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Global Stable Location-Domination in Graphs

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Abstract. In this paper, we introduce and investigate the concept of global stable location-domination in graphs. We also characterize the global stable locating-dominating sets in the join, edge corona, corona, and lexicographic product of graphs and determine the value of the corresponding global stable locating-domination number.

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1. Introduction

Over the past decades, the study of dominating sets have been explored by many mathematicians. A lot of its variants have been introduced. An interesting variant of domination which was introduced by Slater is the location-domination [10, 11]. Locating-dominating sets were first introduced in order to identify the location of fires or intruders in a building [7].

In a building, a fire alarm system is placed on a wall or a ceiling. Each alarm will send a signal when it detects a fire in any adjacent vertices and the activated signal will determine the location of the fire. Suppose that exactly one of the alarm fails or will not function. When this situation happens, the fire alarm system may not function precisely. Moreover, if this fire alarm system fails, a backup placement of fire alarms will be useful in a more precise detection of fires in the building. To address these problems, some additional conditions can be imposed to the concept of location-domination. Some variations of location-domination that incorporate the concepts of globality and stability are introduced and investigated in [1–6, 8, 9].

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2. Terminology and Notation

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The *open neighborhood* of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of $v \in V(G)$, denoted by $deg_G(v)$, is equal to the cardinality of $N_G(v)$ and the maximum degree of G is $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. A vertex w of G is a *leaf* if $deg_G(w) = 1$. A vertex u is a support vertex if $uw \in E(G)$ for some leaf w of G. The sets $L(G)$ annd $S(G)$ denote the sets of leaves and support vertices of G, respectively.

A set $D \subseteq V(G)$ is a *dominating set* in G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N[D] = V(G)$. The minimum cardinality of a dominating set in G, denoted by $\gamma(G)$, is the *domination number* of G.

A subset S of $V(G)$ is a *locating set* in a graph G if every two vertices u and v of $V(G)\backslash S$, $N_G(u) \cap S \neq N_G(v) \cap S$. A subset S of $V(G)$ is a *strictly locating set* if it is locating and $N_G(u) \cap S \neq S$ for all $u \in V(G)\backslash S$. A locating (resp. strictly locating) subset S of $V(G)$ which is also dominating is called a *locating-dominating* (resp. *strictly locating-dominating*) set in a graph G. The minimum cardinality of a locating-dominating (resp. strictly locating-dominating) set in G, denoted by $\gamma_L(G)$ (resp. $\gamma_{SL}(G)$), is called the *L-domination* (resp. *SL-domination*) *number* of G.

A locating (resp. strictly locating) set S in G is a *stable locating* (resp. *stable strictly locating*) set in G if $S_v = S\backslash\{v\}$ is a locating (resp. strictly locating) set of G for each $v \in S$. The minimum cardinality of a stable strictly locating set, denoted by $\eta_{ssls}(G)$, is called the *stable strictly location number* of G. Any stable strictly locating set with cardinality η_{ssls} is called a minimum stable strictly locating set or an η_{ssls}-set.

A locating (resp. strictly locating) set S in G is a *stable locating* (resp. *stable strictly locating*) set in G if $S_v = S\backslash\{v\}$ is a locating (resp. strictly locating) set of G for each $v \in S$. A locating-dominating (resp. stricty locating-dominating) set S of G is a *stable locating-dominating* (resp. *stable strictly locating-dominating*) set of G if $S_v = S\backslash\{v\}$ is a locating-dominating (resp. strictly locating-dominating) set of G for each $v \in S$. The minimum cardinality of a stable locating dominating (resp. stable strictly locating-dominating) set of G, denoted by $\gamma_l^s(G)$ (resp. $\gamma_{sl}^s(G)$), is called the *stable locating-domination* (resp. *stable strictly locating-domination*) *number* of G. A stable locating-dominating (resp. stable strictly locating-dominating) set of G with cardinality $\gamma_l^s(G)$ (resp. $\gamma_{sl}^s(G)$) is called γ_l^s*-set* (resp. γ_{sl}^s*-set*) of G.

