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Hesitant Fuzzy Compactness and Hesitant Fuzzy Regularity in Hesitant Fuzzy Topological Spaces

A. Swaminathan[1], Cenap Ozel[2], Ibtesam Alshammari[3,*]

[1] *Department of Mathematics, Government Arts College(A), Kumbakonam, Tamil Nadu-612 002, India*
[2] *Department of Mathematics, King Abdulaziz University, Jeddah-21589, Saudi Arabia*
[3] *Department of Mathematics, University of Hafr Al-Batin, Hafr Al-Batin, Saudi Arabia*

Abstract. We define hesitant fuzzy maximal open cover to establish hesitant fuzzy m-compactness and discuss its properties. Further we obtain few more results on hesitant fuzzy minimal c-regular and minimal c-normal spaces. We have proved that a hesitant fuzzy Haussdorff m-compact space is hesitant fuzzy minimal c-normal.

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1. Introduction

Origination of fuzzy sets by Zadeh[11] emerged many branches of mathematics for many decades. Chang[1] introduced fuzzy topology in 1968. As an addendum to fuzzy sets, the notion hesitant fuzzy set introduced by Torra[4] in 2010. Deepak et. al. [2] introduced hesitant fuzzy topological space and extended the study to hesitant connectedness and compactness in hesitant fuzzy topological space. The notions of hesitant fuzzy minimal, maximal open[9] and hesitant fuzzy minimal, maximal clopen[7] sets introduced by Swaminathan and Sivaraja. Also the idea of hesitant fuzzy mean open and closed sets[8] investigated by Swaminathan and Sivaraja.

In section 2, we define a new notion hesitant fuzzy maximal open cover in hesitant fuzzy topological space. Section 3 of this paper, the concept of hesitant fuzzy m-compact space and some properties are discussed. In section 4, the notion of hesitant fuzzy minimal c-regular (resp.c-normal spaces) are extended from which it is showed that a hesitant fuzzy Haussdorff m-compact space is hesitant fuzzy minimal c-normal.

*Corresponding author.
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Email addresses: asnathanway@gmail.com (A. Swaminathan),
cenap.ozel@gmail.com (C. Ozel),
iealshamri@uhb.edu.sa, iealshamri@hotmail.com (I. Alshammari)

The following terminologies, "hesitant fuzzy minimal open set, hesitant fuzzy maximal open set, hesitant fuzzy mean open set,hesitant fuzzy clopen set, hesitant fuzzy cut-point space, hesitant fuzzy connected topological space,hesitant fuzzy disconnected topological space and hesitant fuzzy topological space"are respectively abbreviated as "HFMIO, HFMAO, HFMEO, HFCLO, HFCS, HFCTS,HFDTS and HFTS."

2. Preliminaries

Definition 2.1. [4] A HFS h in X is a function $h : X \to P[0,1]$, where $P[0,1]$ represents the power set of $[0,1]$.

We define the hesitant fuzzy empty set h^0(resp. whole set h^1) is a HFS in X as follows: $h^0(x) = \phi$ (resp. $h^1(x) = [0,1]), \forall x \in X$. $HS(X)$ stands for collection of HFS in X.

Definition 2.2. [5] Let X be a nonempty set. A HFT τ of subsets X is said to be HFT on X if
(i) $h^0, h^1 \in \tau$.
(ii) $\bigcup_{i\in J} h_i \in \tau$ for each $(h_i)_{i\in J} \subset \tau$.
(iii) $h_1 \cap h_2 \in \tau$ for any $h_1,\ h_2 \in \tau$.

"The pair (X,τ) is called HFTS. The members of τ are called HFO sets in X. A HFS h in X is HFC set (in short HFC) in (X,τ) if $h^c \in \tau$."

Definition 2.3. [3] Two HFS h_1 and h_2 of X are said to be equal if $h_1 \subset h_2$ and $h_2 \subset h_1$.

Definition 2.4. [4] Let $h \in HS(X)$ for any nonempty set X. Then h^c is the complement of h which is HFS in X such that $h^c(x) = [h(x)]^c = [0,1]\backslash h(x)$.

