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Vertex Cover Hop Dominating Sets in Graphs

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Abstract. Let G be a graph. Then a subset C of vertices of G is called a vertex cover hop dominating if C is both a vertex cover and a hop dominating of G. The vertex cover hop domination number of G, denoted by $\gamma_{vch}(G)$, is the minimum cardinality among all vertex cover hop dominating sets in G. In this paper, we initiate the study of vertex cover hop domination in a graph and we determine its relations with other parameters in graph theory. We characterize the vertex cover hop dominating sets in some special graphs, join, and corona of two graphs and we finally obtain the exact values or bounds of the parameters of these graphs.

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Key Words and Phrases: Vertex cover, vertex cover hop dominating set, vertex cover hop domination number

1. Introduction

One of the rapidly developing areas of research in graph theory is the study of domination. The idea was first introduced by Claude Berge [1] in 1958 and Ore [6] in 1962. Moreover, graph theory has also witnessed a surge of interest in recent research, with hop domination as an intriguing area of investigation which is one of the variations of domination in graphs. First introduced and investigated by Natarajan et al. [5], this concept can be used in modelling social networks. Over time, researchers have extensively studied hop domination and its numerous variants, as evidenced by a range of notable studies found in [2–4, 7–11].

In this study, we will introduce the concept of vertex cover hop domination in a graph. We will investigate and characterize these sets in some special graphs and graphs obtained from some binary operations. Moreover, some bounds or exact values of these graphs will be presented.

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2. Terminology and Notation

Let G be a graph. A subset B of $V(G)$ is an *independent* if for every pair of distinct vertices $v, w \in B$, $d_G(v,w) \neq 1$. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the independence number of G. Any independent set B with cardinality equal to $\alpha(G)$ is called an α-set of G.

A vertex a in G is a *hop neighbor* of a vertex b in G if $d_G(a,b) = 2$. The set $N^2_G(a) = \{b \in V(G) : d_G(a,b) = 2\}$ is called the *open hop neighborhood* of a. The *closed hop neighborhood* of a in G is given by $N^2_G[a] = N^2_G(a) \cup \{a\}$. The *open hop neighborhood* of $S \subseteq V(G)$ is the set $N^2_G(S) = \bigcup_{a \in S} N^2_G(a)$. The *closed hop neighborhood* of S in G is the set $N^2_G[S] = N^2_G(S) \cup S$.

A subset S of $V(G)$ is a *hop dominating* of G if for every $a \in V(G) \backslash S$, there exists $b \in S$ such that $d_G(a,b) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A subset U of vertices of a graph G is called a *vertex cover* of G if every edge in G is incident with a vertex in U. The minimum cardinality of such set is the *vertex covering number* of G and is denoted by $\beta(G)$.

A subset C of $V(G)$ is a *pointwise non-dominating set* if for every $v \in V(G) \setminus C$, there exists $u \in C$ such that $v \notin N_G(u)$. The minimum cardinality of a pointwise non-dominating set of G, denoted by $pnd(G)$, is called a *pointwise non-domination number* of G. Any pointwise non-dominating set of G with cardinality $pnd(G)$ is called a pnd-set of G.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle\{v\}\rangle + H^v$.

3. Results

We shall define the concept of vertex cover hop domination in a graph as follows:

Definition 1. Let G be a graph. A subset C of vertices $V(G)$ of G is said to be a *vertex cover hop dominating* if C is both a vertex cover and a hop dominating set in G. The *vertex cover hop domination number* of G, denoted by $\gamma_{vch}(G)$, is the minimum cardinality among all vertex cover hop dominating sets in G. Any vertex cover hop dominating set with cardinality equal to $\gamma_{vch}(G)$ is called a *γ_{vch}-set* of G.

Example 1. Consider the graph G in Figure 1. Let $C = \{e, f, g, h\}$. Notice that every edge of G is incident to atleast one vertex in C. Thus, C is a vertex cover set of G. Observe that $a, d \in N_G^2(f)$, $b \in N_G^2(g)$ and $c \in N_G^2(e)$. It follows that $N_G^2[C] = V(G)$, showing that C is a hop dominating set in G. Therefore, C is a vertex cover hop dominating set of G. Moreover, it can be verified that $\gamma_{vch}(G) = 4$.