A locating-dominating set S of a graph G is a *global locating-dominating set* if it is a locating-dominating set of both G and its complement, $\overline{G}$. The *global locating-domination number* $\lambda_g(G)$ is the minimum cardinality of a global locating-dominating set of G.

A set S in G is a *global stable locating-dominating* (resp. *global stable strictly locating-dominating*) set in G if S is a stable locating-dominating (resp. stable strictly locating-dominating) set in G and in its complement, $\overline{G}$. The minimum cardinality of a global stable locating-dominating (resp. global stable strictly locating-dominating) set of G, denoted by $\lambda_{gl}^s(G)$ (resp. $\lambda_{gsl}^s(G)$), is called the *global stable locating-domination* (resp. *global stable strictly locating-domination*) number of G. A global stable locating-dominating (resp.

global stable strictly locating-dominating) set of G with cardinality $\lambda^s_{gl}(G)$ (resp. $\lambda^s_{gsl}(G)$) is called a λ^s_{gl}*-set* (resp. λ^s_{gsl}*-set*) of G.

Let G and H be two graphs. The join $G+H$ of G and H is the graph with vertex-set $V(G+H) = V(G) \overset{\bullet}{\cup} V(H)$ and edge-set $E(G+H) = E(G) \overset{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The edge corona $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of end vertices u and v of every edge uv of G to every vertex of the copy H^{uv} of H (that is, forming the join $\langle\{u,v\}\rangle + H^{uv}$ for each $uv \in E(G)$). The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i-th vertex of G to every vertex in the i-th copy of H. For every $v \in V(G)$, we denote by H^v the copy of H whose vertices are joined or attached to the vertex v. For each $v \in V(G)$, the subgraph $\langle v\rangle + H^v$ of $G \circ H$ will be denoted by $v + H^v$. The *lexicographic product* $G[H]$ of G and H is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

3. Results

The first result gives the correct version of the one found in [1].

Proposition 1. *Let G be a non-trivial graph. Then G admits a stable locating-dominating set if and only if G has no isolated vertices. If G has no isolated vertices, then $2 \le \gamma^s_l(G) \le |V(G)|$. Moreover, the following statements hold:*

(i) $\gamma^s_l(G) = 2$ *if and only if* $G = K_2$.

(ii) *If S is a stable locating-dominating set of G, then $L(G) \cup S(G) \subseteq S$.*

(iii) $\gamma^s_l(G) = |V(G)|$ *if and only if for every $v \in V(G)$, $v \in L(G) \cup S(G)$ or there exists $w \in V(G) \setminus \{v\}$ such that $N_G(v)\backslash\{w\} = N_G(w)\backslash\{v\}$.*

Theorem 1. *Let G be a any graph. Then G admits a global stable locating-dominating set if and only if G and $\overline{G}$ have no isolated vertices.*

Proof. Suppose that G admits a global stable locating-dominating set. Then G and $\overline{G}$ admit a stable locating-dominating set. Thus, by Proposition 1, G and $\overline{G}$ have no isolated vertices. □

Corollary 1. *Let G be a non-trivial connected graph. Then G admits a global stable locating-dominating set if and only if $\gamma(G) \neq 1$.*

Remark 1. *For any non-trivial connected graph G with $\gamma(G) \neq 1$, $\lambda_g(G) \le \lambda^s_{gl}(G) \le |V(G)|$ and $\gamma^s_l(G) \le \lambda^s_{gl}(G)$.*

Remark 2. *Let G be a graph that admits a global stable locating-dominating set. Then $\lambda^s_{gl}(G) = \lambda^s_{gl}(\overline{G})$.*

Proposition 2. *For any non-trivial graph G of order $n \geq 4$, $4 \leq \lambda^s_{gl}(G) \leq n$.*

