\| \le \theta_2 \|x - y\|.$$

Since $\theta_2 < 1$ by (11), it follows that S is a contraction operator on A. By the Banach contraction mapping principle, S has a fixed point $x \in A$, and x is a positive solution of (1). Thus, the proof is completed.

Theorem 3. *Assume that (3)-(5) hold and $-1 < -p \le p(t) \le 0$. Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$, the case $d < 0$ can be treated similarly. Let X be the set as in the proof of Theorem 1. Set

$$A = \{x \in X : N_3 \le x(t) \le d, \quad t \ge t_0\},$$

where N_3 is a positive constant such that $N_3 + pd < d$. Clearly, A is a closed, bounded and convex subset of X. In view of (3)-(5), there exists a $t_1 > t_0$ sufficiently large such that $t - \tau \ge t_0$, $\sigma_1(t) \ge t_0$, $\sigma_2(t) \ge t_0$ for $t \ge t_1$ and

$$p + \frac{2}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} q_i(u) du ds \le \theta_3 < 1, \quad i = 1, 2, \tag{14}$$

where θ_3 is a constant,

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} [f_1(u, d) + |g(u)|] du ds \le d - \alpha, \tag{15}$$

$$\frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} [f_2(u, d) + |g(u)|] du ds \le \alpha - N_3 - pd, \tag{16}$$

where α is a positive constant such that $N_3 + pd < \alpha < d$. Define the operator S on A by

$$(Sx)(t) = \begin{cases} \alpha + p(t)x(t-\tau) + \frac{1}{(n-2)!} \int_t^{\infty} \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^{s} [f_1(u, x(\sigma_1(u))) \\ -f_2(u, x(\sigma_2(u))) - g(u)] du ds, \quad t \ge t_1 \\ (Sx)(t_1), \quad t_0 \le t \le t_1. \end{cases}$$

Clearly, Sx is continuous. First, we shall show that $SA \subset A$. For every $x \in A$ and $t \ge t_1, by$ using (15), we have

$$\begin{aligned} (Sx)(t) &= \alpha + p(t)x(t-\tau) \\ &+ \frac{1}{(n-2)!} \int_t^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} [f_1(u, x(\sigma_1(u))) - f_2(u, x(\sigma_2(u))) - g(u)] du ds \\ &\le \alpha + \frac{1}{(n-2)!} \int_{t_1}^{\infty} \int_{t_1}^{s} \frac{(s-t)^{n-2}}{r(s)} [f_1(u, d) + |g(u)|] du ds \end{aligned}$$

$$\leq \quad d$$

and applying (16), we have

$$\begin{aligned}(Sx)(t) &= \alpha + p(t)x(t-\tau) \\ &+ \frac{1}{(n-2)!}\int_t^\infty \int_{t_1}^s \frac{(s-t)^{n-2}}{r(s)}[f_1(u,x(\sigma_1(u))) - f_2(u,x(\sigma_2(u))) - g(u)]duds \\ &\geq \alpha - pd - \frac{1}{(n-2)!}\int_{t_1}^\infty \int_{t_1}^s \frac{(s-t)^{n-2}}{r(s)}[f_2(u,d) + |g(u)|]duds \\ &\geq N_3.\end{aligned}$$

Hence, $SA \subset A$. Finally, we show that S is a contraction operator on A. In fact, for $x, y \in A$ and $t \geq t_1$, using (2) and (14), we obtain

$$\begin{aligned}|(Sx)(t) - (Sy)(t)| &\leq p|x(t-\tau) - y(t-\tau)| \\ &+ \frac{1}{(n-2)!}\sum_{i=1}^2 \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^s |f_i(u,x(\sigma_i(u))) - f_i(u,y(\sigma_i(u)))|duds \\ &\leq p|x(t-\tau) - y(t-\tau)| \\ &+ \frac{1}{(n-2)!}\sum_{i=1}^2 \int_{t_1}^\infty \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^s q_i(u)|x(\sigma_i(u)) - y(\sigma_i(u)))|duds \\ &\leq \|x-y\|\Big[p + \frac{1}{(n-2)!}\sum_{i=1}^2 \int_{t_1}^\infty \int_{t_1}^s \frac{(s-t)^{n-2}}{r(s)} q_i(u)duds\Big] \\ &\leq \theta_3\|x-y\|.\end{aligned}$$

This implies that

$$\|Sx - Sy\| \leq \theta_3\|x-y\|.$$

Since $\theta_3 < 1$ by (14), it follows that S is a contraction operator on A. By the Banach contraction mapping principle, S has a fixed point $x \in A$, which is obviously a positive solution of (1). This completes the proof.

