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Solving Fuzzy System of Boundary Value Problems by Homotopy Perturbation Method with Green's Function

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Abstract. In this research, we successfully demonstrated the use of the homotopy perturbation method with Green's function to find approximate solutions for the fuzzy system of boundary value problems. Our results showcase the effectiveness of this method in providing accurate and reliable solutions. The consistent way to reduce the size of the computation gives to reach the exact solution is one of the best methods adopted to determine the behavior of the solution directly in order to determine the approximate solution analytically, Finally, the problems that have been addressed confirmed the validity of the method applied analytically in this research using comparison with some numerical problems.

2020 Mathematics Subject Classifications: 03E72, 34A07, 34B05, 34B15

Key Words and Phrases: Fuzzy system, Boundary value problems, Homotopy perturbation method, He's polynomials, Green's function

1. Introduction

A multitude of dynamic processes can be represented mathematically as systems of partial or ordinary differential equations. Differential equations have been widely recognized as a successful modeling approach. Meanwhile, fuzzy group theory serves as a potent tool in handling unknown variables and processing fuzzy or subjective information in mathematical models [1, 5, 9, 18]. It may be necessary to use such mathematical modeling, and to use fuzzy differential equations (FDEs) that involve a system of elementary value problems. Fuzzy systems of differential equations appear when modeling these problems are incomplete and questionable in the case of inaccurate data. Fuzzy differential equation systems are applicable mathematical models [6]. Dynamic models of dynamic systems with uncertainties or ambiguous models, represented by fuzzy differential equations, are widely utilized across a spectrum of fields, such as geometry and population modeling. These models have proven to be versatile and applicable in a diverse range of

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applications, making them a valuable tool for data analysis and modeling [24].
In recent years, the homotopy perturbation method (HPM) has been studied and successfully applied by many engineers, scientists and researchers, and used to solve the integral equations and differential equations. This HPM was introduced first by Dr. Ji Huan He in 1998 [15, 16]. The HPM is a coupling of the homotopy method, a basic concept of topology, and the classical perturbation technique. This coupling will provide with a suitable way to obtain approximate or analytic solution for different problems arising in various scientific fields [14]. The HPM boasts several advantages, including the ability to obtain accurate approximate solutions with only a few iterations and a rapid convergence of the solution series in many instances. This method has proven to be a valuable tool in solving a wide range of problems, particularly in the realm of integral equations [20].
The integral equations can be used to describe a wide variety of problems in science and engineering [3]. The HPM is a sophisticated and effective approach for solving system of linear and nonlinear differential and integral equations [2]. A system of linear and nonlinear differential equations may also be solved using this approach due to the complexity of finding exact solutions for nonlinear differential equations, any perturbative method that meets certain criteria is deemed acceptable. The HPM offers an analytical solution by utilizing the initial conditions and is noteworthy for its ability to produce highly accurate approximate solutions with only a small number of terms required [4]. In most cases, only approximate solutions, either numerical or analytical, can be obtained for nonlinear problems. The numerous branches of science have a vast array of nonlinear problems for which exact solutions cannot be found. As a result, numerous analytical and numerical approximations have been studied and developed to tackle these complex equations. Thus, alternative methods become crucial for solving these nonlinear equations effectively [14].
The concept of fuzziness has experienced a surge in popularity and application across various fields since its inception, including learning theory, automata, decision-making processes, algorithms, pattern classification, and linguistics. The idea behind fuzzy concepts involves converting differential equations into differential inclusions and accepting the solution as the $\alpha-$cut of the fuzzy solution, leading to the transformation of the fuzzy differential equation (FDE) into a equivalent fuzzy integral equation [6].
In the last years, the researchers Visuvasam et al. [23], Shanthi et al. [19], Vijayalakshmi et al. [21] and Vijayalakshmi et al. [22] have applied HPM to solve the mathematical models as (porous rotating disk electrodes, ECE reactions at rotating disk electrodes, magnetohydrodynamic, etc.).
The main objective of the fuzzy HPM with Green's function, is to develop a more effective and efficient method for solving linear and non-linear fuzzy system of boundary value problems (FSBVPs)

$$\left\{\begin{array}{l} \sum\limits_{i=0}^{2} a_i(x)\,\widetilde{y}_1^{(i)}(x,\alpha) + \sum\limits_{i=0}^{1} b_i(x)\,\widetilde{y}_2^{(i)}(x,\alpha) + N_1\left[\widetilde{y}_1(x,\alpha),\widetilde{y}_2(x,\alpha)\right] = \widetilde{f}_1(x,\alpha), \\ \sum\limits_{i=0}^{2} c_i(x)\,\widetilde{y}_2^{(i)}(x,\alpha) + \sum\limits_{i=0}^{1} d_i(x)\,\widetilde{y}_1^{(i)}(x,\alpha) + N_2\left[\widetilde{y}_1(x,\alpha),\widetilde{y}_2(x,\alpha)\right] = \widetilde{f}_2(x,\alpha), \end{array}\right. \quad x \in [0,1] \tag{1}$$

with the boundary conditions

$$\begin{aligned} \widetilde{y}_1(0,\alpha) = \widetilde{y}_1(1,\alpha) = (0,0), \\ \widetilde{y}_2(0,\alpha) = \widetilde{y}_2(1,\alpha) = (0,0), \end{aligned} \tag{2}$$

where $\widetilde{y}_1(x,\alpha), \widetilde{y}_2(x,\alpha), \widetilde{f}_1(x,\alpha), \widetilde{f}_2(x,\alpha)$ and $a_i(x), b_i(x), c_i(x), d_i(x)$ for $i = 0,1,2$ are analytic functions, and we are seeking the solutions $\widetilde{y}_1(x,\alpha), \widetilde{y}_2(x,\alpha)$ satisfying Equation 1. We assume that for every $\widetilde{f}_1(x,\alpha), \widetilde{f}_2(x,\alpha)$ Equation 1 has one and only one solution.

By combining the advantages of the HPM and Green's function, this study aims to provide a new approach that leverages the strengths of both methods to obtain accurate approximate solutions for fuzzy ODEs. This research can help to deepen our understanding of fuzzy system of ODEs and provide valuable insights into the development of new techniques for solving these complex problems.

2. The fundamental concept of HPM

Now, we demonstrate the fundamental concept of the HPM [8, 13, 14], we assume the following non-linear differential equation.

$$D(y) - g(r) = 0, r \in \Omega \tag{3}$$

with the condition

$$B\left(y, \frac{\partial y}{\partial n}\right) = 0, r \in \Gamma \tag{4}$$

where D is a general differential operator, which can be divided into linear parts L and N nonlinear term of Equation 4, B is a boundary operator, $g(r)$ is a known analytic function, Γ is the boundary of the domain Ω. can be rewritten as

$$L(y) + N(y) - g(r) = 0. \tag{5}$$

By the homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (-p)[L(v) - L(v_0)] + p[A(v) - g(r)] = 0, p \in [0,1], r \in \Omega) \tag{6}$$

or

$$H(v,p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - g(r)] = 0, \tag{7}$$

where $p \in [0,1]$ is an embedding parameter, v_0 is an initial approximation of Equation 6 which satisfies the boundary conditions. Obviously, from Equation 7 we have

$$H(v,0) = L(v) - L(v_0) = 0, \tag{8}$$

$$H(v,1) = D(v) - g(r) = 0. \tag{9}$$

The changing process of p from zero to unity is just that of $y(r,p)$ from $y_0(r)$ to $y(r)$. In topology, this is called deformation, and $L(y) - L(v_0)$, $B(y) - f(r)$ are called homptopic.