Proof. Let S be λ^s_{gl}-set. By Proposition 1(i), $|S| \geq 3$. Now, suppose that $\lambda^s_{gl}(G) = 3$ and let $S = \{a, b, c\}$. Pick any $v \in V(G)\backslash S$. Suppose $n = 4$. Since S is a stable locating-dominating set, $|N_G(v)| \geq 2$. Suppose $a, b \in N_G(v)$. Since $S\backslash\{c\}$ is a dominating set, $ac \in E(G)$ or $bc \in E(G)$. If $ac \notin E(G)$, then $bc \in E(G)$. Since $S\backslash\{c\}$ is a locating set, $ab \in E(G)$. Hence, if $c \notin N_G(v)$, then $N_G(a) \cap S = N_G(v) \cap S$. This implies that $S\backslash\{a\}$ is not a locating set. Similarly, if $ac \in E(G)$, then $S\backslash\{b\}$ or $S\backslash\{c\}$ is not a locating set. Thus, $N_G(v) = \{a, b, c\}$. This is not possible because v is an isolated vertex in $\overline{G}$. If $\langle S\rangle = K_3$, then a and b are isolated vertices in $\overline{G}$. Therefore, $|V(G)| \geq 5$. Let $v, w \in V(G)\backslash S$. Since S is a stable locating-dominating set, we may assume that $|N_G(v) \cap S| = 2$ and $|N_G(w) \cap S| = 3$. Assume that $N_G(v) \cap S = \{a, b\}$. Then $N_G(w) \cap (S\backslash\{c\}) = N_G(v) \cap (S\backslash\{c\})$, implying that $S\backslash\{c\}$ is not a locating set. Therefore, $\lambda^s_{gl}(G) \geq 4$. $\square$

Lemma 1. [2] *Let G be a graph and $S \subseteq V(G)$. If $x, y \in V(G)\backslash S$, then $N_G(x) \cap S \neq N_G(y) \cap S$ if and only if $N_{\overline{G}}(x) \cap S \neq N_{\overline{G}}(y) \cap S$.*

Remark 3. *Let G be a graph of order $n \geq 4$ and suppose it admits a global stable locating-dominating set. If $\gamma^s_l(G) = n$ or $\gamma^s_l(\overline{G}) = n$, then $\lambda^s_{gl}(G) = n$.*

Note that the converse of Remark 3 is not true. To see this, consider the graphs in Figure 1. It can be verified that $S = V(G)\backslash\{4\}$ and $S' = V(\overline{G})\backslash\{1\}$ are γ^s_l-sets in G and $\overline{G}$, respectively. Hence, $\gamma^s_l(G) = \gamma^s_l(\overline{G}) = 4$. However, $\lambda^s_{gl}(G) = 5$.

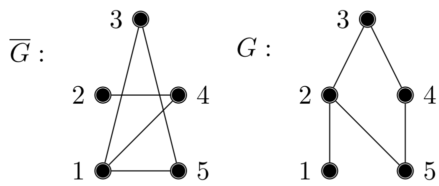

Figure 1: A graph G with $\gamma^s_l(G) = 4$ and $\gamma^s_l(\overline{G}) = 4$ but $\lambda^s_{gl}(G) = 5$

This particular example shows that the concept of stable locating-dominating set is not equivalent to the concept of global stable locating-dominating set.

Theorem 2. *Let G be a non-trivial graph of order $n \geq 5$ such that $\Delta(G) \leq n - 3$. If G admits a global stable locating-dominating set, then $\lambda^s_{gl}(G) = n$ if and only if $\gamma^s_l(G) = n$.*

Proof. Suppose that $\lambda^s_{gl}(G) = n$. Suppose that $\gamma^s_l(G) \neq n$. By Proposition 1, there exists $v \in V(G)$ such that $v \notin L(G) \cup S(G)$ and $N_G(v)\backslash\{w\} \neq N_G(w)\backslash\{v\}$ for all $w \in V(G)\backslash\{v\}$. Let $S = V(G)\backslash\{v\}$. Then S is a locating-dominating set of G. Let $z \in S$ and set $S_z = S\backslash\{z\} = V(G)\backslash\{v, z\}$. Suppose that $vz \notin E(G)$ and choose $x, y \in V(G)$ such that $xz, vy \in E(G)$. Then $x, y \in S_z$. Since $v \notin L(G) \cup S(G)$, $\deg_G(v) \geq 2$. Hence there exists $x \in V(G)\backslash\{v, z\} = S_z$ such that $xz \in E(G)$. Since $v \notin S(G)$, $N_G(z) \neq \{v\}$. Hence, there exists $y \in S_z$ such that $yz \in E(G)$. Thus, S_z is a dominating set of G. Since