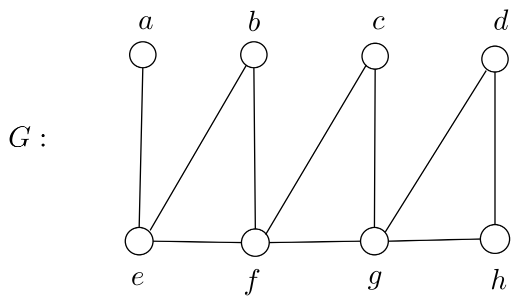

Figure 1: Graph G with $\gamma_{vch}(G) = 4$

Proposition 1. *Let G be any graph. Then each of the following is true.*

(*i*) *A vertex cover may not be a hop dominating.*

(*ii*) *A hop dominating may not be a vertex cover.*

Proof. (i) Consider the graph H given in Figure 2. Let $C' = \{b, d, e, f, g\}$. Observe that every edge of H is incident to at least one vertex in C', and so C' is a vertex cover set of H. However C' is not a hop dominating set of H since $i \notin N_G^2[x]\ \forall x \in C'$.

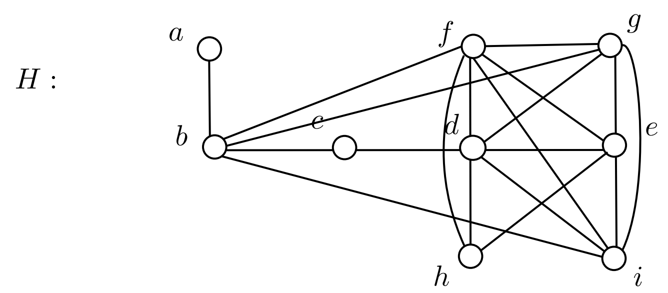

Figure 2: Graph H

(ii) Consider again the graph G in Figure 1. Let $M = \{c, f, g\}$. Notice that $a, d, h \in N_G^2(f)$ and $b, e \in N_G^2(g)$. Thus, $N_G^2[M] = V(G)$, showing that M is a hop dominating set of G. However, M is not a vertex cover of G since edges ae, be and dh are not incident to any vertex in M. □

Proposition 2. *Let G be any graph. Then each of the following holds:*

(i) $\gamma_h(G) \leq \gamma_{vch}(G)$, *and this bound is sharp.*

(ii) $\beta(G) \leq \gamma_{vch}(G)$, *and this bound is sharp.*

Proof. (i) Let C be a minimum vertex cover hop dominating set of G. Then C is a hop dominating in G (by definition). Since $\gamma_h(G)$ is the minimum cardinality among all hop dominating sets in G, it follows that $\gamma_h(G) \leq |C| = \gamma_{vch}(G)$. To see that the bound is sharp, consider $G = K_{1,n}$. Then $\gamma_h(G) = \gamma_{vch}(G) = 2$.

(ii) Let C be a minimum vertex cover hop dominating set of G. Then C is a vertex cover in G (by definition). Since $\beta(G)$ is the minimum cardinality of a vertex cover set in G, it follows that $\beta(G) \leq |C| = \gamma_{vch}(G)$. To see that sharpness is attainable, consider the graph G in Figure 3. Let $T = \{b, d, e\}$. Then T is a vertex cover set of G, and so $\beta(G) \leq 3$. Since $\{x, y\}$ is not a vertex cover of G for any pair of distinct vertices $x, y \in V(G)$, it follows that $\beta(G) \geq 3$. Thus, $\beta(G) = 3$. Now, observe that $N^2_G[b] = \{b, c, f, g\}$, $N^2_G[d] = \{a, d, f, g\}$ and $N^2_G[e] = \{a, c, e\}$. Thus, $N^2_G[T] = V(G)$, showing that T is a vertex cover hop dominating set of G. Since T is a minimum vertex cover of G, it follows that T is a minimum vertex cover hop dominating set of G, and so $\gamma_{vch}(G) = 3$. Consequently, $\gamma_{vch}(G) = 3 = \beta(G)$. □

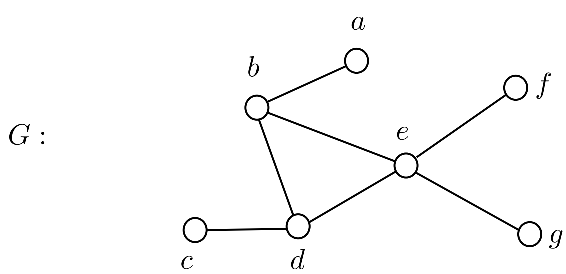

Figure 3: Graph G with $\beta(G) = \gamma_{vch}(G)$

Theorem 1. *Let G be any graph on $n \geq 1$ vertices. Then $1 \leq \gamma_{vch}(G) \leq n$. Moreover,*

