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J^2-Independence Parameters of Some Graphs

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Abstract. Let G be a graph. A subset I' of a vertex-set $V(G)$ of G is called a J^2-independent in G if for every pair of distinct vertices $a, b \in I'$, $d_G(a,b) \neq 1$, $N_G^2[a] \backslash N_G^2[b] \neq \emptyset$ and $N_G^2[b] \backslash N_G^2[a] \neq \emptyset$. The maximum cardinality among all J^2-independent sets in G, denoted by $\alpha_{J^2}(G)$, is called the J^2-independence number of G. Any J^2-independent set I' satisfying $|I'| = \alpha_{J^2}(G)$ is called the maximum J^2-independent set of G or an α_{J^2}-set of G. In this paper, we establish some bounds of this parameter on a generalized graph, join and corona of two graphs. We characterize J^2-independent sets in some families of graphs, and we use these results to derive the exact values of parameters of these graphs. Moreover, we investigate the connections of this new parameter with other variants of independence parameters. In fact, we show that the J^2-independence number of a graph is always less than or equal to the standard independence number.

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Key Words and Phrases: independent set, J^2-independent set, J^2-independence number

1. Introduction

The independent set in graph has been studied excessively and one of the topics in Graph Theory which has been growing rapidly. Moreover, the problem of finding the maximum independent set in graphs is a fundamental problem not just in Graph Theory but also in Theoretical Computer Science. A subset V' of the vertex-set $V(G)$ of a graph G is said to be an independent if no two vertices in V' are adjacent. An independent set is

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called maximum if it is of largest cardinality, that is, if $V' \cup \{v\}$ is not an independent set for any $v \in V(G) \setminus V'$, and it is denoted by $i(G)$ to be the number of maximal independent sets of G.

In 1992, Jiuqiang Liu[8] developed new properties for the number of maximal independent sets $i(G)$ and the number of maximum independent sets $i_m(G)$, as well as determine the largest number of maximal and maximum independent sets possible in a k-connected graph of order n(with n large) and characterize the respective extremal graphs. In [1], established an upper bound as a tool to prove that the disjoint union of complete bipartite graphs $K_{d,d}$ maximises the number of independent sets of a d-regular graph. Some variants of representing the independent sets in graphs were studied by some researchers (see[1–4, 6, 7, 9–11]).

In 2022, J. Hassan et al. [6] introduced the hop independent sets in graphs. A subset S of $V(G)$ is called a *hop independent* if for every pair of distinct vertices $x, y \in S$, $d_G(x, y) \neq 2$. The maximum cardinality of a hop independent set in G, denoted by $\alpha_h(G)$, is called the *hop independence* number of G. Any hop independent set S with cardinality equal to $\alpha_h(G)$ is called an α_h-set of G. They have shown that every maximum hop independent set in a graph is a hop dominating set, that is, the hop independence number of a graph G is always greater than or equal to the hop domination number of a graph. They have characterized this type of set in graphs under some binary operations such join, corona, lexicographic product and Cartesian product of two graphs. These characterizations had been used to derive some formulas of a hop independence numbers of these graphs.

Recently, J. Hassan et al. [5] introduced and investigated new concept called J^2-hop domination. A subset $T = \{v_1, v_2, \cdots, v_m\}$ of vertices of a graph G is called a J^2-set if $N_G^2[v_i] \backslash N_G^2[v_j] \neq \varnothing$ for every $i \neq j$, where $i, j \in \{1, 2, \ldots, m\}$. A J^2-set T is called a J^2-hop dominating in G if for every $a \in V(G) \setminus T$, there exists $b \in T$ such that $d_G(a, b) = 2$. The J^2-hop domination number of G, denoted by $\gamma_{J^2h}(G)$, is the maximum cardinality among all J^2-hop dominating sets in G. They have shown that every maximum hop independent set is a J^2-hop dominating, hence, this parameter is always greater or equal compare to the hop independence parameter on any graph. Moreover, they derived some lower and upper bounds of the parameter for a generalized graph, join and corona of two graphs, respectively.