Definition 2.5. [5] Suppose that (X,τ) is a HFTS such that $x_\lambda \in H_p(X)$ and $N \in HS(\eta)$. Then the hesitant fuzzy neighbourhood N of x_λ is defined as if for an hesitant fuzzy set $U \in \tau$ such that $x_\lambda \in U \subset N$.

Definition 2.6. *[9]A proper nonzero HFO set ξ of X is said to be*
(i)HFMIO set if ξ and h^0 are only HFO sets contained in ξ.
(ii)HFMAO set if h^1 and ξ are only HFO sets containing ξ.

Definition 2.7. *[9]A proper nonzero HFC set η of X is said to be*
(i) HFMAC set if any HFC set which contains η is h^1 or η.
(ii)HFMIC set if any HFC set which is contained in η is h^0 or η.

Definition 2.8. *[7] A proper HFCLO set φ of X is called a HFMICLO set if ϱ is a HFCLO set such that $\varrho < \varphi$, then $\varrho = \varphi$ or $\varrho = h^0$.*

Definition 2.9. *[7] A proper HFCLO set φ of X is called a HFMACLO set if ϱ is a HFCLO set such that $\varphi < \varrho$, then $\varphi = \varrho$ and $\varrho = h^1$.*

Definition 2.10. *[8] In a fts X, ξ is called a HFMEO(resp.γ FMEC) if $\exists\ \lambda, \mu(\neq \xi)$ two distinct proper HFO sets (resp. two distinct proper hesitant fuzzy closed sets $\zeta, \varphi(\neq \gamma)$) such that $\lambda < \xi < \mu$(resp. $\zeta < \gamma < \varphi$)*

Lemma 2.1.. *[6] Each nonzero HFO set γ of a T_1-fcts X is infinite and is not a HFMIO in X.*

Theorem 2.2.. *[6] A proper HFO set γ of a T_1-fcts X is a HFMEO set in X iff $\gamma \neq h^1 - \{x_\alpha\}$ for any $x_\alpha \in X$.*

2.1. Hesitant Fuzzy Maximal Open Cover and Hesitant Fuzzy m-compact Spaces

We now introduce hesitant fuzzy maximal open covers. Further the idea of hesitant fuzzy m-compact space is studied by means of hesitant fuzzy maximal open covers.

A hesitant fuzzy cover $\mathscr{C}$ of X is an hesitant fuzzy refinement of the hesitant fuzzy cover $\mathscr{D}$ of X if $\forall\ \xi \in \mathscr{C}$, $\exists \zeta \in \mathscr{D}$ such that $\xi < \zeta$.

Definition 2.11. Let $\mathscr{C}$ and $\mathscr{D}$ be two hesitant fuzzy covers of a HFTS $X.\mathscr{C}$ is an hesitant fuzzy s-refinement of $\mathscr{D}$ if for each $\xi \in \mathscr{C}\ \exists\ \zeta \in \mathscr{D}$ such that $\xi < \zeta$. A hesitant fuzzy s-refinement $\mathscr{C}$ of $\mathscr{D}$ is said to be a HFO s-refinement of $\mathscr{D}$ if all members of $\mathscr{C}$ and $\mathscr{D}$ are HFO.

It is clear that if $\mathscr{D} = \{h^1\}$ and $\xi \neq h^1$ for each $\xi \in \mathscr{C}$, then $\mathscr{C}$ is an hesitant fuzzy s-refinement of $\mathscr{D}$. If $\mathscr{C}$ is hesitant fuzzy s-refinement of $\mathscr{D}$ then $\mathscr{C}$ is an hesitant fuzzy refinement of $\mathscr{D}$. Further we see that no element of an s-hesitant fuzzy refinement of any hesitant fuzzy cover of X is HFMAO.

Definition 2.12. A HFO cover $\mathscr{C}$ of a HFTS X is called a HFMAO cover of X if $\mathscr{C}$ is not an hesitant fuzzy s-refinement of any other HFO cover of X.

Lemma 2.3.. *A HFO cover containing a HFMAO set is hesitant fuzzy maximal.*

Proof. Obvious.