(i) $\gamma_{vch}(G) = 1$ *if and only if G is trivial.*

(ii) $\gamma_{vch}(G) = 2$ *if and only if* $\gamma_h(G) = 2 = |T|$ *such that* $V(G) \setminus T$ *is an independent set of G.*

(iii) $\gamma_{vch}(G) = n$ *if and only if every component of* G *is complete.*

Proof. Clearly, $1 \leq \gamma_{vch}(G) \leq n$.

(i) Suppose $\gamma_{vch}(G) = 1$. If G is not trivial graph, then $\gamma_h(G) \geq 2$. By Proposition 2, $\gamma_{vch}(G) \geq 2$, a contradiction. Hence, G is trivial.

Conversely, suppose G is trivial graph. Clearly, $\gamma_{vch} = 1$.

(ii) $\gamma_{vch}(G) = 2$, say $T = \{a, b\}$ is a minimum vertex cover hop dominating set of G. Then by (i), G is non-trivial and so $\gamma_h(G) \geq 2$. By assumption and Proposition 2, we have $\gamma_h(G) = 2$. Now, suppose that $V(G) \setminus T$ is not an independent set in G. Then there exist $a, b \in V(G) \setminus T$ such that $d_G(a, b) = 1$, a contradiction to the fact that T is a vertex cover of G. Hence, $V(G) \setminus T$ is an independent set in G.

Conversely, suppose $\gamma_h(G) = 2 = |T|$, say $T = \{a, b\}$ is a γ_h-set of G such that $V(G)\setminus T$ is an independent set in G. Then $\gamma_{vch}(G) \geq 2$ by Proposition 2. Now, suppose T is not a vertex cover of G. Then there exist an edge $e = xy$ such that $x, y \notin T$. It follows that $x, y \in V(G) \setminus T$, a contradiction to the fact that $V(G) \setminus T$ is an independent set in G. Hence, T is a vertex cover of G, and so T is a vertex cover hop dominating of G. Consequently, $\gamma_{vch}(G) = 2$.

(iii) Suppose that $\gamma_{vch}(G) = n$. Suppose there is a component H of G which is non-complete. Then $d_H(a, b) = 2 = d_G(a, b)$ for some $a, b \in V(H)$. Let $C' = V(G) \setminus \{a\}$. Then C' is a vertex cover hop dominating set of G. It follows that $\gamma_{vch}(G) \leq n - 1$, a contradiction. Thus, every component of G is complete.

Conversely, suppose that every component of G is complete. Then $\gamma_h(G) = n$. By Proposition 2, $\gamma_{vch}(G) = n$. □

The next result is a direct consequence of Theorem 1.

Corollary 1. *Let* G *be any graph of order* $n \geq 1$. *Then each of the following statements holds.*

(i) $\gamma_{vch}(G) \geq 2$ *if and only if* G *is non-trivial.*

(ii) $\gamma_{vch}(G) = n$ *if* $G = K_n$.

(iii) $\gamma_{vch}(G) \leq n - 1$ *if* G *has non-complete component.*

(iv) $\gamma_{vch}(G) + \gamma_{vch}(\overline{G}) = 2n$ *if* $G = \overline{K}_n$ *or* $G = K_n$.

(v) $\gamma_{vch}(G) \cdot \gamma_{vch}(\overline{G}) = n^2$ *if* $G = \overline{K}_n$ *or* $G = K_n$.

Theorem 2. *Let* G *be a graph of order* $n \geq 1$. *Then each of the following statements holds.*

(i) Let G *be a connected graph. Then* $\gamma_{vch}(G) = n$ *if and only if* $G = K_n$.

(*ii*) $4 \leq \gamma_{vch}(G) + \gamma_{vch}(\overline{G}) \leq 2n - 1$ *if G has one non-complete component.*

(*iii*) $4 \leq \gamma_{vch}(G) \cdot \gamma_{vch}(\overline{G}) \leq n^2 - n$ *if G has one non-complete component.*

Proof. (*i*) Suppose $\gamma_{vch}(G) = n$. Then by Theorem 1(iii), every component of G is complete. Since G is connected, it follows that $G = K_n$.