In this paper, we initiate the study of new variant of independence called J^2-independence. A certain subset S of a vertex-set $V(G)$ of G is called a J^2-independent if S is both a J^2-set and an independent set of a graph G. We investigate its properties and its relationships with other variants of independence. Further, we characterize J^2-independent sets in some classes of graphs and we use these results to determine the J^2-independence numbers of these graphs. Furthermore, we present some lower bounds of the parameter on the join and corona of two graphs.

We believe that the results of this study would give additional insights to researchers in the field and would help them for more research directions in the future.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. Two vertices x, y of G are *adjacent*, or *neighbors*, if xy is an edge of G. The *open neighborhood* of x in G is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A graph G is *connected* if every pair of its vertices can be joined by a path. Otherwise, G is *disconnected*. A maximal connected subgraph (not a subgraph of any connected subgraph) of G is called a *component* of G.

A *path graph* is a non-empty graph with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$, where the $x_i^{'}s$ are all distinct. The path of order n is denoted by P_n. If G is a graph and u and v are vertices of G, then a path from vertex u to vertex v is sometimes called a *u-v path*. The *cycle graph* $C_n = [x_1, x_2, \ldots, x_n, x_1]$ is the graph of order $n \geq 3$ with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$.

A graph is *complete* if every pair of distinct vertices are adjacent. A complete graph of order n is denoted by K_n.

The *complement* of a graph G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u, v \in V(G) \text{ and } uv \notin E(G)\}$.

Let G and H be any two graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The *corona* G and H, denoted by $G \circ H$, the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then Joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} + H^v \rangle$.

The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

A subset I of $V(G)$ is called an *independent* (resp. *hop independent*) if for every pair of distinct vertices $x, y \in I$, $d_G(x, y) \neq 1$ (resp. $d_G(x, y) \neq 2$). The maximum cardinality of an independent set (resp. hop independent set) in G, denoted by $\alpha(G)$(resp. $\alpha_h(G)$), is called the *independence* (resp. *hop independence*) number of G. Any independent (resp. hop independent) set I with cardinality equal to $\alpha(G)$ (resp. α_h) is called an α-set (resp. α_h-set) of G.

3. Results

We begin this section by introducing the concept of J^2-independence in a graph.

Definition 1. Let G be a simple graph. A subset I' of $V(G)$ is called a J^2-independent in G if for every pair of distinct vertices $a, b \in I'$, $d_G(a,b) \neq 1$, $N_G^2[a] \backslash N_G^2[b] \neq \emptyset$ and $N_G^2[b] \backslash N_G^2[a] \neq \emptyset$. The maximum cardinality among all J^2-independent sets in G, denoted by $\alpha_{J^2}(G)$, is called the J^2-independence number of G. Any J^2-independent set I' satisfying $|I'| = \alpha_{J^2}(G)$ is called the maximum J^2-independent set of G or an α_{J^2}-set of G.

Example 1. Consider the graph K in Figure 1. Let $I = \{a, d, g, h\}$. Clearly I is a maximum independent set of K. Notice that $N_K^2[a] = \{a, d, e\}$, $N_K^2[d] = \{a, d, c, f, g\}$, $N_K^2[g] = \{c, d, g, h\}$, and $N_K^2[h] = \{e, g, h\}$. Thus, $N_K^2[a] \backslash N_K^2[d] = \{e\}$, $N_K^2[a] \backslash N_K^2[g] = \{a, e\}$, $N_K^2[a] \backslash N_K^2[h] = \{a, d\}$, $N_K^2[d] \backslash N_K^2[a] = \{c, f, g\}$, $N_K^2[d] \backslash N_K^2[g] = \{a, f\}$, $N_K^2[d] \backslash N_K^2[h] = \{a, c, d, f\}$, $N_K^2[g] \backslash N_K^2[a] = \{c, g, h\}$, $N_K^2[g] \backslash N_K^2[d] = \{h\}$, $N_K^2[g] \backslash N_K^2[h] = \{c, d\}$, $N_K^2[h] \backslash N_K^2[a] = \{g, h\}$, $N_K^2[h] \backslash N_K^2[d] = \{e, h\}$, $N_K^2[h] \backslash N_K^2[g] = \{e\}$. Therefore, I is a maximum J^2-independent set of K, and so $\alpha_{J^2}(K) = 4$.