Theorem 2.4. (Existence of HFMAO covers)**.** *There exists a HFMAO cover in an infinite T_1-HFTS.*

Proof. Let X be an infinite T_1-HFTS. Then for each $x_\alpha \in X$, $h^1 - \{x_\alpha\}$ is HFMAO set in X. Let $x_\beta \in X$. Consider a finite hesitant fuzzy subset $M = \{x_{\alpha_i} | x_{\alpha_i} \neq \varphi, i \in Z; 1 \leq i \leq n\}$. Also ξ in X is hesitant fuzzy closed as X is T_1-HFTS. Henceforth $\{h^1 - \{x_\beta\}, h^1 - G\}$ is HFO cover of X having HFMAO set $h^1 - \{x_\beta\}$. Hence by Lemma 2.3.,$\{h^1 - \{x_\beta\}, h^1 - M\}$ is HFMAO cover of X.

Theorem 2.5.. *Any HFO cover $\mathscr{M}$ of an infinite T_1-HFTS is a HFMAO cover of X iff $\mathscr{M}$ contains a HFMAO set.*

Proof. Let $\mathscr{M} = \{U_k | k \in V\}$ be a HFMAO cover of X such that no U_k,$k \in V$ is HFMAO. By Theorem 2.2., U_k is not also HFMIO for each $k \in V$ which implies that U_k,$k \in V$ is HFMEO. So $\forall\ k \in V$, $\exists V_k$ a proper HFO set $V_k$such that $U_k \underset{/}{<} = V_k$. Let

$\mathscr{N} = \{V_k | U_k \lneq V_k, U_k \in \mathscr{M}\}$. Clearly $\mathscr{N}$ is hesitant fuzzy cover of X. Therefore $\mathscr{M}$ is an hesitant fuzzy s-refinement of $\mathscr{N}$ a contradiction to tha fact that $\mathscr{M}$ is a HFMAO cover of X. Hence $\mathscr{M}$ has a HFMAO set as one among its members. The converse part follows by Lemma 2.3..

Definition 2.13. A HFTS X is said to be a hesitant fuzzy m-compact if each HFMO cover of X has a finite HFO s-refinement.

Theorem 2.6.. *Every infinite T_1-HFCTS is hesitant fuzzy m-compact.*

Proof. Let $\mathscr{M}$ be HFMAO cover of an infinite T_1-HFCTS X. By Theorem 2.5., $\mathscr{M}$ contains a HFMAO set U. By Theorem 2.5., take $U = h^1 - \{x_\alpha\}$ for some $x_\alpha \in X$. There is an $V \in \mathscr{M}$ such that $x_\alpha \in V$. By Lemma 2.1., for hesitant fuzzy points $x_\alpha, x_\beta \in V$ with $x_\alpha \neq x_\beta$ there are HFO sets $V_1 = h^1 - \{x_\alpha, x_\beta\}$,$V_2 = V - \{x_\alpha\}$,$V_3 = V - \{x_\beta\}$ of X. Then $\{V_1, V_2, V_3\}$ is an hesitant fuzzy s-refinement of $\mathscr{M}$.

Example 2.7.. *Let $\tau = \{h^0, h^1, h_1, h_2, h_3, h_4\}$ and (X, τ) be a hesitant fuzzy topological space where* $h_1 = \begin{cases} [0,1] & if \quad x \neq \frac{1}{4} \\ 0 & if \quad x = \frac{1}{4} \end{cases}$;$h_2 = \begin{cases} 0 & if \quad x \neq \frac{1}{4} \\ [0,1] & if \quad x = \frac{1}{4} \end{cases}$; $h_3 = \begin{cases} [0,\frac{1}{4}] & if \quad x \neq \frac{1}{4} \\ [0,1] & if \quad x = \frac{1}{4} \end{cases}$;$h_4 = \begin{cases} [0,\frac{1}{4}] & if \quad x \neq \frac{1}{4} \\ 0 & if \quad x = \frac{1}{4} \end{cases}$.
Clearly (X, τ) is hesitant fuzzy compact but not hesitant fuzzy m-compact.

Remark 2.8.. *By Theorem 3.4, the real number space with the usual hesitant fuzzy topology is hesitant fuzzy m-compact but generally it is not hesitant fuzzy compact. Since by Theorem 2.6. along with Example 2.7., we conclude that both hesitant fuzzy compactness and hesitant fuzzy m-compactness are independent.*

Definition 2.14. A function $f : X \to Y$ is said to be hesitant fuzzy m-continuous if inverse image of each proper HFO set in Y is HFMAO in X.