S_z is a locating set in G, S_z is a locating set in $\overline{G}$ by Lemma 1. This shows that S is a stable locating-dominating set in $\overline{G}$. Therefore, S is a global stable locating-dominating set in G and $\lambda_{gl}^s(G) \leq |S| = n-1$, a contradiction. Therefore, $\gamma_l^s(G) = n$.

Conversely, suppose that $\gamma_l^s(G) = n$. By Remark 1, $\lambda_{gl}^s(G) = n$. □

Remark 4. *Theorem 2 may not hold if the condition $\Delta(G) \leq n-3$ is removed.*

To see this, consider the graphs G and $\overline{G}$ shown in Figure 1. Note that $\Delta(G) = |V(G)| - 2$ and $\gamma_l^s(G) \neq 5$ but $\lambda_{gl}^s(G) = 5 = |V(G)|$.

Corollary 2. *Let G be a non-trivial connected graph of order $n = 4$. Then $\lambda_{gl}^s(G) = 4$ if and only if $G = C_4$ or $G = P_4$.*

Theorem 3. [1] *Let G and H be non-trivial graphs. A set S is a stable locating-dominating set of $G+H$ if and only if $S = S_G \cup S_H$ and S_G and S_H are stable locating sets of G and H, respectively, and at least one of them is a stable strictly locating set or both of them are strictly locating sets.*

Proposition 3. *Let G and H be non-trivial graphs. Then $G+H$ admits a global stable locating-dominating set if and only if G and H admit a stable strictly locating set. Moreover, $S \subseteq V(G+H)$ is a global stable locating-dominating set in $G+H$ if and only if $S = A \cup B$, where A and B are stable strictly locating sets in G and H, respectively.*

Proof. Suppose that $G+H$ admits a global stable locating-dominating set, say S. Then S is a stable locating-dominating set in $G+H$. By Theorem 3, A and B are stable locating sets in G and H, respectively.

Now, suppose that A is not a stable strictly locating set. Let $w \in A$ and set $A^w = A \backslash \{w\}$. Since A is not a stable strictly locating set, there exists $z \in V(G) \backslash A^w$ such that $N_G(z) \cap A^w = A^w$. This implies that $N_{\overline{G}}(z) \cap A^w = \varnothing$, a contradiction since $\overline{G+H} = \overline{G} \cup \overline{H}$ is a stable locating-dominating set. Thus, A is a stable strictly locating set in G. Similarly, B is a stable strictly locating set in H.

Conversely, suppose that $S = A' \cup B'$, where A' and B' are stable strictly locating sets in G and H, respectively. Since A' is stable strictly locating set in G, A' is a stable locating set and a strictly locating set. Similarly, since B' is stable strictly locating set in H, B' is a stable locating set and a strictly locating set. By Theorem 3, S is a stable locating-dominating set in $G+H$. Since A' and B' are strictly locating sets, A' and B' are dominating sets in $\overline{G}$ and $\overline{H}$, respectively. Hence, S is a dominating set in $\overline{G+H}$.

Let $a, b \in V(\overline{G+H}) \backslash S$ with $a \neq b$. Suppose that $a, b \in V(\overline{G})$. Since A' is a locating set, $N_G(a) \cap A' \neq N_G(b) \cap A'$. Hence, $N_{\overline{G+H}}(a) \cap S = N_{\overline{G}}(a) \cap A' \neq N_{\overline{G}}(b) \cap A' = N_{\overline{G+H}}(b) \cap S$ by Lemma 1. Similarly, $N_{\overline{G+H}}(a) \cap S \neq N_{\overline{G+H}}(b) \cap S$ if $a, b \in V(\overline{H})$. Thus, S is a locating set in $\overline{G+H} = \overline{G} \cup \overline{H}$.