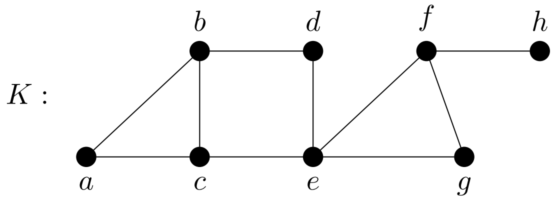

Figure 1: Graph K with $\alpha_{J^2}(K) = 4$

Remark 1. *Let G be a Graph. Then*

(i) any singleton set $\{x\}$, where $x \in V(G)$, is a J^2-indeppendent set of G; and

(ii) an independent set I may not be a J^2-independent in G.

Proposition 1. *Let G be a graph. Then*

(i) $\alpha_{J^2}(G) \leq \alpha(G)$; *and*

(ii) $1 \leq \alpha_{J^2}(G) \leq |V(G)|$.

Proof. (i) Let G be a graph and let I be a maximum J^2-independent set of G. Then I is an independent set in G. Since αG is the maximum cardinality of an independent set in G, it follows that $\alpha_{J^2}(G) = |I| \leq \alpha(G)$.

(ii) Since every singleton set $\{x\}$, where $x \in V(G)$, is a J^2-independent, we have $\alpha_{J^2}(G) \geq 1$. Morever, since any J^2-independent set I of G is always a subset of $V(G)$, it follows that $\alpha_{J^2}(G) \leq |V(G)|$. Therefore, $1 \leq \alpha_{J^2}(G) \leq |V(G)|$. □

Remark 2. *Let G be a graph. Then the difference $\alpha(G) - \alpha_{J^2}(G)$ can be arbitrarily large.*

To see this, let m be any positive integer and consider the graph G in Figure 2. Let $I = \{v_1, v_2, \ldots, v_{m+1}\}$ and $I' = \{u\}$. Then I is a maximum independent set of G. Hence, $\alpha(G) = m + 1$. Now, clearly I' is a J^2-independent set of G. Since $d_G(u, v_i) = 1$ for each $i \in \{1, 2, \ldots, m+1\}$, and $N_G^2[v_s] = N_G^2[v_t]\ \forall\ s \neq t$, where $s, t \in \{1, 2, \ldots, m+1\}$, it follows that I' is a maximum J^2-independent set of G. Consequently,

$$\alpha(G) - \alpha_{J^2}(G) = m + 1 - 1 = m.$$

Since m can be made arbitrarily large, the assertion follows.

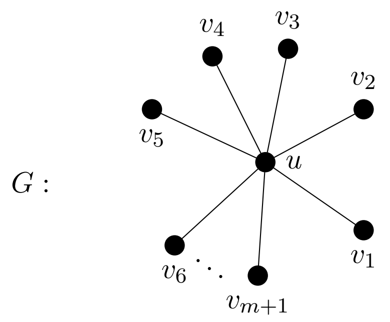

Figure 2: Graph G with $\alpha(G) - \alpha_{J^2}(G) = m$

Theorem 1. *Let G be a graph. Then $\alpha_{J^2}(G) = |V(G)|$ if and only if every component of G is trivial.*

Proof. Suppose that $\alpha_{J^2}(G) = |V(G)|$, say that $I = V(G)$ is the maximum J^2-independent set of G. Since I is an independent set of G, $d_a(a, b) \neq 1\ \forall\ a, b \in V(G)$. Suppose there is a component K of G which is non-trivial. Then there exist $x, y \in V(K) \subseteq V(G)$ such that $d_K(x, y) = d_G(x, y) = 1$, a contradiction. Hence, every component of G is trivial.