Theorem 2.9.. *Let X be a hesitant fuzzy m-compact topological space and $f : X \to Y$ be a bijective hesitant fuzzy m-continuous function. Then Y is hesitant fuzzy m-compact.*

Proof. Let $S(Y)$ be a hesitant fuzzy cover of Y. Then $\mathscr{S}^{(X)} = \{f^{-1}(U_k) | U_k \in \mathscr{S}^{(Y)}\}$ is a HFMAO cover of X . By hesitant fuzzy m-compactness of $X, \mathscr{S}^{(X)}$ has a finite hesitant fuzzy s-refinement ${\mathscr{S}_1}^{(X)} = \{f^{-1}(U_k) | U_k \in \mathscr{S}^{(Y)}, k \in Z^+\}$ which gives ${\mathscr{S}_1}^{(Y)} = \{f(f^{-1}(U_k)) | U_k \in \mathscr{S}^{(Y)}, k \in Z^+\} = \{U_k | U_k \in \mathscr{S}^{(Y)}, k \in Z^+\}$. For each $k \in Z^+$, there exists $U \in \mathscr{S}^{(Y)}$ such that $f^{-1}(U_k) \lneq f^{-1}(U)$ gives $U_k \lneq U$. Hence ${\mathscr{S}_1}^{(Y)}$ is a hesitant fuzzy finite s-refinement of $\mathscr{S}^{(Y)}$.

Definition 2.15. A hesitant fuzzy point x_α of a HFTS X is hesitant fuzzy m-complete accumulation point of any hesitant fuzzy subset M of X if $|U \wedge M| = |M|$ for each HFMAO set U containing x_α.

Theorem 2.10.. *Each infinite hesitant fuzzy subset of a hesitant fuzzy m-compact space has an hesitant fuzzy m-complete accumulation point.*

Proof. Let G be an infinite hesitant fuzzy subset of a hesitant fuzzy m-compact HFTS X. Assume for each $x_\alpha \in X$, there is a HFMAO set W_{x_α} containing x_α and satisfying $|W_{x_\alpha} \wedge \varrho| < |\varrho|$. Since $\{W_{x_\alpha}|x_\alpha \in X\}$ is an HFO cover of X consists of HFMAO sets, by Lemma 2.3., $\{W_{x_\alpha}|x_\alpha \in X\}$ is a HFMAO cover of X. Therefore a finite hesitant fuzzy s-refinement $\{W_{x_\alpha}|x_{\alpha_i} \in X, i \in Z^+\}$ of $\{W_{x_\alpha}|x_\alpha \in X\}$. Now $|\varrho| = |\bigvee_{i=1}^{n}(W_{x_\alpha} \wedge \varrho)| < |\varrho|$, a contradiction.

2.2. Hesitant Fuzzy Minimal c-regular and Hesitant Fuzzy c-normal Spaces

Definition 2.16. A HFTS X is called a hesitant fuzzy minimal c-regular if for each $x_\alpha \in X$ and each HFMIC set γ with $x_\alpha \notin \gamma$, there exists disjoint HFO sets λ,μ such that $x_\alpha \in \lambda$ and $\lambda < \mu$.

Theorem 2.11.. *Let X be a HFTS. Then the following are equivalent:*
(i) X is hesitant fuzzy minimal c-regular.
(ii) Given a hesitant fuzzy point $x_\alpha \in X$ and a HFMAO set ω containing x_α , there is an HFO set ϱ such that $x_\alpha \in \varrho < Cl(\varrho) < \omega$.
(iii) For a hesitant fuzzy point $x_\alpha \in X$ and a HFMIC set γ with $x_\alpha \notin \gamma$, there exists HFO set ω containing x_α such that $Cl(\omega) \wedge \gamma = h^0$.

Proof. $(i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Rightarrow (i)$: Proof follows.

Definition 2.17. A HFTS X is called a hesitant fuzzy minimal c-normal if for each pair of distinct HFMIC sets η, γ there exists disjoint HFO sets λ,μ such that $\eta < \lambda$ and $\gamma < \mu$.