The converse follows from Corollary 1(ii).

(*ii*), (*iii*) Suppose that G has one non-complete component. Then by Corollary 1(iii), $\gamma_{vch}(G) \leq n - 1$. By Theorem 1, $\gamma_{vch}(\overline{G}) \leq n$. Thus, $\gamma_{vch}(G) + \gamma_{vch}(\overline{G}) \leq 2n - 1$ and $\gamma_{vch}(G) \cdot \gamma_{vch}(\overline{G}) \leq n^2 - n$. Since G is non-trivial, $\gamma_{vch}(G) \geq 2$ and $\gamma_{vch}(\overline{G}) \geq 2$ by Corollary 1(i). Therefore, $\gamma_{vch}(G) + \gamma_{vch}(\overline{G}) \geq 4$ and $\gamma_{vch}(G) \cdot \gamma_{vch}(\overline{G}) \geq 4$. Consequently, $4 \leq \gamma_{vch}(G) + \gamma_{vch}(\overline{G}) \leq 2n - 1$ and $4 \leq \gamma_{vch}(G) \cdot \gamma_{vch}(\overline{G}) \leq n^2 - n$. □

Proposition 3. *Let G be any graph on $n \geq 2$ vertices. If $\gamma_{vch}(G) = 2$, then $\gamma_h(G) = 2$. However, the converse of is not always true.*

Proof. Suppose $\gamma_{vch}(G) = 2$. Then $\gamma_h(G) \leq 2$ by Proposition 2. Since $\gamma_h(G) \geq 2$ for any graph of order $n \geq 2$, it follows that $\gamma_h(G) = 2$.

To see the converse is not necessarily true, consider $P_5 = [v_1, v_2, v_3, v_4, v_5]$. Let $C = \{v_2, v_3\}$. Then C is a γ_h-set of P_5 and so $\gamma_h(P_5) = 2$. However, $\gamma_{vch}(P_5) = 3$. □

Definition 2. A pointwise non-dominating set $C \subseteq V(G)$ is called a *vertex cover pointwise non-dominating set* of G if it is a vertex cover of G. The minimum cardinality of a vertex cover pointwise non-dominating set of G, denoted by $vcpnd(G)$, is called a *vertex cover pointwise non-domination number* of G. Any vertex cover pointwise non-dominating set of G with cardinality equal to $vcpnd(G)$ is called a *vcpnd-set* of G.

Example 2. Consider the graph G in Figure 4. Let $C = \{a, b, f\}$. Notice that every edge of G is incident to atleast one vertex in C. Thus, C is a vertex cover set of G. Now, since $c, d, e \notin N_G(a)$, it follows that C is a pointwise non-dominating set of G. Hence, C is a vertex cover pointwise non-dominating set of G. Moreover, it can be verified that $vcpnd(G) = 3$.

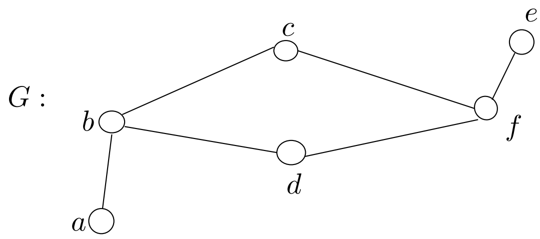

Figure 4: Graph G with $vcpnd(G) = 3$

Theorem 3. *Let G be any graph of order n. Then each of the following is true.*

(i) $1 \leq vcpnd(G) \leq n$

(ii) $pnd(G) \leq vcpnd(G)$.

(iii) $vcpnd(G) = 1$ *if and only if every component of G is trivial.*

(iv) $vcpnd(G) = n$ *if and only if every component of G is a non-trivial complete graph.*

Proof. (i) Since any vertex cover pointwise non-dominating set in G is a non-empty, it follows that $vcpnd(G) \geq 1$. Also, since every vertex cover pointwise non-dominating set C is contained in $V(G)$, we have $vcpnd(G) \leq n$. Therefore, $1 \leq vcpnd(G) \leq n$.

(ii) Let C be a minimum vertex cover pointwise non-dominating set of G. Then C is a pointwise non-dominating (by definition). Since $pnd(G)$ is the minimum cardinality among all pointwise non-dominating sets in G, we have $vcpnd(G) = |C| \geq pnd(G)$.