In this paper, the authors will first use the imbedding parameter p as a "small parameter", and assume that the solution of Equation 7 can be written as a power series in p :

$$y = \sum_{n=0}^{\infty} p^n y_n = y_0 + py_1 + p^2 y_2 + \cdots , \tag{10}$$

and the nonlinear term Ny can be decomposed as

$$Ny = \sum_{n=0}^{\infty} p^n H_n(y), \tag{11}$$

where the H_n are He's polynomials of $y_0, y_1, \ldots, y_n$ and are calculated by the definitional formula [11]

$$H_n(y_0, y_1, \ldots, y_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i y_i\right)\right]_{p=0}, \quad n = 0, 1, 2, \ldots .$$

Setting $p = 1$ results in the approximate solution of Equation 3

$$y = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n y_n = y_0 + y_1 + y_2 + y_3 + \cdots . \tag{12}$$

The convergence of the series solution (10) is given in [12].

3. HPM with Green's function

The Green's function $G(x, \xi)$ corresponding to linear operator $\frac{d^2}{dx^2}\widetilde{y}_i(x, \alpha), i = 1, 2$ in Eq. (1) is

$$G(x,\xi) = \begin{cases} C_1(\xi)x + C_2(\xi), & 0 \le \xi \le x, \\ C_3(\xi)x + C_4(\xi), & x \le \xi \le 1, \end{cases} \tag{13}$$

where $x \neq \xi$, $C_1(\xi)$, $C_2(\xi)$, $C_3(\xi)$ and $C_4(\xi)$ are linearly independent solutions of $\frac{d^2}{dx^2}\widetilde{y}_i(x,\alpha)$ with $a_2(x) = 1$ and $c_2(x) = 1$. We follow three conditions to find the Green's function of the second-order BVP.

- $G(x, \xi)$ is continuous at $x = \xi$:

$$[C_1(\xi)x + C_2(\xi)]_{x=\xi} = [C_3(\xi)x + C_4(\xi)]_{x=\xi} \tag{14}$$

- $\frac{\partial}{\partial x}G(x, \xi)$ is discontinuous at $x = \xi$:

$$\left\{\frac{\partial}{\partial x}[C_1(\xi)x + C_2(\xi)] - \frac{\partial}{\partial x}[C_3(\xi)x + C_4(\xi)]\right\}_{x=\xi} = -1 \tag{15}$$

− $G(x, \xi)$ satisfies the homogeneous boundary conditions (2):

$$[C_1(\xi)\, x + C_2(\xi)]_{x=0} = 0, \quad [C_3(\xi)\, x + C_4(\xi)]_{x=1} = 0. \tag{16}$$

By solving the above three equations (14)-(16) to find $C_1(\xi), C_2(\xi), C_3(\xi)$ and $C_4(\xi)$, therefore, the Green's function (13) of Eqs. (1) and (2) can be written as

$$G(x,\xi) = \begin{cases} \xi(1-x), & 0 \le \xi \le x, \\ x(1-\xi), & x \le \xi \le 1. \end{cases} \tag{17}$$

Now, we costruct the HPM with Green's function, we have

$$\begin{aligned} \widetilde{y}_1(x,\alpha) = \widetilde{h}_1(x,\alpha) &- \int_0^1 G(x,\xi)\, \widetilde{f}_1(\xi,\alpha)\, d\xi \\ &+ p\int_0^1 G(x,\xi) \sum_{i=0}^{1} a_i(\xi)\, \widetilde{y}_1^{(i)}(\xi,\alpha)\, d\xi \\ &+ p\int_0^1 G(x,\xi) \sum_{i=0}^{1} b_i(\xi)\, \widetilde{y}_2^{(i)}(\xi,\alpha)\, d\xi \\ &+ p\int_0^1 G(x,\xi)\, N_1[\widetilde{y}_1(\xi,\alpha), \widetilde{y}_2(\xi,\alpha)]\, d\xi, \\ \widetilde{y}_2(x,\alpha) = \widetilde{h}_2(x,\alpha) &- \int_0^1 G(x,\xi)\, \widetilde{f}_2(\xi,\alpha)\, d\xi \\ &+ p\int_0^1 G(x,\xi) \sum_{i=0}^{1} c_i(\xi)\, \widetilde{y}_2^{(i)}(\xi,\alpha)\, d\xi, \\ &+ p\int_0^1 G(x,\xi) \sum_{i=0}^{1} d_i(\xi)\, \widetilde{y}_1^{(i)}(\xi,\alpha)\, d\xi \\ &+ p\int_0^1 G(x,\xi)\, N_2[\widetilde{y}_1(\xi,\alpha), \widetilde{y}_2(\xi,\alpha)]\, d\xi, \end{aligned} \tag{18}$$

where $\widetilde{h}_i(x,\alpha)$ is the solution of $y_i''(x,\alpha) = 0$, $i = 1, 2$ with the boundary conditions (2).

Substituting Eqs. 10 and 11 into Equation 18, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n \widetilde{y}_{1,n}(x,\alpha) &= \widetilde{h}_1(x,\alpha) - \int_0^1 G(x,\xi)\, \widetilde{f}_1(\xi,\alpha)\, d\xi \\
&\quad + p\int_0^1 G(x,\xi) \sum_{i=0}^{1} a_i(\xi) \sum_{n=0}^{\infty} p^n \widetilde{y}_{1,n}^{(i)}(\xi,\alpha)\, d\xi \\
&\quad + p\int_0^1 G(x,\xi) \sum_{i=0}^{1} b_i(\xi) \sum_{n=0}^{\infty} p^n \widetilde{y}_{2,n}^{(i)}(\xi,\alpha)\, d\xi \\
&\quad + p\int_0^1 G(x,\xi) \sum_{n=0}^{\infty} p^n \widetilde{H}_{1,n} d\xi \\
\widetilde{y}_2(x,\alpha) &= \widetilde{h}_2(x,\alpha) - \int_0^1 G(x,\xi)\, \widetilde{f}_2(\xi,\alpha)\, d\xi \\
&\quad + p\int_0^1 G(x,\xi) \sum_{i=0}^{1} c_i(\xi) \sum_{n=0}^{\infty} p^n \widetilde{y}_{2,n}^{(i)}(\xi,\alpha)\, d\xi, \\
&\quad + p\int_0^1 G(x,\xi) \sum_{i=0}^{1} d_i(\xi) \sum_{n=0}^{\infty} p^n \widetilde{y}_{1,n}^{(i)}(\xi,\alpha)\, d\xi \\
&\quad + p\int_0^1 G(x,\xi) \sum_{n=0}^{\infty} p^n \widetilde{H}_{2,n} d\xi.
\end{aligned} \tag{19}$$

Using the fuzzy HPM, according to Equation 19, the lower iterations (L) are then determined in the following recursive way:

$$p^0: \begin{cases} \underline{y}_{1,0}(x,\alpha) = \underline{h}_1(x,\alpha) - \int_0^1 G(x,\xi)\, \underline{f}_1(\xi,\alpha)\, d\xi, \\ \underline{y}_{2,0}(x,\alpha) = \underline{h}_2(x,\alpha) - \int_0^1 G(x,\xi)\, \underline{f}_2(\xi,\alpha)\, d\xi, \end{cases}$$

$$p^n: \begin{cases} \underline{y}_{1,n+1}(x,\alpha) = \int_0^1 G(x,\xi) \sum_{i=0}^{1} a_i(\xi)\, \underline{y}_{1,n}^{(i)}(\xi,\alpha)\, d\xi, \\ \qquad + \int_0^1 G(x,\xi) \left[\sum_{i=0}^{1} b_i(\xi)\, \underline{y}_{2,n}^{(i)}(\xi,\alpha) + \underline{H}_{1,n} \right] d\xi, \\ \underline{y}_{2,n+1}(x,\alpha) = \int_0^1 G(x,\xi) \sum_{i=0}^{1} c_i(\xi)\, \underline{y}_{2,n}^{(i)}(\xi,\alpha)\, d\xi, \\ \qquad + \int_0^1 G(x,\xi) \left[\sum_{i=0}^{1} d_i(\xi)\, \underline{y}_{1,n}^{(i)}(\xi,\alpha) + \underline{H}_{2,n} \right] d\xi, \end{cases} , n \geq 1 \tag{20}$$