Theorem 4. *Assume that (3)-(5) hold and* $-\infty < -p_1 \leq p(t) \leq -p_2 < -1$. *Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$, the case $d < 0$ can be treated similarly. Let X be the set as in the proof of Theorem 1. Set

$$A = \{x \in X : N_4 \leq x(t) \leq d, \quad t \geq t_0\},$$

where N_4 is a positive constant such that $p_1 N_4 + d < p_2 d$. It is clear that A is a closed, bounded and convex subset of X. By (3)-(5), we can choose a $t_1 > t_0$ sufficiently large such that $\sigma_1(t+\tau) \geq t_0$, $\sigma_2(t+\tau) \geq t_0$ for $t \geq t_1$ and

$$\frac{1}{p_2}\Big[1 + \frac{2}{(n-2)!}\int_{t_1}^\infty \int_{t_1}^s \frac{(s-t)^{n-2}}{r(s)} q_i(u)duds\Big] \leq \theta_4 < 1, \quad i = 1, 2, \tag{17}$$

where θ_4 is a constant,

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}[f_1(u,d)+|g(u)|]duds \le p_2 d-\alpha \tag{18}$$

and

$$\frac{1}{(n-2)!}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}[f_2(u,d)+|g(u)|]duds \le \alpha-p_1N_4-d, \tag{19}$$

where α is a positive constant such that $p_1N_4+d<\alpha<p_2d$. Define the operator S on A by

$$(Sx)(t)=\begin{cases} -\frac{1}{p(t+\tau)}\Big[\alpha-x(t+\tau)+\frac{1}{(n-2)!}\int_{t+\tau}^{\infty}\frac{(s-t-\tau)^{n-2}}{r(s)}\int_{t_1+\tau}^{s}[f_1(u,x(\sigma_1(u))) \\ -f_2(u,x(\sigma_2(u)))-g(u)]duds\Big], \quad t\ge t_1 \\ (Sx)(t_1), \quad t_0\le t\le t_1. \end{cases}$$

Clearly, Sx is continuous. We shall show that $SA\subset A$. For each $x\in A$ and $t\ge t_1$, by using (18), we have

$$\begin{aligned}(Sx)(t) &= -\frac{1}{p(t+\tau)}\Big[\alpha-x(t+\tau)+\frac{1}{(n-2)!}\int_{t+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{(s-t-\tau)^{n-2}}{r(s)}[f_1(u,x(\sigma_1(u))) \\ &- f_2(u,x(\sigma_2(u)))-g(u)]duds\Big] \\ &\le \frac{1}{p_2}\Big[\alpha+\frac{1}{(n-2)!}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}[f_1(u,d)+|g(u)|]duds\Big] \\ &\le d\end{aligned}$$

and applying (19), we obtain

$$\begin{aligned}(Sx)(t) &= -\frac{1}{p(t+\tau)}\Big[\alpha-x(t+\tau)+\frac{1}{(n-2)!}\int_{t+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{(s-t-\tau)^{n-2}}{r(s)}[f_1(u,x(\sigma_1(u))) \\ &- f_2(u,x(\sigma_2(u)))-g(u)]duds\Big] \\ &\ge -\frac{1}{p(t+\tau)}\Big[\alpha-d-\frac{1}{(n-2)!}\int_{t_1+\tau}^{\infty}\int_{t_1+\tau}^{s}\frac{(s-t-\tau)^{n-2}}{r(s)}[f_2(u,d)+|g(u)|]duds\Big] \\ &\ge \frac{1}{p_1}\Big[\alpha-d-\frac{1}{(n-2)!}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}[f_2(u,d)+|g(u)|]duds\Big] \\ &\ge N_4.\end{aligned}$$