Next, let $w \in S$ and set $S_w = S \backslash \{w\}$. Suppose $w \in A'$ and let $A^{w'} = A \backslash \{w\}$. Since A is a stable strictly locating set in G, $A^{w'}$ is a strictly locating set. Hence, $A^{w'}$ is a dominating set in $\overline{G}$. Now, let $a, b \in V(\overline{G+H}) \backslash A^{w'}$ with $a \neq b$. Suppose that

$a, b \in V(\overline{G})\backslash A^{w'}$. Since $A^{w'}$ is a stable locating set, $N_G(a) \cap A^{w'} \neq N_G(b) \cap A^{w'}$. Hence, $N_{\overline{G+H}}(a) \cap S_w = N_{\overline{G}}(a) \cap A^{w'} \neq N_{\overline{G}}(b) \cap A^{w'} = N_{\overline{G+H}}(b) \cap S_w$ by Lemma 1. Thus, $A^{w'}$ is a locating-dominating set in $\overline{G}$. Similarly, $B^{w'} = B\backslash\{w\}$ is a locating-dominating set in $\overline{H}$ if $w \in B$.

Thus, S is a stable locating-dominating set in $\overline{G+H} = \overline{G} \cup \overline{H}$. Therefore, S is a global stable locating-dominating set in $G + H$. □

Corollary 3. *Let G and H be non-trivial connected graphs. A subset S of $V(G+H)$ is a minimum global stable locating-dominating set in $G+H$ if and only if $S = A \cup B$, where A and B is a minimum stable strictly locating set in G and H, respectively. In particular, $\lambda^s_{gl}(G+H) = \eta_{ssls}(G) + \eta_{ssls}(H)$.*

Corollary 4. *Let G be a graph and let $K_m = \langle\{v_1, v_2, ..., v_m\}\rangle$ where $m \geq 2$. A subset S of $V(\overline{K}_m + G)$ is a global stable locating-dominating set in $\overline{K}_m + G$ if and only if $S = \{v_1, v_2, ..., v_m\} \cup S_G$, where S_G is a stable strictly locating set in G. In particular, $\lambda^s_{gl}(\overline{K}_m + G) = m + \eta_{ssls}(G)$.*

Proof. The only stable strictly locating set in $\overline{K}_m$ is the $V(\overline{K}_m)$. Thus, by Proposition 3, S is a global stable locating-dominating set in $\overline{K}_m + G$. □

Theorem 4. [4] *Let G be a connected graph of order $m \geq 3$ and let H be any non-trivial connected graph. Then C is a stable locating-dominating set of $G \diamond H$ if and only if $C = A \cup \left(\cup_{uv \in E(G)} S_{uv}\right)$ and satisfies the following conditions:*

(i) $A \subseteq V(G)$,

(ii) *For each $uv \in E(G)$,*

- *(a)* *S_{uv} is a stable locating set of H^{uv};*
- *(b)* *S_{uv} is a stable locating-dominating set of H^{uv} whenever $u, v \notin A$;*
- *(c)* *S_{uv} is a stable strictly locating set of H^{uv} for each $v \in L(G)$ with $v \notin A$; and*
- *(d)* *S_{uv} is a stable strictly locating-dominating set of H^{uv} whenever $u, v \notin A$ and $\{u, v\} \cap L(G) \neq \varnothing$.*

(iii) *For each $w \in A$ and for each $z \in N_G(w)$, we have:*

- *(a)* *S_{zw} is a strictly locating set of H^{zw} whenever $w \in L(G)$ and*
- *(b)* *S_{zw} is a strictly locating-dominating set of H^{zw} whenever $z \notin A$ and $\{z, w\} \cap L(G) \neq \varnothing$.*