Conversely, suppose that every component K of G is trivial. Let $V(G) = \{a_1, a_2, \ldots, a_m\}$, $m \in \mathbb{N}$. Then $d_G(a_i, a_j) \neq 1$ and $a_i \in N_G^2[a_i] \backslash N_G^2[a_j]\ \forall\ i \neq j$, where $i, j \in \{1, 2, \ldots, m\}$. Thus, $\mathbb{N}_G^2[a_i] \backslash N_G^2[a_j] \neq \emptyset\ \forall\ i \neq j$, $i, j \in \{1, 2, \ldots, m\}$. Therefore, $V(G)$ is a J^2- independent set of G, and so $\alpha_{J^2}(G) = |V(G)|$. □

Theorem 2. *Let G be a graph. If G is complete, then $\alpha_{J^2}(G) = 1$. However, the converse is not true.*

Proof. Let G be a complete graph. Then $\alpha(G) = 1$. Hence $\alpha_{J^2}(G) = 1$ by Proposition 1. To see that the converse is not true, consider P_3 which is not complete grph. Let $V(P_3) = \{u_1, u_2, u_3\}$. Observe that $N^2_{P_3}[u_1] = N^2_{P_3}[u_3]$. Thus, u_1 and u_3 cannot be both in any J^2-independent set I of G. Since $d_{P_3}(u_1, u_2) = 1 = d_{P_3}(u_2, u_3)$, either $\{u_1\}, \{u_2\}$ or $\{u_3\}$ is a maximum J^2-independent set of P_3. Therefore, in either case, $\alpha_{J^2}(P_3) = 1$, and so the assertion follows. □

Theorem 3. *Let G be a graph. Then $\alpha_{J^2}(G) = \alpha(G)$ if and only if G has an α-set Q such that Q forms a J^2-set in G.*

Proof. Suppose that $\alpha_{J^2}(G) = \alpha(G) = k$, say $Q = \{w_1, w_2, \ldots, w_k\}$ is a maximum J^2-independent set of G. Then Q is an independent set of G. Since $\alpha_{J^2}(G) = \alpha(G)$, it follows that Q is an α-set of G. Since Q is a J^2-independent set of G, Q is a J^2-set of G.

Conversely, suppose G has an α-set Q of G. Then Q is a maximum independent set of G. Since Q forms a J^2-set in G, it follows that Q is a maximum J^2-independent set of G. Hence, $\alpha(G) = |Q| = \alpha_{J^2}(G)$. □

Theorem 4. *Let q be a positive integer. Then* $\alpha_{J^2}(C_q) = \begin{cases} 1, q = 3, 4 \\ 2, q = 5, 6 \\ \alpha(C_q), q \geq 7. \end{cases}$

Proof. Clearly, $\alpha_{J^2}(C_3) = 1$. For $q = 4$, let $V(C_4) = \{a_1, a_2, a_3, a_4\}$ and $L = \{a_1\}$. Then, L is a J^2-independent set of C_q. Since $d_{C_4}(a_1, a_2) = 1 = d_{C_4}(a_1, a_4)$ and $N^2_{C_4}[a_1] = N^2_{C_4}[a_3]$, it follows that $L = \{a_1\}$ is a maximum J^2-independent set of C_4. Thus, $\alpha_{J^2}(C_4) = 1$. For $q = 5$, let $V(C_5) = \{a_1, a_2, a_3, a_4, a_5\}$. Consider $N = \{a_1, a_3\}$. Then N is a maximum independent set of C_5. Note that $N^2_{C_5}[a_1] = \{a_1, a_3, a_4\}$ and $N^2_{C_3}[a_3] = \{a_1, a_3, a_5\}$. Thus, $N^2_{C_5}[a_1] \backslash N^2_{C_5}[a_3] = \{a_4\} \neq \varnothing$ and $N^2_{C_3}[a_3] \backslash N^2_{C_5}[a_1] = \{a_5\} \neq \varnothing$. Hence, N is a maximum J^2-independent set in C_5, and so $\alpha_{J^2}(C_5) = 2$. Similarly, $\alpha_{J^2}(C_6) = 2$. Suppose that $q \geq 7$. Let $V(C_q) = \{v_1, v_2, \ldots, v_q\}$, and consider the following two cases:

Case 1. q is odd

Let $Q = \{v_1, v_3, \ldots, v_{n-4}, v_{n-2}\}$. Then Q is a maximum independent set of C_q, and so $\alpha(C_q) = |Q|$. Observe that $v_{n-1} \in N^2_{C_q}[v_1] \backslash N^2_{C_q}[v_j]$ $\forall$ $j \neq 1$, $v_{r-2} \in N^2_{C_q}[v_r] \backslash N^2_{C_q}[v_q]$ $\forall$ $r < q$, where $r, q \in \{3, 5, \ldots, n-2\}$, $v_{s+2} \in N^2_{C_q}[v_s] \backslash N^2_{C_q}[v_t]$ $\forall$ $s < t$, where $s, t \in \{3, 5, \ldots, n-2\}$. Thus, $N^2_{C_q}[v_i] \backslash N^2_{C_q}[v_j] \neq \emptyset$ $\forall$ $i \neq j$, where $i, j \in \{1, 3, \ldots, n-4, n-2\}$, showing that Q is a J^2-set in C_q. Hence, Q is a maximum J^2-independent set of C_q, and so $\alpha_{J^2}(C_q) = |Q| = \alpha(C_q)$.

Case 2. q is even

Let $R = \{v_1, v_3, ..., v_{n-3}, v_{n-1}\}$. R is a maximum independent set of C_q, and so $\alpha(C_q) = |R|$. Notice that $v_{n-1} \in N^2_{C_q}[v_1] \backslash N^2_{C_q}[v_i]$ $\forall$ $i \neq n-3, n-1$, $v_1 \in N^2_{C_q}[v_1] \backslash N^2_{C_q}[v_{n-3}]$, $v_3 \in N^2_{C_q}[v_1] \backslash N^2_{C_q}[v_{n-1}]$, $v_{j-2} \in N^2_{C_q}[v_j] \backslash N^2_{C_q}[v_i]$. $\forall$ $i > j$, $i \neq n-1$ $v_{t+2} \in N^2_{C_q}[v_t] \backslash N^2_{C_q}[v_s]$ $\forall$ $s < t$, $s \neq 1$, $t \neq n-1$, $v_s \in N^2_{C_q}[v_s] \backslash N^2_{C_q}[v_{n-1}]$ $\forall$ $s \neq n-3$, $v_{n-5} \in N^2_{C_q}[v_{n-3}] \backslash N^2_{C_q}[v_{n-1}]$,

$v_{n-1} \in N^2_{C_q}[v_{n-1}]\backslash N^2_{C_q}[v_m]$ $\forall$ $m \neq 1, n-3$, $v_{n-3} \in N^2_{C_q}[v_{n-1}]\backslash N^2_{C_q}[v_1]$ and $v_1 \in N^2_{C_q}[v_{n-1}]\backslash N^2_{C_q}[v_{n-3}]$. Thus, $N^2_{C_q}[v_i]\backslash N^2_{C_q}[v_j]$ $\forall$ $i \neq j$, $i,j \in \{1,3,\ldots,n-3,n-1\}$ and so R is a J^2-set in C_q. Consequently, $\alpha_{J^2}(C_q) = |R| = \alpha(C_q)$ for all $n \geq 7$. □

Theorem 5. *Let m and n be positive integers. Then $\alpha_{J^2}(K_{m,n}) = 1$.*

Proof. Let $V(K_{m,n}) = \{u_1,u_2,\ldots,u_m,v_1,v_2,\ldots,v_n\}$, where $V(\overline{K}_m) = \{u_1,u_2,\ldots,u_m\}$ and $V(\overline{K}_n) = \{v_1,v_2,\ldots,v_n\}$. Consider $M = \{u_1\}$. Then M is a J^2-independent set of $K_{m,n}$. Observe that $N^2_{K_{m,n}}[u_i] = N^2_{K_{m,n}}[u_j]$ $\forall$ $i \neq j$, where $i,j \in \{1,2,\ldots,m\}$ and $N^2_{K_{m,n}}[v_s] = N^2_{K_{m,n}}[v_t]$ $\forall$ $i \neq j$, $i,j \in \{1,2,\ldots,n\}$. Since u_r and v_q are adjacent for all $r \in \{1,2,\ldots,m\}$ and $q \in \{1,2,\ldots,n\}$, it follows that M is a maximum J^2-independent set of $K_{m,n}$. Therefore, $\alpha_{J^2}(K_{m,n}) = 1$ $\forall$ $m,n \geq 1$. □