and the upper iterations (U) are

$$p^0: \begin{cases} \overline{y}_{1,0}(x,\alpha) = \overline{h}_1(x,\alpha) - \int_0^1 G(x,\xi)\overline{f}_1(\xi,\alpha)\,d\xi, \\ \overline{y}_{2,0}(x,\alpha) = \overline{h}_2(x,\alpha) - \int_0^1 G(x,\xi)\overline{f}_2(\xi,\alpha)\,d\xi, \end{cases}$$

$$p^n: \begin{cases} \overline{y}_{1,n+1}(x,\alpha) = \int_0^1 G(x,\xi)\sum_{i=0}^{1} a_i(\xi)\overline{y}_{1,n}^{(i)}(\xi,\alpha)\,d\xi, \\ \quad + \int_0^1 G(x,\xi)\left[\sum_{i=0}^{1} b_i(\xi)\overline{y}_{2,n}^{(i)}(\xi,\alpha) + \overline{H}_{1,n}\right]d\xi, \\ \overline{y}_{2,n+1}(x,\alpha) = \int_0^1 G(x,\xi)\sum_{i=0}^{1} c_i(\xi)\overline{y}_{2,n}^{(i)}(\xi,\alpha)\,d\xi, \\ \quad + \int_0^1 G(x,\xi)\left[\sum_{i=0}^{1} d_i(\xi)\overline{y}_{1,n}^{(i)}(\xi,\alpha) + \overline{H}_{2,n}\right]d\xi. \end{cases}, n \geq 1 \tag{21}$$

By solving the system (20) and (21), we obtain the iterations $\widetilde{y}_{i,0}(x,\alpha)$, $\widetilde{y}_{i,1}(x,\alpha),\ldots,\widetilde{y}_{i,n}(x,\alpha)$, $i = 1,2$. Thus, the approximated solution in a series form is given by

$$\widetilde{y}_i(x,\alpha) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n \widetilde{y}_{i,n}(x,\alpha) = \sum_{n=0}^{\infty}\widetilde{y}_{i,n}(x,\alpha).$$

4. Basic Concepts

Definition 1. *[10] A fuzzy number $\widetilde{u}:\mathbb{R}\to[0,1]$ satisfying the properties:*

(i) $\widetilde{u}$ *is normal, if* $\widetilde{u}(x_0) = 1$, $x_0 \in \mathbb{R}$.

(ii) $\widetilde{u}$ *is fuzzy convex set if* $\widetilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$, $\forall\, x,y \in \mathbb{R}$, $\lambda \in [0,1]$.

(iii) $\widetilde{u}$ *is upper semi-continuous on* $\mathbb{R}$.

(iv) $\overline{\widetilde{u}(x)}$*,closure of* $\widetilde{u}(x)$ $x \in \mathbb{R}$ *and is called a compact set.*

The set of all fuzzy numbers is denoted by $\mathbb{R}_F$. For $0 < \alpha \leq 1$, denote $[\widetilde{u}]_\alpha = \{x \in \mathbb{R}, \widetilde{u}(x) \geq \alpha\}$ and $[\widetilde{u}]_0 = \{x \in \mathbb{R}, \widetilde{u}(x) > 0\}$. Then, it is well-known that for any $\alpha \in [0,1]$, $[\widetilde{u}]_\alpha$ is a bounded closed interval. A triangular fuzzy numbers are very popular and denoted by $A = (a_1, a_2, a_3)$ and defined by

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x-a_1}{a_{2}-a_1}, & a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases}$$

where $a_1, a_3 > 0$.

Definition 2. *[10, 24] An arbitrary fuzzy number* $\widetilde{u}$ *in the parametric form is represented by an ordered pair of functions* $\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right)$, $0 \leq \alpha \leq 1$*; which satisfy the following requirements .*

(i) $\underline{y}(x,\alpha)$ *is a bounded left continuous non-decreasing function over* $[0,1]$.

(ii) $\overline{y}(x,\alpha)$*is a bounded left continuous non-increasing function over* $[0,1]$.

(iii) $\underline{y}(x,\alpha) \leq \overline{y}(x,\alpha)$, $0 \leq \alpha \leq 1$.

Definition 3. *[24] For arbitrary fuzzy numbers* $\widetilde{u}(x,\alpha) = (\underline{u}(x,\alpha), \overline{u}(x,\alpha))$ *and* $\widetilde{v}(x,\alpha) = (\underline{v}(x,\alpha), \overline{v}(x,\alpha))$,
we define the distance between $\widetilde{u}(x,\alpha)$ *and* $\widetilde{v}(x,\alpha)$ *by the quantity*

$$D(\widetilde{u}(x,\alpha), \widetilde{v}(x,\alpha)) = \sup_{\alpha\in[0,1]} \max\left\{\left|\underline{u}(x,\alpha) - \underline{v}(x,\alpha)\right|, \left|\overline{u}(x,\alpha) - \overline{v}(x,\alpha)\right|\right\}$$

5. Applications

In this section, we demonstrate the exceptional precision and accuracy of the results obtained through this method. By showcasing the solutions of two distinct problems.we give a comparison the approximate solution by the (HPM with Green's function) with the exact solution given. Moreover, the absolute errors between them are defined as follows:

$$\begin{aligned} AE_L &= \left|\underline{y}_{i,E}(x,\alpha) - \underline{y}_{i,n}(x,\alpha)\right| \\ AE_U &= \left|\overline{y}_{i,E}(x,\alpha) - \overline{y}_{i,n}(x,\alpha)\right| \end{aligned}, \quad \begin{aligned} i &= 1,2 \\ n &= 1,2,\ldots \end{aligned}$$

where

$\underline{y}_{i,E}(x,\alpha)$: Lower exact solution,

$\underline{y}_{i,n}(x,\alpha)$: Lower approximate solution (HPM),

$\overline{y}_{i,E}(x,\alpha)$: Upper exact solution,

$\overline{y}_{i,n}(x,\alpha)$: Upper approximate solution (HPM).

The computations associated with the problems were performed using a Maple 18 package with a precision of 20 digits.

Problem 1. We first consider the linear FSBVPs [7, 9, 17]:

$$\begin{cases} \widetilde{y}_1''(x,\alpha) + (2x-1)\,\widetilde{y}_1'(x,\alpha) + \cos(\pi x)\,\widetilde{y}_2'(x,\alpha) = \widetilde{f}_1(x,\alpha), \\ \widetilde{y}_2''(x,\alpha) + x\widetilde{y}_1(x,\alpha) = \widetilde{f}_2(x,\alpha), \end{cases} \tag{22}$$

with the boundary conditions

$$\begin{aligned} \widetilde{y}_1(0,\alpha) &= \widetilde{y}_1(1,\alpha) = (0,0), \\ \widetilde{y}_2(0,\alpha) &= \widetilde{y}_2(1,\alpha) = (0,0), \end{aligned} \tag{23}$$

where $\widetilde{f}_1(x,\alpha) = \left[\underline{f}_1(x,\alpha), \overline{f}_1(x,\alpha)\right]$ and $\widetilde{f}_2(x,\alpha) = \left[\underline{f}_2(x,\alpha), \overline{f}_2(x,\alpha)\right]$ are given by

$$\begin{aligned} \underline{f}_1(x,\alpha) &= -\alpha\sin(\pi x)\pi^2 + [(2\alpha\pi+\alpha+1)x]\cos(\pi x), \\ &\quad -\left[\alpha\pi + \frac{1}{2}\alpha + \frac{1}{2}\right]\cos(\pi x), \\ \overline{f}_1(x,\alpha) &= (\alpha-2)\pi^2\sin(\pi x) + [(4\pi - 2\alpha\pi - 4\alpha + 6)x]\cos(\pi x), \\ &\quad + [\alpha\pi + 2\alpha - 2\pi - 3]\cos(\pi x), \\ \underline{f}_2(x,\alpha) &= x\alpha\sin(\pi x) + \alpha + 1, \\ \overline{f}_2(x,\alpha) &= (2-\alpha)x\sin(\pi x) - 4\alpha + 6. \end{aligned} \tag{24}$$