(iv) *For each $zw \in E(G)$ with $z \in A$ and $w \notin A$, if $x \in V(H^{zw})\backslash[S_{zw}\backslash\{p\}]$ for $p \in S_{zw}$ and $N_{H^{zw}} \cap (S_{zw}\backslash\{p\}) = \varnothing$, then for each $y \in N_G(z)\backslash\{w\}$ and for each $q \in V(H^{yz})\backslash S_{yz}$, it holds that $y \in A$ or $N_{H^{yz}}(q) \cap S_{yz} \neq \varnothing$.*

Lemma 2. *Let G be a connected graph such that $G \neq K_2$ and H be any non-trivial connected graph. Then $\gamma(G \diamond H) = 1$ if and only if $\gamma(G) = 1$.*

Proof. Suppose $\gamma(G \diamond H) = 1$, say $\{p\}$ is a dominating set in $G \diamond H$. If $p \in V(G)$, then $\gamma(G) = 1$. Suppose that $p \in V(H^{uv})$ for some $uv \in E(G)$. If $G \neq K_2$, there exists an $xy \in E(G)\backslash\{uv\}$. Then $pq \notin E(G \diamond H)$ for all $q \in V(H^{xy})$, a contradiction. Thus, $G = K_2$. Therefore, $\gamma(G) = 1$.

Conversely, if $\gamma(G) = 1$, say $\{q\}$ is a dominating set in G. Then for every vertex $u \in V(G \diamond H)$, $qu \in E(G \diamond H)$. Thus, $\gamma(G \diamond H) = 1$. □

Theorem 5. *Let G be a connected graph of order $n \geq 4$ and $\Delta(G) \leq n-2$ and H be any non-trivial connected graph. Then $G \diamond H$ admits a global stable locating-dominating set. Moreover, $S \subseteq V(G \diamond H)$ is a global stable locating-dominating set in $G \diamond H$ if and only if S is a stable locating-dominating set in $G \diamond H$. In particular, $\lambda^s_{gl}(G \diamond H) = \gamma^s_l(G \diamond H)$.*

Proof. Since $\gamma(G) \neq 1$, $\gamma(G \diamond H) \neq 1$ by Lemma 2. Therefore since $G \diamond H$ is connected and $\gamma(G \circ H) \neq 1$, $G \circ H$ admits a global stable locating-dominating set by Corollary 1. Let S be a global stable locating-dominating set in $G \circ H$. Then S is a stable locating-dominating set in $G \circ H$.

Conversely, suppose that S is a stable locating-dominating set in $G \diamond H$. Let $A \subseteq V(G)$ and $D_{uv} = V(H^{uv}) \cap S$. By Theorem 4, $S = A \cup \left(\cup_{uv \in E(G)} D_{uv}\right)$ is a stable locating-dominating set in $G \diamond H$. Let $x \in V(\overline{G \diamond H})\backslash S$. If $x \in V(G)\backslash A$, then pick $v \in V(G)\backslash N_G(x)$. Since G is connected, we may choose $u \in N_G(v)$. It follows that $xp \in E(G \diamond H)$ for all $p \in D_{uv}$. By Theorem 4 $(ii)(a)$, $D_{uv} \neq \varnothing$ and $xp \in E(\overline{G \diamond H})$ for all $p \in S_{uv}$. Suppose $x \in V(H^{yz})\backslash D_{yz}$. Choose $wt \in E(G)\backslash\{yz\}$. Then $xq \in E(\overline{G \diamond H})$ for all $q \in D_{wt}$. Thus, S is a dominating set in $\overline{G \diamond H}$.

Now, let $a, b \in V(\overline{G \diamond H})\backslash S$ where $a \neq b$. Since S is a locating set in $G \diamond H$, $N_{G \diamond H}(a) \cap S \neq N_{G \diamond H}(b) \cap S$. Hence, $N_{\overline{G \diamond H}}(a) \cap S \neq N_{\overline{G \diamond H}}(b) \cap S$ by Lemma 1. Thus, S is a locating set in $\overline{G \diamond H}$.