Theorem 2.12.. *Let X be a HFTS. Then the following are equivalent:*
(i) X is hesitant fuzzy minimal c-normal.
(ii) For each HFMIC set ξ and each HFMAO set ω with $\xi < \omega$,there is a HFO set ϱ such that $\xi < \varrho < Cl(\varrho) < \omega$.
(iii) For each pair of distinct HFMIC sets ξ,ζ ,there exists disjoint HFO sets ω, ϱ disjoint HFO sets such that $\xi < \omega$, $Cl(\omega) \wedge \zeta = h^0$ and $\zeta < \varrho$, $Cl(\varrho) \wedge \xi = h^0$.
(iv) For each pair of distinct HFMIC sets ξ,ζ,there exists a pair of disjoint HFO sets ω, ϱ such that $\xi < \omega$, $\zeta < \varrho$ and $Cl(\omega) \wedge Cl(\varrho) = h^0$.

Proof. $(i) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (iii)$:Suppose that $\xi < h^1 - \zeta$ for any HFMAO set $h^1 - \zeta$.By (ii) there exists HFO set ω such that $\xi < \omega < Cl(\omega) < h^1 - \zeta$.Clearly $Cl(\omega) \wedge \zeta = h^0$ as $Cl(\omega) < h^1 - \zeta$. By assuming $\varrho = h^1 - Cl(\omega)$,we get $\zeta < \varrho < h^1 - \omega < h^1 - \xi$. Since $h^1 - \omega$ is HFC set $\zeta < Cl(\varrho) < h^1 - \omega < h^1 - \xi$.Clearly,$Cl(\varrho) \wedge \xi = h^0$ as $Cl(\varrho) < h^1 - \xi$. It is evident that $\omega \vee \varrho = h^0$.

$(iii) \Rightarrow (iv)$: By (iii) for any distinct HFO sets ω,ϱ such that $\xi < \omega$,$Cl(\omega) \wedge \zeta = h^0$ and $\zeta < \varrho$,$Cl(\varrho) \wedge \xi = h^0$. As $Cl(\omega) \wedge \zeta = h^0$, $Cl(\varrho) \wedge \xi = h^0$ imply that $Cl(\omega) \wedge Cl(\varrho) = h^0$.

$(iv) \Rightarrow (i)$: Proof is easy and hence omitted.

Theorem 2.13.. *Every hesitant fuzzy Hausdorff m-compact space is hesitant fuzzy minimal hesitant fuzzy c-regular.*

Proof. Let X be a hesitant fuzzy Hausdorff m-compact. Suppose $\gamma \in X$ is HFMIC set and $x_\alpha \in X$ such that $x_\alpha \notin \lambda$. Since X is hesitant fuzzy Hausdorff, for each $x_\beta \in G$, we have G_{x_β}, H_{x_β} disjoint HFO sets such that $x_\alpha \in G_{x_\beta}$,$x_\beta \in H_{x_\beta}$. Let $\mathscr{G} = \{H_{x_\beta} | x_\beta \in \lambda\} \vee \{h^1 - \lambda\}$. Then $\mathscr{G}$ is HFMAO cover of X by Lemma 2.3.. By hesitant fuzzy m-compactness of X, then we have a finite hesitant fuzzy s-refinement $\mathscr{H}$ of $\mathscr{G}$. Let $M = \vee\{\Lambda \in \mathscr{H} | \Lambda \wedge \lambda \neq h^0\}$. So M is an HFO set which contains λ. Let $\Lambda_1, \Lambda_2 ... \Lambda_n$ be the only hesitant fuzzy members of $\mathscr{H}$ such that $\Lambda_k \wedge \lambda \neq h^0$,$k \in Z^+$. For each $k \in Z^+, \exists\ x_\gamma \in \lambda$ such that $\Lambda_k \not\leq H_{x_{\beta_\gamma}}$,$k \in Z^+$. We put $H = \bigwedge_{k=1}^{n} G_{x_{\beta_\gamma}}$. Then $x_\alpha \in \mu$. It is easy to show that $G \wedge H = h^0$.