The exact solutions of the FSBVPs (22) and (23) are

$$\begin{aligned} \widetilde{y}_{1,E}(x,\alpha) &= [\alpha\sin(\pi x), (2-\alpha)\sin(\pi x)], \\ \widetilde{y}_{2,E}(x,\alpha) &= \left[\left(\frac{\alpha}{2} + \frac{1}{2}\right)\left(x^2 - x\right), (3-2\alpha)\left(x^2-x\right)\right]. \end{aligned}$$

Constructing the HPM with Green's function, we have

$$\begin{aligned} \widetilde{y}_1(x,\alpha) &= \widetilde{h}_1(x,\alpha) - \textstyle\int_0^1 G(x,\xi)\widetilde{f}_1(\xi,\alpha)\,d\xi \\ &\quad + p\textstyle\int_0^1 G(x,\xi)\left[(2\xi-1)\widetilde{y}_1'(\xi,\alpha) + \cos(\pi x)\widetilde{y}_2'(\xi,\alpha)\right]d\xi, \\ \widetilde{y}_2(x,\alpha) &= \widetilde{h}_2(x,\alpha) - \textstyle\int_0^1 G(x,\xi)\widetilde{f}_2(\xi,\alpha)\,d\xi \\ &\quad + p\textstyle\int_0^1 G(x,\xi)\left[\xi\widetilde{y}_1(\xi,\alpha)\right]d\xi, \end{aligned} \tag{25}$$

where $\widetilde{h}_i(x,\alpha)$ is the solution of $y_i''(x,\alpha) = 0$, $i = 1,2$ with the boundary conditions (23) and $G(x,\xi)$ is the Green's function given by (17). Substituting Equation 10 into Equation 25, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} p^n\widetilde{y}_{1,n}(x,\alpha) &= \widetilde{h}_1(x,\alpha) - \textstyle\int_0^1 G(x,\xi)\widetilde{f}_1(\xi,\alpha)\,d\xi \\ &\quad + p\textstyle\int_0^1 G(x,\xi)\left[(2\xi-1)\sum_{n=0}^{\infty} p^n\widetilde{y}_{1,n}'(\xi,\alpha) + \cos(\pi x)\sum_{n=0}^{\infty} p^n\widetilde{y}_{2,n}'(\xi,\alpha)\right]d\xi, \\ \sum_{n=0}^{\infty} p^n\widetilde{y}_{2,n}(x,\alpha) &= \widetilde{h}_2(x,\alpha) - \textstyle\int_0^1 G(x,\xi)\widetilde{f}_2(\xi,\alpha)\,d\xi \\ &\quad + p\textstyle\int_0^1 G(x,\xi)\left[\xi\sum_{n=0}^{\infty} p^n\widetilde{y}_{1,n}(\xi,\alpha)\right]d\xi. \end{aligned} \tag{26}$$

Using the fuzzy HPM, according to Equation 26, the lower iterations (L) are then determined in the following recursive way:

$$\begin{aligned}
&p^0: \begin{cases} \underline{y}_{1,0}(x,\alpha) = \underline{h}_1(x,\alpha) - \int_0^1 G(x,\xi)\,\underline{f}_1(\xi,\alpha)\,d\xi, \\ \underline{y}_{2,0}(x,\alpha) = \underline{h}_2(x,\alpha) - \int_0^1 G(x,\xi)\,\underline{f}_2(\xi,\alpha)\,d\xi, \end{cases} \\
&p^n: \begin{cases} \underline{y}_{1,n+1}(x,\alpha) = \int_0^1 G(x,\xi)\left[(2\xi-1)\,\underline{y}'_{1,n}(\xi,\alpha) + \cos(\pi x)\,\underline{y}'_{2,n}(\xi,\alpha)\right] d\xi, \\ \underline{y}_{2,n+1}(x,\alpha) = \int_0^1 G(x,\xi)\left[\xi \underline{y}_{1,n}(\xi,\alpha)\right] d\xi, \end{cases}
\end{aligned} \tag{27}$$

and the upper iterations (U) are

$$\begin{aligned}
&p^0: \begin{cases} \overline{y}_{1,0}(x,\alpha) = \overline{h}_1(x,\alpha) - \int_0^1 G(x,\xi)\,\overline{f}_1(\xi,\alpha)\,d\xi, \\ \overline{y}_{2,0}(x,\alpha) = \overline{h}_2(x,\alpha) - \int_0^1 G(x,\xi)\,\overline{f}_2(\xi,\alpha)\,d\xi, \end{cases} \\
&p^n: \begin{cases} \overline{y}_{1,n+1}(x,\alpha) = \int_0^1 G(x,\xi)\left[(2\xi-1)\,\overline{y}'_{1,n}(\xi,\alpha) + \cos(\pi x)\,\overline{y}'_{2,n}(\xi,\alpha)\right] d\xi, \\ \overline{y}_{2,n+1}(x,\alpha) = \int_0^1 G(x,\xi)\left[\xi \overline{y}_{1,n}(\xi,\alpha)\right] d\xi. \end{cases}
\end{aligned} \tag{28}$$

By solving the system (27) and (28), we obtain the iterations $\widetilde{y}_{i,0}(x,\alpha)$, $\widetilde{y}_{i,1}(x,\alpha), \ldots, \widetilde{y}_{i,n}(x,\alpha)$, $i = 1,2$. Thus, the approximated solution in a series form is given by

$$\widetilde{y}_i(x,\alpha) = \sum_{n=0}^{5} \widetilde{y}_{i,n}(x,\alpha).$$

We compare the approximated solution using the HPM ($n = 5$) with the exact solution and the error produced in Tables 1–4. Table 5 demonstrate the maximum errors on the interval $0 \le x \le 1$.

Table 1: Comparison of the numerical results for $\underline{y}_1(x,\alpha)$ (Problem 1)

α	x	$\underline{y}_{1,E}(x,\alpha)$	$\underline{y}_{1,5}(x,\alpha)$	AE_L
0.25	0.1	0.0772542	0.0772587	$4.531E-06$
	0.3	0.2022542	0.2022542	$6.846E-06$
	0.5	0.2500000	0.2500069	$6.923E-06$
	0.7	0.2022542	0.2022610	$6.771E-06$
	0.9	0.0772542	0.0772589	$4.728E-06$
0.50	0.1	0.1545084	0.1545163	$7.849E-06$
	0.3	0.4045084	0.4045203	$1.185E-05$
	0.5	0.5000000	0.5000119	$1.198E-05$
	0.7	0.4045084	0.4045202	$1.172E-05$
	0.9	0.1545084	0.1545166	$8.188E-06$
0.75	0.1	0.2317627	0.2317739	$1.116E-05$
	0.3	0.6067627	0.6067796	$1.685E-05$
	0.5	0.7500000	0.7500170	$1.704E-05$
	0.7	0.6067627	0.6067794	$1.667E-05$
	0.9	0.2317627	0.2317743	$1.164E-05$
1.00	0.1	0.3090169	0.3090314	$1.448E-05$
	0.3	0.8090169	0.8090388	$2.186E-05$
	0.5	1.0000000	1.0000221	$2.210E-05$
	0.7	0.8090169	0.8090386	$2.162E-05$
	0.9	0.3090169	0.3090321	$1.510E-05$

Table 2: Comparison of the numerical results for $\overline{y}_1(x,\alpha)$ (Problem 1)