Finally, let $w \in S$ and set $S_w = S\backslash\{w\}$. Again, since $D_{uv} \neq \varnothing$, for every $uv \in E(G)$, S_w is a dominating set in $\overline{G \diamond H}$. Since S is a stable locating set in $G \diamond H$, S_w is a locating set in $G \diamond H$. Thus, S_w is a locating set in $\overline{G \diamond H}$ by Lemma 1. Therefore, S is a stable locating-dominating set in $\overline{G \diamond H}$.

Accordingly, S is a global stable locating-dominating in $G \diamond H$. Consequently, $\lambda^s_{gl}(G \diamond H) = \gamma^s_l(G \diamond H)$. □

Theorem 6. [1] *Let G be a connected non-trivial graph and let H be any graph without isolated vertices. Then $S \subseteq V(G \circ H)$ is a stable locating-dominating set of $G \circ H$ if and only if $S = A \cup [\cup_{v \in V(G)} D_v]$ and satisfies the following properties:*

(i) $A \subseteq V(G)$.

(ii) *D_v is a stable locating-dominating set of H^v for each $v \in (V(G) \setminus A)$, and, in addition, strictly locating when $|N_G(v) \cap A| = 1$.*

(*iii*) *D_v is a stable strictly locating-dominating set of H^v for each $v \in V(G) \setminus N_G(A)$.*

(*iv*) *D_v is a dominating stable locating set for each $v \in A$ and, in addition, strictly locating when $N_G(v) \cap A = \varnothing$.*

Proposition 4. *Let G be a connected non-trivial graph and let H be any graph. Then $G \circ H$ admits a global stable locating-dominating set. Moreover, $S \subseteq V(G \circ H)$ is a global stable locating-dominating set in $G \circ H$ if and only if S is a stable locating-dominating set in $G \circ H$. In particular, $\lambda^s_{gl}(G \circ H) = \gamma^s_l(G \circ H)$.*

Proof. Since $G \circ H$ is connected and $\gamma(G \circ H) \neq 1$, $G \circ H$ admits a global stable locating-dominating set by Corollary 1. Let S be a global stable locating-dominating set in $G \circ H$. Then S is a stable locating-dominating set in $G \circ H$.

Conversely, suppose that S is stable locating-dominating set in $G \circ H$. Let $A \subseteq V(G)$ and $D_v = V(H^v) \cap S$. Then $S = A \cup \left(\cup_{v \in V(G)} D_v\right)$ and $D_v \neq \varnothing$ by Theorem 6. If $v \in V(G) \backslash A$, then we may choose $w \in V(G) \backslash N_G(v)$. Since $D_w \neq \varnothing$, $vp \in E(\overline{G \circ H})$ for all $p \in D_w$. Suppose $v \in V(H^u) \backslash D_u$. Then $vs \in E(\overline{G \circ H})$ for all $s \in D_z \backslash \{u\}$. Thus, S is a dominating set in $\overline{G \circ H}$.

Since S is a locating set in $G \circ H$, $N_{G \circ H}(a) \cap S \neq N_{G \circ H}(b) \cap S$. Hence, $N_{\overline{G \circ H}}(a) \cap S \neq N_{\overline{G \circ H}}(b) \cap S$ by Lemma 1. Thus, S is a locating-dominating set in $\overline{G \circ H}$.

Finally, let $y \in S$ and set $S_y = S \backslash \{y\}$. Again, since $D_v \neq \varnothing$ for each $v \in V(G)$, S_y is a dominating set in $\overline{G \circ H}$. Since S is a stable locating-dominating set in $G \circ H$, S_y is a locating set in $\overline{G \circ H}$. Thus, S_y is a locating-dominating set in $\overline{G \circ H}$. Therefore, S is a stable locating-dominating set in $\overline{G \circ H}$.