Corollary 2.14.. *A hesitant fuzzy Hausdorff m-compact space is hesitant fuzzy minimal c-normal.*

Proof. Let ξ,ζ be distinct HFMIC sets in hesitant fuzzy Hausdorff m-compact space X. By Theorem 2.13., X is hesitant fuzzy minimal c-regular. Hence for each $x_\varphi \in \xi$, $\exists$ G, H HFO sets such that $x_\varphi \in G$,$\zeta < H$ and $G \wedge H = h^0$. The collection $\mathscr{G} = \{G_{x_\varphi} | x_\varphi \in \xi\} \vee \{h^1 - \xi\}$ is a HFMAO cover of X by Lemma 2.3.. Now proceeding like the proof of Theorem 4.3, we get two HFO sets η and μ such that $\xi < \lambda$, $\zeta < \mu$ and $G \wedge H = h^0$.

Lemma 2.15.. *If Y is a HFC(resp.HFO) subset of a HFTS X, then HFMIC (resp.HFMIO) sets in the subspace Y of X are HFMIC (resp.HFMIO) sets in X.*

Proof. Let ξ be a HFMIC set in Y, a HFC subset of a HFTS X. Evidently ξ is also HFC in X as $\xi = \eta \wedge Y$ for any HFC set η in X . If possible, suppose we have a HFC set β in X such that $\beta < \xi$. Clearly $\beta \wedge Y$ is HFC in Y such that $\beta \wedge Y < \beta < \xi$; either $\beta \wedge Y = \xi$ or $\beta \wedge Y = h^0$ as ξ is HFMIC in Y. $\beta \wedge Y = \xi$ implies that $\beta \wedge Y = \beta = \xi$. Now it is enough to prove that $\beta = h^0$ for $\beta \wedge Y = h^0$. We see that $\beta < \xi < Y$ as ξ is a hesitant fuzzy subset of Y. So we have $\beta \wedge Y = \beta \neq h^0$ if $\xi \neq h^0$. Hence $\beta = h^0$.
Similarly we can prove for HFO sets.

Definition 2.18. A hesitant fuzzy subspace Y of a HFTS X is said to be hesitant fuzzy minimally closed(resp. hesitant fuzzy minimally open) invariant if HFMIC(resp.HFMIO) sets of Y are also HFMIC (resp.HFMIO) sets of X.

Theorem 2.16.. *Hesitant fuzzy minimally closed invariant subspaces of hesitant fuzzy minimal c-normal spaces are hesitant fuzzy minimal c-normal.*

Proof. Let ξ,ζ be two distinct HFMIC sets in Y, where Y is hesitant fuzzy minimally invariant subspace of a hesitant fuzzy minimal c-normal space X. Hence ξ,ζ are HFMIC sets in X. As X is hesitant fuzzy minimal c-normal space, $\exists\ \eta,\mu$ distinct HFO sets in X such that $\xi < \eta, \zeta < \mu$ and $(Y \wedge \eta) \wedge (Y \wedge \mu) = h^0$.That is $Y \wedge \eta$; $Y \wedge \mu$ are distinct HFO sets in Y such that $\xi < (Y \wedge \eta)$ and $\zeta < (Y \wedge \mu)$.

Corollary 2.17.. *Each HFC subspace of a hesitant fuzzy minimal c-normal space is hesitant fuzzy minimal c-normal.*

Proof. Using Lemma 2.15., we have to proceed like that of Theorem 2.16..

3. Conclusion

In recent times the notion of hesitant fuzzy minimal and maximal open sets have been important concepts in the literature. There are some family between hesitant fuzzy maximal and hesitant fuzzy minimal sets which is called as hesitant fuzzy mean open sets. When we are dealing with hesitant fuzzy compactness, we may have various covers to h^1. In this paper, we have used particularly hesitant fuzzy maximal open cover for compactness. Further another new ideas namely hesitant fuzzy minimal c-regular and hesitant fuzzy minimal c-normal are extended with various properties. In future one can conclude and study numerous properties of connectedness and compactness in hesitant fuzzy topology. Therefore the hesitant fuzzy minimal, maximal and mean open sets play dominant role and in further study these notions can be investigated via various kinds of hesitant open sets.

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