Theorem 6. *Let S and T be two connected graphs. A subset L of vertices of $S+T$ is a J^2-independent set of $S+T$ if one of the following holds;*

(i) L is a J^2-independent set in S

(ii) L is a J^2-independent set in T

Proof. Suppose that L is a J^2-independent set in S. Then L is an independent set in S. Let $a,b \in L$. Then $d_S(a,b) \neq 1$. If $d_S(a,b) = 2$, then $d_{S+T}(a,b) = 2 \neq 1$, and we are done. If $d_s(a,b) \geq 3$, then $d_{S+T}(a,b) = 2 \neq 1$. Therefore, L is an independent set of $S+T$. It suffices to show that L is a J^2-set in $S+T$. Let $x,y \in L$. Since L is a J^2-independent set is S, it follows $N^2_S[x]\backslash N^2_S[y] \neq \emptyset$ and $N^2_S[y]\backslash N^2_S[x] \neq \emptyset$. Assume that $d_s(x,y) = 2$. Since L is a J^2-independent set in S, there exist $w,z \in V(S)$ such that $w \in N^2_S[x]\backslash N^2_S[y]$ and $z \in N^2_S[y]\backslash N^2_S[x]$. Let $s \in N_S(w) \cap N_S(x)$ and $t \in N_S(z) \cap N_S(y)$. Then $s \in N^2_{S+T}[y]\backslash N^2_{S+T}[x]$ and $t \in N^2_{S+T}[x]\backslash N^2_{S+T}[y]$. Thus, L is a J^2- set in $S+T$.

Next, suppose that $d_S(x,y) \geq 3$. Let $u \in N_S(x)$ and $v \in N_S(y)$, then $u \in N^2_{S+T}[y]\backslash N^2_{S+T}[x]$ and $v \in N^2_{S+T}[x]\backslash N^2_{S+T}[y]$. Hence, L is a J^2-set in $S+T$, showing that L is a J^2-independent set in $S+T$. Similarly, if L is a J^2-independent set in T, then L is a J^2-independent set in $S+T$. □

Corollary 1. *Let S and T be two connected graphs. Then*

$$\alpha_{J^2}(S+T) \geq max\ \{\alpha_{J^2}(S), \alpha_{J^2}(T)\}.$$

Proof. Let L be a maximum J^2-independent set of S. Then by Theorem 6, L is a J^2-independent set of $S+T$. Since $\alpha_{J^2}(S+T)$ is the maximum cardinality among all J^2-independent sets of $S+T$, it follows that

$$\alpha_{J^2}(S+T) \geq |L| = \alpha_{J^2}(S).$$

Similarly, if L' is a maximum J^2-independent set of T, then

$$\alpha_{J^2}(S+T) \geq \left|L'\right| = \alpha_{J^2}(T).$$

Consequently,

$$\alpha_{J^2}(S+T) \geq \max\{\alpha_{J^2}(S), \alpha_{J^2}(T)\}.$$

□

Remark 3. *The Theorem 6 does not hold if either S or T is disconnected.*

Consider the graph $\overline{K}_3 + P_3$ in Figure 3, where $S = \overline{K}_3$ is disconnected and $T = P_3$. Let $S' = \{d, e\}$. Then $d_S(d,e) \neq 1$, $d \in N_S^2[d] \setminus N_S[e]$ and $e \in N_S^2[e] \setminus N_S[d]$. Thus, S' is a J^2-independent set in S. However, $N_{S+T}^2[d] = N_{S+T}^2[e] = \{d, e, f\}$. Hence, S' is not a J^2-set in $S+T$. Consequently, S' is not a J^2-independent set in $S+T$.

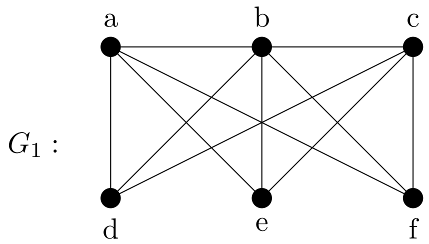

Figure 3: Graph $\overline{K}_3 + P_3$

Theorem 7. *Let S and T be connected graphs. If $W = \bigcup_{a \in V(S)} T_a$, where T_a is a J^2-independent set of T for each $a \in V(S)$, then W is a J^2-independent set of $S \circ T$. Moreover,*

$$\alpha_{J^2}(S \circ T) \geq \alpha_{J^2}(T) \cdot |V(S)|.$$

Proof. Let $W = \bigcup_{a \in V(S)} T_a$, where T_a is a J^2-independent set of T for each $a \in V(S)$.