α	x	$\overline{y}_{1,E}(x,\alpha)$	$\overline{y}_{1,5}(x,\alpha)$	AE_U
0.25	0.1	0.5407797	0.5408069	$2.716E-05$
	0.3	1.4157797	1.4158207	$4.101E-05$
	0.5	1.75000000	1.7500414	$4.147E-05$
	0.7	1.4157797	1.4158203	$4.056E-05$
	0.9	0.5407797	0.5408080	$2.834E-05$
0.50	0.1	0.4635254	0.4635484	$2.294E-05$
	0.3	1.2135254	1.2135601	$3.463E-05$
	0.5	1.5000000	1.5000350	$3.501E-05$
	0.7	1.2135254	1.2135597	$3.425E-05$
	0.9	0.4635254	0.4635494	$2.393E-05$
0.75	0.1	0.3862712	0.3862899	$1.871E-05$
	0.3	1.0112712	1.0112994	$2.824E-05$
	0.5	1.2500000	1.2500285	$2.855E-05$
	0.7	1.0112712	1.0112991	$2.793E-05$
	0.9	0.3862712	0.3862907	$1.952E-05$
1.00	0.1	0.3090169	0.3090314	$1448E-05$
	0.3	0.8090169	0.8090388	$2.186E-05$
	0.5	1.0000000	1.0000221	$2.210E-05$
	0.7	0.8090169	0.8090386	$2.162E-05$
	0.9	0.3090169	0.3090321	$1.510E-05$

Table 3: Comparison of the numerical results for $\underline{y}_2(x,\alpha)$ (Problem 1)

α	x	$\underline{y}_{2,E}(x,\alpha)$	$\underline{y}_{2,5}(x,\alpha)$	AE_L
0.25	0.1	-0.056250	-0.0562509	$9.101E-07$
	0.3	-0.131250	-0.1312525	$2.542E-06$
	0.5	-0.156250	-0.1562534	$3.486E-06$
	0.7	-0.131250	-0.1312532	$3.256E-06$
	0.9	-0.056250	-0.0562514	$1.452E-06$
0.50	0.1	-0.067500	-0.6750158	$1.583E-05$
	0.3	-0.157500	-0.1575044	$4.423E-06$
	0.5	-0.187500	-0.1875060	$6.065E-06$
	0.7	-0.157500	$-.01575056$	$5.665E-06$
	0.9	-0.067500	-0.0675025	$2.527E-06$
0.75	0.1	-0.787500	-0.7875225	$2.256E-06$
	0.3	-0.183750	-0.1837563	$6.304E-06$
	0.5	-0.218750	-0.2187586	$8.644E-06$
	0.7	-0.183750	-0.1837580	$8.074E-06$
	0.9	-0.787500	-0.7875360	$3.601E-06$
1.00	0.1	-0.900000	-0.9000292	$2.929E-06$
	0.3	-0.210000	-0.2100081	$8.186E-06$
	0.5	-0.250000	-0.2500112	$1.122E-05$
	0.7	-0.210000	-0.2100104	$1.048E-05$
	0.9	-0.900000	-0.9000467	$4.675E-06$

Table 4: Comparison of the numerical results for $\overline{y}_2(x,\alpha)$ (Problem 1)

α	x	$\overline{y}_{2,E}(x,\alpha)$	$\overline{y}_{2,5}(x,\alpha)$	AE_U
0.25	0.1	-0.2250	-0.2250054	$5.048E-06$
	0.3	-0.5250	-0.5250153	$1.531E-05$
	0.5	-0.6250	-0.6250210	$2.100E-05$
	0.7	-0.5250	-0.5250196	$1.961E-05$
	0.9	-0.2250	-0.2250087	$8.750E-06$
0.50	0.1	-0.1800	-0.1800046	$4.631E-06$
	0.3	-0.3200	-0.3200090	$9.056E-06$
	0.5	-0.5000	-0.5000177	$1.774E-05$
	0.7	-0.4200	$-.04200165$	$1.657E-05$
	0.9	-0.1800	-0.1800073	$7.392E-06$
0.75	0.1	-0.1350	-0.1350037	$3.780E-06$
	0.3	-0.3150	-0.3150105	$1.056E-05$
	0.5	-0.3750	-0.3750144	$1.448E-05$
	0.7	-0.3150	-0.3150135	$1.352E-05$
	0.9	-0.1350	-0.1350060	$6.034E-06$
1.00	0.1	-0.9000	-0.9000292	$2.929E-06$
	0.3	-0.2100	-0.2100081	$8.186E-06$
	0.5	-0.2500	-0.2500112	$1.122E-05$
	0.7	-0.2100	-0.2100104	$1.048E-05$
	0.9	-0.9000	-0.9000467	$4.675E-06$

Table 5: Maximum errors (Problem 1)

α	n	$\widetilde{y}_1(x,\alpha)$		$\widetilde{y}_2(x,\alpha)$	
		$\underline{y}_1(x,\alpha)$	$\overline{y}_1(x,\alpha)$	$\underline{y}_2(x,\alpha)$	$\overline{y}_2(x,\alpha)$
0.25	5	$6.957E-06$	$4.167E-05$	$3.546E-06$	$2.136E-05$
0.50	5	$1.204E-05$	$3.519E-05$	$6.169E-06$	$1.804E-05$
0.75	5	$1.712E-05$	$2.870E-05$	$8.793E-06$	$1.473E-05$
1.00	5	$2.221E-05$	$2.221E-05$	$1.141E-05$	$1.141E-05$

In Figures 1–4, a very good agreement is shown between the exact solution $\big(\widetilde{y}_{i,E}(x,0.5), i=1,2\big)$ with a continuous line and the approximate solution by the HPM with Green's function $\big(\widetilde{y}_{i,5}(x,0.5), i=1,2\big)$ with the symbol o.

We present the contour plot in $2D$ on the (x,α)-plane for the exact solutions $\big(\widetilde{y}_{i,E}(x,0.5), i=1,2\big)$ with a continuous line and the approximate solution by the HPM with Green's function $(\widetilde{y}_{i,5}(x,0.5),\ i=1,2)$ with the symbol o in Figures 5–8.

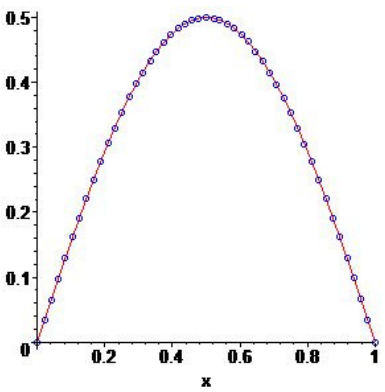

Figure 1: Plot of $\underline{y}_{1,E}(x,0.5)$ and $\underline{y}_{1,5}(x,0.5)$.

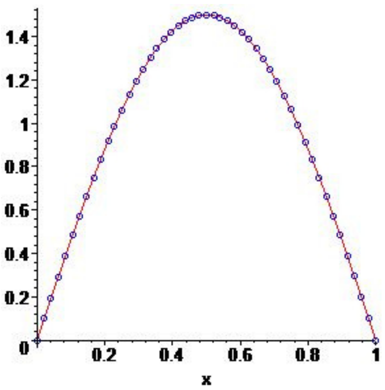

Figure 2: Plot of $\overline{y}_{1,E}(x,0.5)$ and $\overline{y}_{1,5}(x,0.5)$.

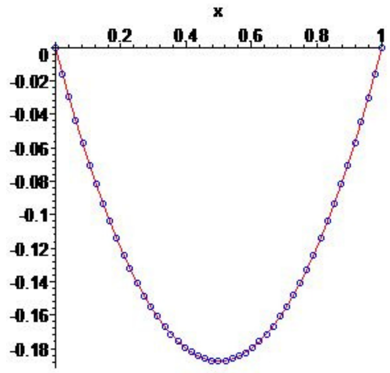

Figure 3: Plot of $\underline{y}_{2,E}(x, 0.5)$ and $\underline{y}_{2,5}(x, 0.5)$.