Hence, we proved that $SA\subset A$. Now we shall show that S is a contraction operator on A. In fact, for $x,y\in A$ and $t\ge t_1$, in view of (2) and (17), we have

$$|(Sx)(t)-(Sy)(t)| \le \frac{1}{|p(t+\tau)|}\Big[|x(t+\tau)-y(t+\tau)|$$

$$+\frac{1}{(n-2)!}\sum_{i=1}^{2}\int_{t+\tau}^{\infty}\frac{(s-t-\tau)^{n-2}}{r(s)}\int_{t_1+\tau}^{s}|f_i(u,x(\sigma_i(u)))-f_i(u,y(\sigma_i(u)))|duds\Bigg]$$
$$\le \frac{\|x-y\|}{p_2}\left[1+\frac{1}{(n-2)!}\sum_{i=1}^{2}\int_{t_1}^{\infty}\int_{t_1}^{s}\frac{(s-t)^{n-2}}{r(s)}q_i(u)duds\right]$$
$$\le \theta_4\|x-y\|.$$

This implies that

$$\|Sx-Sy\|\le\theta_4\|x-y\|.$$

Since $\theta_4 < 1$ by (17), S is a contraction operator on A. By the Banach contraction mapping principle, S has a fixed point $x \in A$, and x is a positive solution of (1). Thus, the proof is completed.

Example 1. *Consider the equation*

$$\begin{aligned}&(e^t(x(t)-e^{-t-4}x(t-4))''')'+e^{-t-5}x(t-5)\\&-e^{-t-6}x^3(t-2)-e^{-2t}+e^{-4t}+8e^{-t}=0,\quad t_0>5,\end{aligned}\tag{20}$$

where $n=4$, $r(t)=e^t$, $p(t)=e^{-t-4}$, $\tau=4$, $\sigma_1(t)=t-5$, $\sigma_2(t)=t-2$, $f_1(t,x)=e^{-t-5}x$, $f_2(t,x)=e^{-t-6}x^3$ *and* $g(t)=e^{-2t}-e^{-4t}-8e^{-t}$. *Thus,*

$$|f_1(t,x)-f_1(t,y)|=|e^{-t-5}x-e^{-t-5}y|=e^{-t-5}|x-y|,\quad \textit{where}\quad x,y\in[a,b],a>0,$$
$$|f_2(t,x)-f_2(t,y)|=|e^{-t-6}x^3-e^{-t-6}y^3|=e^{-t-6}|x^2+xy+y^2||x-y|\le 3b^2e^{-t-6}|x-y|,$$

where $x,y\in[a,b],a>0$. *Letting* $q_1(t)=e^{-t-5}$ *and* $q_2(t)=3b^2e^{-t-6}$, *then*

$$\frac{1}{(n-2)!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}q_1(u)duds=\frac{1}{2!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^2}{e^s}e^{-u-5}duds<\infty$$

and

$$\frac{1}{(n-2)!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}q_2(u)duds=\frac{1}{2!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^2}{e^s}3b^2e^{-u-6}duds<\infty.$$

Furthermore,

$$\frac{1}{(n-2)!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}|f_1(u,d)|duds=\frac{1}{2!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^2}{e^s}e^{-u-5}|d|duds<\infty,\quad d\ne 0,$$

$$\frac{1}{(n-2)!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}|f_2(u,d)|duds=\frac{1}{2!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^2}{e^s}e^{-u-6}|d|^3duds<\infty,\quad d\ne 0,$$

and

$$\frac{1}{(n-2)!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}|g(u)|duds=\frac{1}{2!}\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^2}{e^s}e^{-u-5}|e^{-2u}-e^{-4u}-8e^{-u}|duds<\infty.$$

We see that all conditions of Theorem 1 are satisfied. In fact, $x(t)=e^{-t}$ *is a nonoscillatory solution of (20).*

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