Let $x, y \in w$. If $x, y \in T_c$ for some $c \in V(S)$, then $d_{S \circ T}(x,y) \neq 1$ because T_c is an independent set of T.

Claim: $N_{S \circ T}^2[x] \backslash N_{S \circ T}^2[y] \neq \emptyset$ and $N_{S \circ T}^2[y] \backslash N_{S \circ T}^2[x] \neq \emptyset$.

Since $N_T^2[x] \backslash N_T^2[y] \neq \emptyset$, there exists $w \in V(T)$ such that $d_T(x,w) = 2$ and $d_T(y,w) \neq 2$. If $d_T(x,y) = 2$,, then $d_T(y,w) \neq 1$. Thus, $d_T(y,w) \geq 3$. Let $t \in N_T(x) \cap N_T(w)$. Then $t \in N_{S \circ T}^2[y] \backslash N_{S \circ T}^2[x]$. Hence, $N_{S \circ T}^2[y] \backslash N_{S \circ T}^2[x] \neq \emptyset$.

Assume that $d_T(x,y) \geq 3$. Suppose that $d_T(y,w) = 1$. Let $v \in N_T(x) \cap N_T(w)$. Then $v \in N_{S \circ T}^2[y] \backslash N_{S \circ T}^2[x]$. Thus, $N_{S \circ T}^2[y] \backslash N_{S \circ T}^2[x] \neq \emptyset$. If $d_T(y,w) \geq 3$, then by preceding argument, $N_{S \circ T}^2[y] \backslash N_{S \circ T}^2[x] \neq \emptyset$. Similarly, if $N_T^2[y] \backslash N_T^2[x] \neq \emptyset$, then

$N^2_{S\circ T}[x]\backslash N^2_{S\circ T}[y] \neq \emptyset$. Therefore, W is a J^2-independent set of $S \circ T$. Consequently,

$$\alpha_{J^2}(S \circ T) \geq \alpha_{J^2}(T) \cdot |V(S)|.$$

□

Remark 4. *The Theorem 7 does not hold if T is disconnected.*

Consider the graph $S \circ T$ in Figure 4, where T is disconnected. Let $B = \{u_1, u_3\}$. Then $d_T(u_1, u_3) \neq 1$. Hence, B is an independent set of T. Observe that $N^2_T[u_1] = \{u_1\}$ and $N^2_T[u_3] = \{u_3\}$. Thus, $N^2_T[u_1]\backslash N^2_T[u_3] = \{u_1\} \neq \emptyset$ and $N^2_T[u_3]\backslash N^2_T[u_1] = \{u_3\} \neq \emptyset$. Therefore, B is a J^2-independent set of T. Now, notice that

$$N^2_{T+S}[u_1] = \{u_1, u_3, y\} \subseteq \{u_1, u_2, u_3, y\} = N^2_{T+S}[u_3].$$

It follows that B is not a J^2-set of $S + T$. Consequently, B is not a J^2-independent set of $S + T$.

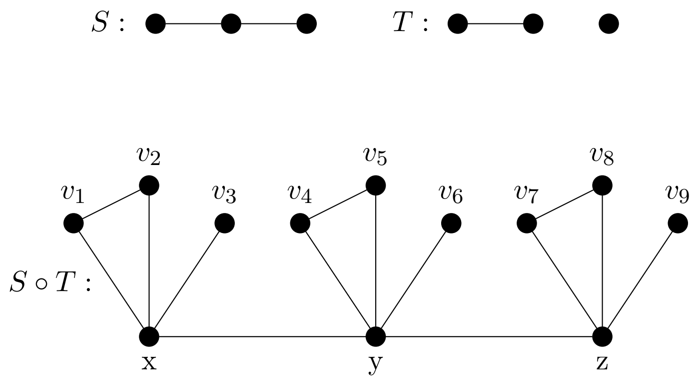

Figure 4: Graph $S \circ T$

Theorem 8. *Let G be a graph. Then the hop independence and J^2-independence parameters are incomparable.*