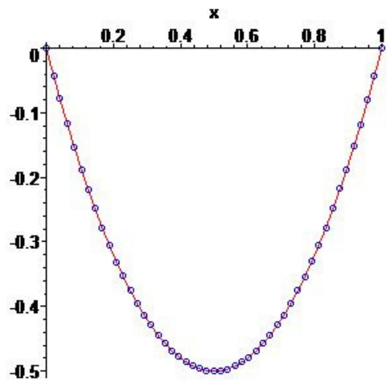

Figure 4: Plot of $\overline{y}_{2,E}(x, 0.5)$ and $\overline{y}_{2,5}(x, 0.5)$.

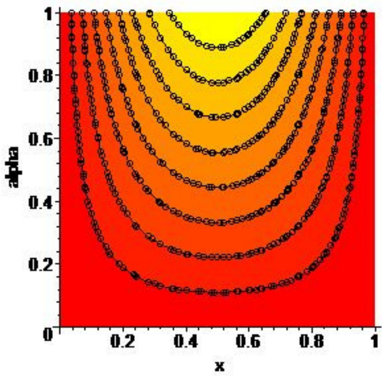

Figure 5: The contour plot in $2D$ (x, α)-plane for $\underline{y}_{1,E}(x, \alpha)$ and $\underline{y}_{1,5}(x, \alpha)$.

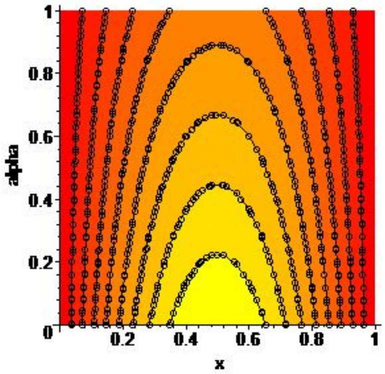

Figure 6: The contour plot in $2D$ (x, α)-plane for $\overline{y}_{1,E}(x, \alpha)$ and $\overline{y}_{1,5}(x, \alpha)$.

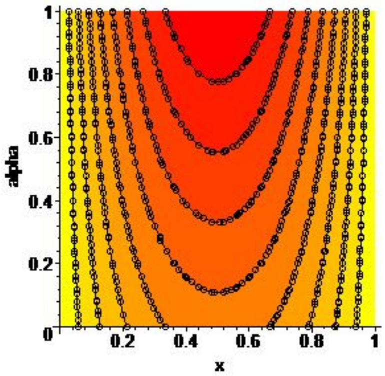

Figure 7: The contour plot in $2D$ (x, α)-plane for $\underline{y}_{2,E}(x, \alpha)$ and $\underline{y}_{2,5}(x, \alpha)$.

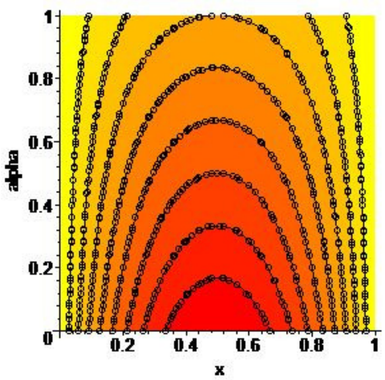

Figure 8: The contour plot in $2D$ (x, α)-plane for $\overline{y}_{2,E}(x, \alpha)$ and $\overline{y}_{2,5}(x, \alpha)$.

Problem 2 Now, we turn to a non-linear FSBVPs [7, 9, 17]:

$$\begin{cases} \widetilde{y}_1''(x,\alpha) + x\widetilde{y}_1'(x,\alpha) + \cos(\pi x)\,\widetilde{y}_2'(x,\alpha) = \widetilde{f}_1(x,\alpha), \\ \widetilde{y}_2''(x,\alpha) + x\widetilde{y}_1'(x,\alpha) + xy_1^2(x,\alpha) = \widetilde{f}_2(x,\alpha), \end{cases} \tag{29}$$

with the boundary conditions

$$\begin{aligned} \widetilde{y}_1(0,\alpha) = \widetilde{y}_1(1,\alpha) = (0,0), \\ \widetilde{y}_2(0,\alpha) = \widetilde{y}_2(1,\alpha) = (0,0), \end{aligned} \tag{30}$$

where $\widetilde{f}_1(x,\alpha) = \left[\underline{f}_1(x,\alpha), \overline{f}_1(x,\alpha)\right]$ and $\widetilde{f}_2(x,\alpha) = \left[\underline{f}_2(x,\alpha), \overline{f}_2(x,\alpha)\right]$ are given by

$$\begin{aligned} \underline{f}_1(x,\alpha) &= \alpha^2 \sin(x) + (\alpha^2x^2 - \alpha^2x + 2\alpha^2)\cos(x) \\ &\quad + (-\alpha^3x - x + \frac{\alpha^3}{2} + \frac{1}{2})\cos(\pi x), \\ \overline{f}_1(x,\alpha) &= (3-2\alpha)\sin(x) + (-2\alpha x^2 + 3x^2 + 2\alpha x - 3x - 4\alpha + 6)\cos(x) \\ &\quad + (2\alpha^3x - 4x - \alpha^3 + 2)\cos(\pi x), \\ \underline{f}_2(x,\alpha) &= \left(\alpha^4x^3 - 2\alpha^4x^2 + \alpha^4x\right)\sin(x)^2 + \alpha^2x\sin(x) \\ &\quad + (\alpha^2x^2 - \alpha^2x)\cos(x) - \alpha^3 - 1, \\ \overline{f}_2(x,\alpha) &= +\left(4\alpha^2 - 12\alpha + 9\right)x^3\sin(x)^2 + (-8\alpha^2 + 24\alpha - 18)x^2\sin(x)^2 \\ &\quad + \left(4\alpha^2 - 12\alpha + 9\right)x\sin(x)^2 + (-2\alpha+3)x\sin(x) \\ &\quad + (-2\alpha+3)x^2\cos(x) + (2\alpha-3)x\cos(x) - 4 + 2\alpha^3. \end{aligned} \tag{31}$$

The exact solutions of the FSBVPs (29) and (30) are

$$\begin{aligned} \widetilde{y}_{1,E}(x,\alpha) &= \left[\alpha^2(x-1)\sin(x), (3-2\alpha)(x-1)\sin(x)\right], \\ \widetilde{y}_{2,E}(x,\alpha) &= \left[\frac{(\alpha^3+1)}{2}\left(x-x^2\right), \left(2-\alpha^3\right)\left(x-x^2\right)\right]. \end{aligned}$$

Constructing the HPM with Green's function, we have

$$\begin{aligned} \widetilde{y}_1(x,\alpha) &= \widetilde{h}_1(x,\alpha) - \textstyle\int_0^1 G(x,\xi)\,\widetilde{f}_1(\xi,\alpha)\,d\xi \\ &\quad + p\textstyle\int_0^1 G(x,\xi)\left[\xi\widetilde{y}_1'(\xi,\alpha) + \cos(\pi x)\,\widetilde{y}_2'(\xi,\alpha)\right]d\xi, \\ \widetilde{y}_2(x,\alpha) &= \widetilde{h}_2(x,\alpha) - \textstyle\int_0^1 G(x,\xi)\,\widetilde{f}_2(\xi,\alpha)\,d\xi \\ &\quad + p\textstyle\int_0^1 G(x,\xi)\left[\xi\widetilde{y}_1'(\xi,\alpha) + \xi\widetilde{y}_1^2(\xi,\alpha)\right]d\xi, \end{aligned} \tag{32}$$

where $\widetilde{h}_i(x,\alpha)$ is the solution of $y_i''(x,\alpha) = 0$, $i = 1,2$ with the boundary conditions (30) and $G(x,\xi)$ is the Green's function given by (17). Substituting Eqs. (10) and (11) into