(iii) Suppose $vcpnd(G) = 1$, say, $\{v\}$ is a $vcpnd$-set of G. Suppose there is a component W of G which is non-trivial. Then there exist $a, b \in V(W)$ such that $ab \in E(W) \subseteq E(G)$. If $v \neq a, b$, then $vcpnd(G) \geq 2$, a contradiction. Suppose $v = a$. Then b must also be in the vertex cover pointwise non-dominating set S of G. Hence, $vcpnd(G) \geq 2$, a contradiction. Therefore, every component of G is trivial.

The converse is clear.

(iv) Suppose $vcpnd(G) = n$, say, $C = V(G)$ is the $vcpnd$-set of G. Then by (iii), every component of G is a non-trivial graph. Now, suppose there is a component C of G which is non-complete. Then there exist $u, v \in V(C) \subseteq V(G)$ such that $d_C(u, v) = 2 = d_G(u, v) = 2$. Clearly, $V(G) \setminus \{u\}$ is a vertex cover pointwise non-dominating set in G. Hence, $vcpnd(G) \leq n - 1$, a contradiction. Therefore, every component of G is non-trivial complete graph.

Conversely, suppose that every component of G is a non-trivial complete graph. Then $pnd(G) = n$. By (i) and (ii), we have $vcpnd(G) = n$. $\square$

Observation 1. *Let n be any positive integer. Then*

$$(i)\ \ vcpnd(P_n) = \begin{cases} 2 & if\ n = 2,3 \\ 3 & if\ n = 5 \\ \frac{n}{2} & if\ n \geq 4\ and\ even \\ \lfloor \frac{n}{2} \rfloor & if\ n \geq 7\ and\ odd. \end{cases}$$

$$(ii)\ \ vcpnd(C_n) = \begin{cases} 3 & if\ n = 3 \\ \frac{n}{2} & if\ n \geq 4\ and\ even \\ \lceil \frac{n}{2} \rceil & if\ n \geq 5\ and\ odd. \end{cases}$$

Theorem 4. *[10] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets of G and H, respectively.*

Theorem 5. *Let G and H be two non-complete graphs. A subset $C = C_G \cup C_H$ of $V(G+H)$ is a vertex cover hop dominating set of $G+H$ if and only if C satisfies one of the following conditions:*

(i) $C_G = V(G)$ and C_H is a vertex cover pointwise non-dominating set of H.

(ii) $C_H = V(H)$ and C_G is a vertex cover pointwise non-dominating set of G.

Proof. Suppose C is a vertex cover hop dominating set of $G + H$. Suppose $C_G = \varnothing$. Observe that $V(G) \nsubseteq N^2_G[C]$. Since C is a hop dominating set of $G + H$, it follows that $V(G) \subseteq N^2_G[C]$, a contradiction. Thus, $C_G \neq \varnothing$. Similarly, $C_H \neq \varnothing$. If $C_G = V(G)$ and $C_H = V(H)$, then we are done. Suppose that $C_H \neq V(H)$. Then there exists $a \in V(H) \setminus C_H$ such that $a \notin C$. If $C_G \neq V(G)$, then there exists $b \in V(G) \setminus C_G$ such that $ab \in E(G+H)$. However, $a, b \notin C$, a contradiction to the fact that C is a vertex cover of $G+H$. Hence, $C_G = V(G)$. Since C is a hop dominating set in $G+H$, it follows that C_H is a pointwise non-dominating set of H by Theorem 4. Also, since C is a vertex cover of $G+H$, C_H is a vertex cover of H. Thus, (i) holds. Similarly, (ii) holds.

Conversely, suppose that (i) holds. Then $C = V(G) \cup C_H$ is a vertex cover of $G+H$. Since C_H is a pointwise non-dominating set, it follows that $C = V(G) \cup C_H$ is a hop dominating set in $G+H$ by Theorem 4. Therefore, $C = C_G \cup C_H$ is a vertex cover hop dominating set of $G+H$. Similarly, if (ii) holds, then $C = C_G \cup C_H$ is a vertex cover hop dominating set of $G+H$. □

Theorem 6. *Let G and H be two non-complete graphs. Then*

$$\gamma_{vch}(G+H) = min\{|V(G)| + vcpnd(H), |V(H)| + vcpnd(G)\}.$$

Proof. Let $C = C_G \cup C_H$ be a minimum vertex cover hop dominating set of $G+H$. Then by Theorem 5, C satisfies one of the following:

(i) $C_G = V(G)$ and C_H is a vertex cover pointwise non-dominating set of H.