Proof. Consider the graph G in Figure 5. Let $Q = \{a, e\}$, Then Q is an independent set of G. Observe that $N^2_G[a] = \{a, h\}$ and $N^2_G[e] = \{d, e, g\}$. Thus, $N^2_G[a]\backslash N^2_G[e] = \{a, h\} \neq \emptyset$ and $N^2_G[e]\backslash N^2_G[a] = \{d, e, g\} \neq \emptyset$ and so Q is a J^2 independent set of G. Since, $d_G(a, b) = d_G(a, d) = d_G(a, c) = 1$, $N^2_G[a] \subseteq N^2_G[h]$, $d_G(e, f) = 1 = d_G(e, h)$ and $N^2_G[e] = N^2_G[g]$, it follows that Q is a maximum J^2-independent set of G. Hence, $\alpha_{J^2}(G) = 2$. Now, let $Q' = \{a, b, c, d\}$. Then Q' is a maximum hop independent set of G. Therefore, $\alpha_h(G) = 4$.

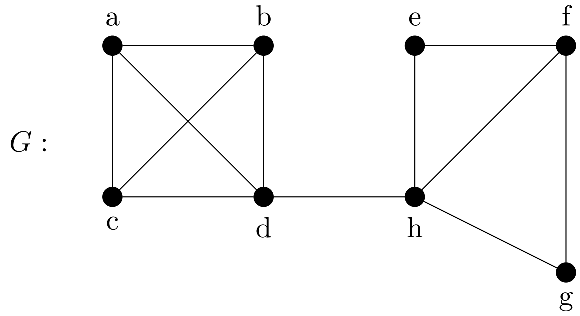

Figure 5: Graph G with $\alpha_h(G) = 4$ and $\alpha_J^2(G) = 2$

Next consider the graph $K_2 + P_{13}$ in Figure 6. Let $R = \{a, d, f, h, j, m\}$. Then, R is a maximum J^2-independent set of $K_2 + P_{13}$, and so $\alpha_J^2(K_2 + P_{13}) = 6$.

Now, let $R' = \{a, b, x, y\}$. Then, R' is a maximum hop independent set $K_2 + P_{13}$. Hence, $\alpha_h(K_2 + P_{13}) = 4$. □

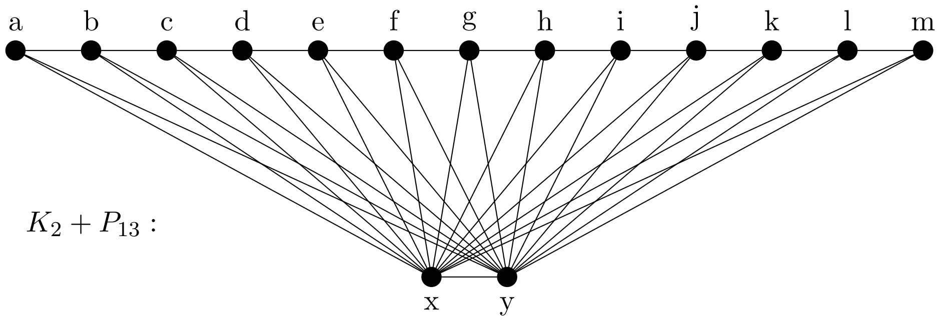

Figure 6: Graph $K_2 + P_{13}$ with $\alpha_h(K_2 + P_{13}) = 4$ and $\alpha_J^2(K_2 + P_{13}) = 6$

4. Conclusion

The concept of J^2-independence has been introduced and investigated in this study. Its bounds with respect to the order of a graph and other parameters have been determined. It was shown that any graph G admits a J^2-independence. Moreover, characterizations of J^2-independent sets in some classes of graphs have been presented and used to determine the exact values of the parameter. Some graphs that were not considered in this study could be an interesting topic to consider for further investigation of the concept.

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