Equation 32, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p^n \widetilde{y}_{1,n}(x,\alpha) &= \widetilde{h}_1(x,\alpha) - \int_0^1 G(x,\xi)\widetilde{f}_1(\xi,\alpha)\,d\xi \\ &+p\int_0^1 G(x,\xi)\left[\xi\sum_{n=0}^{\infty} p^n \widetilde{y}'_{1,n}(\xi,\alpha) + \cos(\pi x)\sum_{n=0}^{\infty} p^n \widetilde{y}'_{2,n}(\xi,\alpha)\right]d\xi, \\ \sum_{n=0}^{\infty} p^n \widetilde{y}_{2,n}(x,\alpha) &= \widetilde{h}_2(x,\alpha) - \int_0^1 G(x,\xi)\widetilde{f}_2(\xi,\alpha)\,d\xi \\ &+p\int_0^1 G(x,\xi)\left[\xi\sum_{n=0}^{\infty} p^n \widetilde{y}'_{1,n}(\xi,\alpha) + \xi\sum_{n=0}^{\infty} p^n H_{1,n}\right]d\xi.\end{aligned} \tag{33}$$

Using the fuzzy HPM, according to Equation 33, the lower iterations (L) are then determined in the following recursive way:

$$\begin{aligned}&p^0: \begin{cases} \underline{y}_{1,0}(x,\alpha) = \underline{h}_1(x,\alpha) - \int_0^1 G(x,\xi)\underline{f}_1(\xi,\alpha)\,d\xi, \\ \underline{y}_{2,0}(x,\alpha) = \underline{h}_2(x,\alpha) - \int_0^1 G(x,\xi)\underline{f}_2(\xi,\alpha)\,d\xi, \end{cases} \\ &p^n: \begin{cases} \underline{y}_{1,n+1}(x,\alpha) = \int 0^1 G(x,\xi)\left[\xi\underline{y}'_{1,n}(\xi,\alpha) + \cos(\pi x)\underline{y}'_{2,n}(\xi,\alpha)\right]d\xi, \\ \underline{y}_{2,n+1}(x,\alpha) = \int 0^1 G(x,\xi)\left[\xi\underline{y}'_{1,n}(\xi,\alpha) + \xi\underline{H}_{1,n}\right]d\xi, \end{cases}\end{aligned} \tag{34}$$

and the upper iterations (U) are

$$\begin{aligned}&p^0: \begin{cases} \overline{y}_{1,0}(x,\alpha) = \overline{h}_1(x,\alpha) - \int_0^1 G(x,\xi)\overline{f}_1(\xi,\alpha)\,d\xi, \\ \overline{y}_{2,0}(x,\alpha) = \overline{h}_2(x,\alpha) - \int_0^1 G(x,\xi)\overline{f}_2(\xi,\alpha)\,d\xi, \end{cases} \\ &p^n: \begin{cases} \overline{y}_{1,n+1}(x,\alpha) = \int_0^1 G(x,\xi)\left[\xi\overline{y}'_{1,n}(\xi,\alpha) + \cos(\pi x)\overline{y}'_{2,n}(\xi,\alpha)\right]d\xi, \\ \overline{y}_{2,n+1}(x,\alpha) = \int_0^1 G(x,\xi)\left[\xi\overline{y}'_{1,n}(\xi,\alpha) + \xi\overline{H}_{1,n}\right]d\xi. \end{cases}\end{aligned} \tag{35}$$

where the nonlinear terms defined by the series

$$\underline{y}^2_{1,n}(\xi,\alpha) = \sum_{n=0}^{\infty}\underline{H}_{1,n}, \quad \overline{y}^2_{1,n}(\xi,\alpha) = \sum_{n=0}^{\infty}\overline{H}_{1,n},$$

and the corresponding He's polynomials $\left[\underline{H}_{1,n}, \overline{H}_{1,n}\right]$ [11] are

$$\underline{H}_{1,n} = \sum_{n=0}^{\infty}\underline{y}_{1,i}\underline{y}_{1,n-i}, \quad \overline{H}_{1,n} = \sum_{n=0}^{\infty}\overline{y}_{1,i}\overline{y}_{1,n-i}, \quad n \geq i, \quad n = 0,1,2,\ldots$$

By solving the system (34) and (35), we obtain the iterations $\widetilde{y}_{i,0}(x,\alpha)$, $\widetilde{y}_{i,1}(x,\alpha),\ldots,\widetilde{y}_{i,n}(x,\alpha)$, $i = 1,2$. Thus, the approximated solution in a series form is given by

$$\widetilde{y}_i(x,\alpha) = \sum_{n=0}^{2}\widetilde{y}_{i,n}(x,\alpha).$$

We compare the approximated solution using the HPM ($n = 2$) with the exact solution and the error produced in Tables 6–9. Table 10 demonstrates the maximum errors on the interval $0 \leq x \leq 1$.

Table 6: Comparison of the numerical results for $\underline{y}_1(x, \alpha)$ (Problem 2)

α	x	$\underline{y}_{1,E}(x, \alpha)$	$\underline{y}_{1,2}(x, \alpha)$	AE_L
0.25	0.1	-0.0056156	-0.0056463	$3.067E-05$
	0.3	-0.0129290	-0.0128439	$8.503E-05$
	0.5	-0.0149820	-0.0146152	$3.667E-04$
	0.7	-0.0120790	-0.0113873	$6.917E-04$
	0.9	-0.0048957	-0.0042341	$6.615E-04$
0.50	0.1	-0.0224625	-0.0226306	$1.681E-04$
	0.3	-0.0517160	-0.0519550	$2.389E-04$
	0.5	-0.0599281	-0.0598771	$5.100E-05$
	0.7	-0.0483163	-0.0477882	$5.280E-03$
	0.9	-0.0195831	-0.0189384	$6.447E-04$
0.75	0.1	-0.0505406	-0.0509607	$4.200E-04$
	0.3	-0.1163610	-0.1171638	$8.027E-04$
	0.5	-0.1348384	-0.1353070	$4.685E-04$
	0.7	-0.1087117	-0.1084163	$2.953E-04$
	0.9	-0.0440621	-0.0433940	$6.680E-04$
1.00	0.1	-0.0898500	-0.0906715	$8.214E-04$
	0.3	-0.2068641	-0.2085212	$1.657E-03$
	0.5	-0.2397127	-0.2409320	$1.219E-03$
	0.7	-0.1932653	-0.1932705	$5.257E-06$
	0.9	-0.0783326	-0.0775769	$7.557E-04$

Table 7: Comparison of the numerical results for $\overline{y}_1(x,\alpha)$ (Problem 2)

α	x	$\overline{y}_{1,E}(x,\alpha)$	$\overline{y}_{1,2}(x,\alpha)$	AE_U
0.25	0.1	-0.2246251	-0.2272738	$2.648E-03$
	0.3	-0.5171603	-0.5226018	$5.441E-03$
	0.5	-0.5992819	-0.6039739	$4.691E-03$
	0.7	-0.4831632	-0.4850771	$1.913E-03$
	0.9	$-.01958317$	-0.1952747	$5.569E-04$
0.50	0.1	-0.1797001	-0.1816565	$1.956E-03$
	0.3	-0.4137282	-0.4176773	$3.949E-03$
	0.5	-0.4794255	-0.4825976	$3.172E-03$
	0.7	-0.3865306	-0.3873276	$7.970E-04$
	0.9	-0.1566653	-0.1556608	$1.004E-03$
0.75	0.1	-0.1347751	-0.1361216	$1.346E-03$
	0.3	-0.3102962	-0.3129718	$2.675E-03$
	0.5	-0.3595691	-0.3615538	$1.984E-03$
	0.7	-0.2898979	-0.2900114	$1.134E-04$
	0.9	-0.1174990	-0.1163858	$1.113E-03$
1.00	0.1	-0.0898500	-0.0906715	$8.214E-04$
	0.3	-0.2068641	-0.2085212	$1.657E-03$
	0.5	-0.2397127	-0.2409320	$1.219E-03$
	0.7	-0.1932653	-0.1932705	$5.257E-06$
	0.9	-0.0783326	-0.0775769	$7.557E-04$

Table 8: Comparison of the numerical results for $\underline{y}_2(x,\alpha)$ (Problem 2)