Accordingly, S is a global stable locating-dominating in $G \circ H$. Consequently, $\lambda^s_{gl}(G \circ H) = \gamma^s_l(G \circ H)$. □

Theorem 7. [4] *Let G and H be a non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$. Then $C = \bigcup_{x \in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a global stable locating-dominating set of $G[H]$ if and only if each of the following holds:*

(*i*) *$S = V(G)$.*

(*ii*) *T_x is a stable locating set in H for every $x \in V(G)$.*

(*iii*) *If x and y are adjacent vertices with $N_G[x] = N_G[y]$ and one, say T_x is not strictly locating, then T_y is a stable strictly locating set of H.*

(*iv*) *If x and y are distinct non-adjacent vertices of G with $N_G(x) = N_G(y)$ and one, say T_x is not a dominating set, then T_y is a stable dominating set of H.*

Proposition 5. *Let G and H be a non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$. Then $G[H]$ admits a global stable locating-dominating set. Moreover, $C = \bigcup_{u \in S} (\{u\} \times T_u)$, where $S \subseteq V(G)$ and $T_u \subseteq V(H)$ for each $u \in S$, is a global stable locating-dominating set of $G[H]$ if and only if each of the following holds:*

(i) $S = V(G)$.

(ii) T_u *is a stable locating set in* H *for every* $u \in V(G)$.

(iii) *If* u *and* v *are adjacent vertices with* $N_G[u] = N_G[v]$ *and one, say* T_u *is not strictly locating, then* T_v *is a stable strictly locating set of* H.

(iv) *If* u *and* v *are distinct non-adjacent vertices of* G *with* $N_G(u) = N_G(v)$ *and one, say* T_u *is not a dominating set, then* T_v *is a stable dominating set of* H.

(v) *For each* $u \in V(G)$ *with* $deg_G(u) = |V(G)| - 1$, *it hold that* $T_u \backslash N_H(x) \neq \varnothing$ *for all* $x \in V(H)\backslash T_u$, *that is,* T_u *is dominating in* $\overline{H}$.

Proof. Since $\gamma(H) \neq 1$, $\gamma(G[H]) \neq 1$. By Corollary 1, $G[H]$ admits a global stable locating-dominating set. Let C be a global stable locating-dominating set in $G[H]$. Then C is a stable locating-dominating set in $G[H]$. By Theorem 7, (i), (ii), (iii), and (iv) hold. Now, let $u \in V(G)$ with $\deg_G(u) = |V(G)| - 1$. Suppose that T_u is not a dominating set of $\overline{H}$. Then there exists $w \in V(\overline{H})\backslash N_{\overline{H}}[T_u]$ and $N_H(u) \cap T_u = T_u$. Thus, there exists $(u, w) \in V(G[H])\backslash C$ such that $N_{G[H]}((u, w)) \cap C = C$ which implies that $N_{\overline{G[H]}}((u, w)) \cap C = \varnothing$. Hence, C is not a dominating set of $G[H]$, a contradiction. Therefore, T_u is a dominating set of $\overline{H}$.

Conversely, suppose that (i), (ii), (iii), (iv), and (v) hold. Then by Proposition 7, C is a stable locating-dominating set of $G[H]$. By (i) and (v), C is a dominating set of $\overline{G[H]}$. By Lemma 1 and (ii), C is a stable locating-dominating set of $\overline{G[H]}$. Thus, C is a stable locating-dominating set of $\overline{G[H]}$. Therefore, C is a global stable locating-dominating set of $G[H]$. □

The following result is a consequence of Proposition 5.

Corollary 5. *Let* G *and* H *be a non-trivial connected graphs with* $\Delta(H) \leq |V(H)| - 2$. *If* $\gamma(G) \neq 1$, *then* $\lambda^s_{gl}(G[H]) = \gamma^s_l(H)$.

Conclusion

This study introduced and initially investigated the concept of global stable locating-dominating set. This concept was shown to be different from the previously defined concept of stable locating-dominating set. Graph which attains the order as its global stable locating-domination number was characterized. Global stable locating-dominating sets in the join, edge corona, corona, and lexicographic product of graphs were characterized and global stable locating-domination number of each of these graphs were determined. In this paper, the authors did not provide any realization result involving global stable locating-domination number and any locating-related parameters. Thus, it is recommended that a realization result involving global stable locating-domination number and stable locating-domination number be constructed. Further, the concept can be studied for other graphs not considered in this study.

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