(ii) $C_H = V(H)$ and C_G is a vertex cover pointwise non-dominating set of G.

Thus,

$$\gamma_{vch}(G+H) = |C| = |C_G| + |C_H| \geq V(G)| + vcpnd(H)$$

and

$$\gamma_{vch}(G+H) = |C| = |C_G| + |C_H| \geq V(H)| + vcpnd(G).$$

Consequently,

$$\gamma_{vch}(G+H) \geq \min\{|V(G)| + vcpnd(H), |V(H)| + vcpnd(G)\}.$$

On the other hand, suppose that $C = V(G) \cup C_H$, where C_H is a minimum vertex cover pointwise non-dominating set of H. Then C is a vertex cover hop dominating set in $G+H$ by Theorem 5. Thus, $|V(G)| + vcpnd(H) = |C| \geq \gamma_{vch}(G+H)$. Next, suppose that $C = V(H) \cup C_G$, where C_G is a minimum vertex cover pointwise non-dominating set of G. Then C is a vertex cover hop dominating set in $G+H$ by Theorem 5. Thus, $|V(H)| + vcpnd(G) = |C| \geq \gamma_{vch}(G+H)$. Therefore,

$$\gamma_{vch}(G+H) = \min\{|V(G)| + vcpnd(H), |V(H)| + vcpnd(G)\}.$$

□

The following result follows from Observation 1 and Theorem 6.

Corollary 2. *Let n be any positive integer. Then each of the following is true.*

(i) $$\gamma_{vch}(P_n + P_n) = n + vcpnd(P_n) = \begin{cases} n+2 & if\ n = 3 \\ n+3 & if\ n = 5 \\ n + \frac{n}{2} & if\ n \geq 4 \text{ and even} \\ n + \lfloor \frac{n}{2} \rfloor & if\ n \geq 7 \text{ and odd.} \end{cases}$$

(ii) $$\gamma_{vch}(C_n + C_n) = n + vcpnd(C_n) = \begin{cases} n + \frac{n}{2} & if\ n \geq 4 \text{ and even} \\ n + \lceil \frac{n}{2} \rceil & if\ n \geq 5 \text{ and odd.} \end{cases}$$

Theorem 7. *Let G be any complete graph and H be any graph. A set $C \subseteq V(G+H)$ is vertex cover hop dominating set of $G+H$ if and only if $C = V(G) \cup C_H$, C_H is a vertex cover pointwise non-dominating set in H.*

Proof. Suppose C is a vertex cover hop dominating set of $G+H$. Suppose $C_G = \varnothing$. Observe that $V(G) \nsubseteq N^2_G[C]$. Since C is a hop dominating set of $G+H$, it follows that $V(G) \subseteq N^2_G[C]$, a contradiction. Thus, $C_G \neq \varnothing$. Similarly, $C_H \neq \varnothing$. Since C is a hop dominating set in $G+H$, it follows that C_H is a pointwise non-dominating set of H by Theorem 4. Also, since C is a vertex cover of $G+H$, C_H is a vertex cover of H. Now, suppose $C_G \neq V(G)$. Then there exists $a \in V(G) \setminus C_G$ such that $a \notin C$. Since

G is complete, it follows that $a \notin N^2_{G+H}[C]$, a contradiction to the fact that C is a hop dominating set in $G+H$. Hence, $C_G = V(G)$.

Conversely, suppose that $C = V(G) \cup C_H$, where C_H is a vertex cover pointwise non-dominating set in H. Then C is a vertex cover set of $G+H$. By Theorem 4, C is a hop dominating of $G+H$. Thus, C is a vertex cover hop dominating set of $G+H$. □

Theorem 8. *Let G be complete graph and H be any graph. Then.*

$$\gamma_{vch}(G+H) = |V(G)| + vcpnd(H).$$

Proof. Let $C = V(G) \cup C_H$ be a minimum vertex cover hop dominating set of $G+H$. Then by Theorem 7, $C = V(G) \cup C_H$, where C_H is a vertex cover pointwise non-dominating set of H. Thus,

$$\gamma_{vch}(G+H) = |C| = |V(G)| + |C_H| \geq |V(G)| + vcpnd(H).$$

Next, suppose that $C = V(G) \cup C_H$, where C_H is a minimum vertex cover pointwise non-dominating set of H. Then C is a vertex cover hop dominating set in $G+H$ by Theorem 7. Hence, $|V(G)| + vcpnd(H) = |C| \geq \gamma_{vch}(G+H)$. Therefore,

$$\gamma_{vch}(G+H) = |V(G)| + vcpnd(H).$$

□

The following result follows that from Observation 1 and Theorem 7.