α	x	$\underline{y}_{2,E}(x,\alpha)$	$\underline{y}_{2,2}(x,\alpha)$	AE_L
0.25	0.1	0.0457031	0.0457084	$5.327E-06$
	0.3	0.1066406	0.1068146	$1.740E-04$
	0.5	0.1269531	0.1274189	$4.657E-04$
	0.7	0.1066406	0.1074269	$7.862E-04$
	0.9	0.0457031	0.0464363	$7.332E-04$
0.50	0.1	0.0506250	0.0506059	$1.900E-05$
	0.3	0.1181250	0.1182502	$1.252E-04$
	0.5	0.1406250	0.1410804	$4.554E-04$
	0.7	0.1181250	0.1190405	$9.155E-04$
	0.9	0.0506250	0.0515624	$9.374E-04$
0.75	0.1	0.0639843	0.0639516	$3.277E-05$
	0.3	0.1492968	0.1494366	$1.397E-04$
	0.5	0.1777343	0.1783339	$5.995E-04$
	0.7	0.1492968	0.1506143	$1.317E-03$
	0.9	0.0639843	0.0653802	$1.395E-03$
1.00	0.1	0.0900000	0.0900165	$1654E-05$
	0.3	0.2100000	0.2103843	$3843E-04$
	0.5	0.2500000	0.2511554	$1155E-03$
	0.7	0.2100000	0.2122592	$2259E-03$
	0.9	0.0900000	0.0922510	$2251E-03$

Table 9: Comparison of the numerical results for $\overline{y}_2(x,\alpha)$ (Problem 2)

α	x	$\overline{y}_{2,E}(x,\alpha)$	$\overline{y}_{2,2}(x,\alpha)$	AE_U
0.25	0.1	0.1785937	0.1793249	$7.311E-04$
	0.3	0.4167187	0.4195254	$2.806E-03$
	0.5	0.4960937	0.5014287	$5.335E-03$
	0.7	0.4167187	0.4243124	$7.593E-03$
	0.9	0.1785937	0.1846205	$6.026E-03$
0.50	0.1	0.16875	0.1692397	$4.897E-04$
	0.3	0.39375	0.3958039	$2.053E-03$
	0.5	0.46875	0.4728440	$4.094E-03$
	0.7	0.39375	0.3998311	$6.081E-03$
	0.9	0.16875	0.1737862	$5.036E-03$
0.75	0.1	0.1420312	0.1422721	$2.408E-04$
	0.3	0.3314062	0.3326319	$1.225E-03$
	0.5	0.3945312	0.3972166	$2.685E-03$
	0.7	0.3314062	0.3357163	$4.310E-03$
	0.9	0.1420312	0.1458354	$3.804E-03$
1.00	0.1	0.0900000	0.0900165	$1654E-05$
	0.3	0.2100000	0.2103843	$3843E-04$
	0.5	0.2500000	0.2511554	$1155E-03$
	0.7	0.2100000	0.2122592	$2259E-03$
	0.9	0.0900000	0.0922510	$2251E-03$

Table 10: Maximum errors (Problem 2)

		$\widetilde{y}_1(x,\alpha)$		$\widetilde{y}_2(x,\alpha)$	
α	n	$\underline{y}_1(x,\alpha)$	$\overline{y}_1(x,\alpha)$	$\underline{y}_2(x,\alpha)$	$\overline{y}_2(x,\alpha)$
0.25	2	$7.879E-04$	$5.445E-03$	$8.800E-04$	$7.809E-03$
0.50	2	$7.074E-04$	$3.949E-03$	$1.084E-03$	$6.395E-03$
0.75	2	$8.027E-04$	$2.675E-03$	$1.595E-03$	$4.694E-03$
1.00	2	$1.657E-03$	$1.657E-03$	$2.639E-03$	$2.639E-03$

In Figures 9–12, a very good agreement is shown between the exact solution $\big(\widetilde{y}_{i,E}(x,0.5), i=1,2\big)$ with a continuous line and the approximate solution by the HPM with Green's function $(\widetilde{y}_{i,2}(x,0.5), i=1,2)$ with the symbol o.

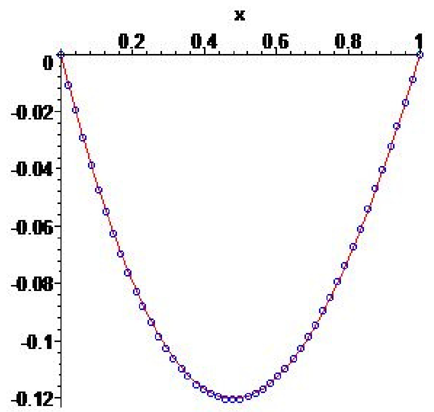

Figure 9: Plot of $\underline{y}_{1,E}(x, 0.5)$ and $\underline{y}_{1,2}(x, 0.5)$.

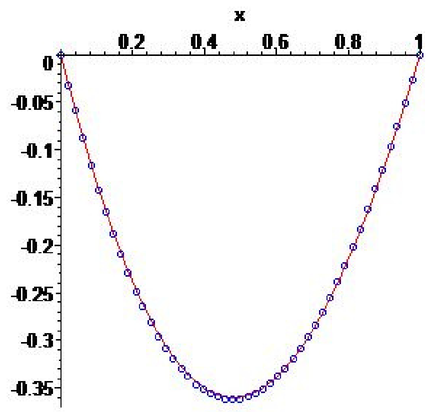

Figure 10: Plot of Plot of $\overline{y}_{1,E}(x, 0.5)$ and $\overline{y}_{1,2}(x, 0.5)$.

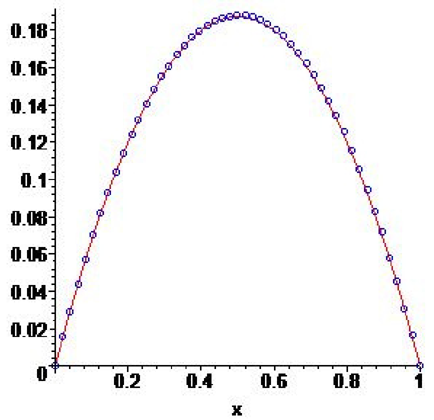

Figure 11: Plot of $\underline{y}_{2,E}(x, 0.5)$ and $\underline{y}_{2,2}(x, 0.5)$.

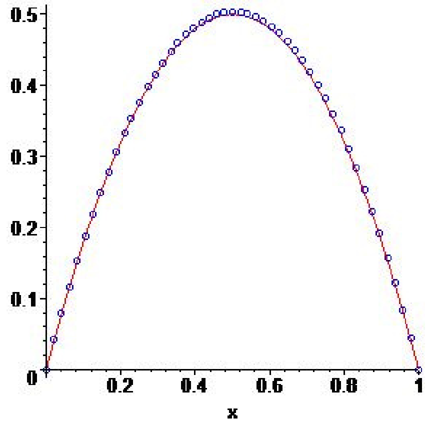

Figure 12: Plot of $\overline{y}_{2,E}(x, 0.5)$ and $\overline{y}_{2,2}(x, 0.5)$.

6. Conclusions

The HPM with Green's function has been used to solve linear and nonlinear fuzzy system of BVPs in this paper. The HPM we proved to be highly effective in addressing problems we describe an efficient method for solving fuzzy system of boundary value problem. We employed the HPM technique using Green functions to simplify the computation of the problem into linear and nonlinear fuzzy system of boundary value problem. This method is computationally efficient, as demonstrated by illustrative in two problems. Moreover, we tested our approach through numerical simulations, and the results show that our method effectively solve the problems at hand. Overall, our findings suggest that the HPM approach with Green functions can be a valuable tool for addressing similar problems in the future. The results obtained when $(\alpha = 1)$ are compatible with those methods in the literature such as the Variational iteration method, Sinc-collocation method, and Boundary value methods.

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