Corollary 3. *Let n and m be positive integers. Then each of the following is true.*

$$(i)\ \ \gamma_{vch}(K_n + P_m) = n + vcpnd(P_m) = \begin{cases} n+2 & if\ m = 3 \\ n+3 & if\ m = 5 \\ n + \frac{m}{2} & if\ m \geq 4 \ and\ even \\ n + \lfloor \frac{m}{2} \rfloor & if\ m \geq 7\ and\ odd. \end{cases}$$

$$(ii)\ \ \gamma_{vch}(K_n + C_m) = n + vcpnd(C_m) = \begin{cases} n + \frac{m}{2} & if\ m \geq 4\ and\ even \\ n + \lceil \frac{m}{2} \rceil & if\ m \geq 5\ and\ odd. \end{cases}$$

$$(iv)\ \ \gamma_{vch}(W_n) = 1 + vcpnd(C_m) = \begin{cases} 1 + \frac{m}{2} & if\ m \geq 4\ and\ even \\ 1 + \lceil \frac{m}{2} \rceil & if\ m \geq 5\ and\ odd. \end{cases}$$

$$(v)\ \ \gamma_{vch}(F_n) = 1 + vcpnd(P_m) = \begin{cases} 3 & if\ m = 3 \\ 4 & if\ m = 5 \\ 1 + \frac{m}{2} & if\ m \geq 4\ and\ even \\ 1 + \lfloor \frac{m}{2} \rfloor & if\ m \geq 7\ and\ odd. \end{cases}$$

Theorem 9. *Let G be a non-trivial connected graph and let H be any graph. If $C = V(G) \cup (\cup_{v \in V(G)} C_v)$, where $C_v \subseteq V(H^v)$ is a vertex cover pointwise non-dominating set of H for each $v \in V(G)$, then C is a vertex cover hop dominating set of $G \circ H$.*

Proof. Let $C = V(G) \cup (\cup C_v)$, where C_v is a vertex cover pointwise non-dominating set H^v for each $v \in V(G)$. Since C_v is a pointwise non-dominating set in H^v for each $v \in V(G)$, it follows that $N^2_{\langle v+H^v \rangle}[C_v] = V(H^v)$ for each $v \in V(G)$. Thus,

$$N^2_{G \circ H}[C] = N^2_{G \circ H}[V(G) \cup (\bigcup_{v \in V(G)} C_v)] = V(G \circ H),$$

and so C is a hop dominating set of $G \circ H$. Since C_v is a vertex cover of H^v for each $v \in V(G)$, $C = V(G) \cup (\cup_{v \in V(G)} C_v)$ is a vertex cover of $G \circ H$. Consequently, C is a vertex cover hop dominating set of $G \circ H$. □

Corollary 4. *Let G be a non-trivial connected graph and let H be any graph. Then*

$$\gamma_{vch}(G \circ H) \leq |V(G)| + |V(G)| \cdot vcpnd(H).$$

Proof. Let $C = V(G) \cup (\bigcup_{v \in V(G)} C_v)$, where C_v is a minimum vertex cover pointwise non-dominating set of H. By Theorem 9, C is a vertex cover hop dominating set $G \circ H$. Thus,

$$\gamma_{vch}(G \circ H) \leq |C| = |V(G)| + |V(G)| \cdot vcpnd(H).$$

□

4. Conclusion

The concept of vertex cover hop domination has been introduced and initially investigated in this study. Graphs that attained some specific vertex cover hop domination number have been characterized. The vertex cover hop domination number of the join and corona of two graphs have been obtained. These characterizations have been used to obtain bounds or exact values of the vertex cover hop domination number of each of these graphs. Exploring bounds for this newly introduced parameter in relation to other known parameters possibly provides insightful information.

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