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Köthe dual of some vector-valued sequence spaces

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Abstract. We study some properties of the spaces $\lambda(E)$ of weakly λ-summable sequences and $\lambda\langle E\rangle$ of strongly λ-summable sequences of a locally convex space E. For example, after proving results on bounded sets of these spaces, we express the elements of their Köthe duals in terms of sequences in the continuous dual E' of E, then we prove that these spaces possess the AK property if and only if the Köthe dual coincides with the continuous dual.

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Introduction

In order to characterize the nuclearity of a locally convex space E, A. Pietsch [9] introduced the spaces $\ell_p\{E\}$ and $\ell_p[E]$ of absolutely ℓ_p-summable and weakly ℓ_p-summable sequences in E, respectively. This allowed the author also to introduce and study the absolutely p-summing operators. Later, J. S. Cohen [2] introduced the space $\ell_p\langle E\rangle$ of strongly p-summable sequences and used this space together with the spaces $\ell_p[E]$ and $\ell_p\{E\}$ to define the strongly and the nuclear p-summing operators. H. Apiola [1], in order to get new conditions for the nuclearity of E, generalized to an arbitrary locally convex space E, the definition of $\ell_p\langle E\rangle$. On the other hand, A. Pietsch [9], dealing again with a perfect sequence space λ equipped with its Köthe normal topology, introduced the space $\lambda(E)$ of weakly λ-summable sequences in E. We note that, considering the general case where λ is no longer endowed with its Köthe normal topology, but with a general polar topology, M. Florencio and P. J. Paúl [3] studied $\lambda(E)$ and clarified the relationship between $\lambda(E)$ and the completion of the injective tensor product $\lambda\otimes_\epsilon E$. They determined conditions on E that make $\lambda(E)$ an AK space.
Let us mention here that the authors, in [7, 8, 10–13], studied many aspects of the space $\lambda(E)$ such as the reflexivity, the nuclearity and the representation of the continuous dual

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in terms of strongly λ^*-summable sequences in E', where λ^* is the Köthe dual of λ and E' the continuous dual of E.
In this note, we consider on $\lambda(E)$ and $\lambda\langle E\rangle$ locally convex topologies defined in a natural way as bellow, and then, we study some aspects of their properties such as, bounded sets and the generalized Köthe dual. We introduce, in the preliminary section, the notations and background that will be needed in the sequel. In section 2, a fundamental family of bounded sets in $\lambda(E)$ and some bounded sets of $\lambda\langle E\rangle$ are exhibited. Section 3 is devoted to the determination of the Köthe dual of $\lambda(E)$ and $\lambda\langle E\rangle$ in terms of sequences of continuous linear forms on E. In particular, we extend to these spaces the well known result that, the continuous dual of a scalar sequence space λ with respect to a polar topology, coincides with its Köthe dual, if and only if λ has the AK property.

1. Notations and background

Throughout this note, if V is a normed space then V', $\|\cdot\|_V$ and B_V will denote the continuous dual, the norm and the closed unit ball of V, respectively.
We will stand by λ a perfect Banach sequence space and by λ^* its Köthe dual. Although many of results presented here are valid for more general setting, we will assume that the norm of λ satisfies the conditions:
(1) If $\alpha, \beta \in \lambda$, with $\alpha \leq \beta$, then $\|\alpha\|_\lambda \leq \|\beta\|_\lambda$, and
(2) $(\lambda, \|\cdot\|_\lambda)$ is an AK space, i.e., every $\alpha = (\alpha_n)_n \in \lambda$ is the $\|\cdot\|_\lambda$-limit of its sections $(\alpha_1, \ldots, \alpha_n, 0, \ldots), n \in \mathbb{N}$.
This condition is satisfied if and only if $\lambda^* = \lambda'$. So, λ will be reflexive whenever $(\lambda^*, \|\cdot\|_{\lambda^*})$ is also an AK space. The results proved here are then applicable to many cases of the Orlicz sequence spaces ℓ_M (see for example [12]) and, in particular, to the ℓ_p spaces.
Further, we mean by E a sequentially complete Hausdorff locally convex space, E' its continuous dual and by $\mathcal{M}$ the collection of all absolutely convex, $\sigma(E', E)$-closed and equicontinuous subsets of E'. The topology of E is then defined by the family of seminorms $(P_M)_{M\in\mathcal{M}}$ such that, for all $x \in E$,

$$P_M(x) = \sup\{|a(x)| : \ a \in M\}, \text{ for all } M \in \mathcal{M}.$$

Define the following space

$$\lambda(E) = \left\{x = (x_n)_n \subset E : \sum \alpha_n x_n \text{ converges in } E, \text{ for all } (\alpha_n)_n \in \lambda^*\right\}.$$

Following [3], a locally convex topology on $\lambda(E)$ is defined by the family of seminorms $(\epsilon_M)_{M\in\mathcal{M}}$, where

$$\epsilon_M(x) := \sup\left\{\sum_{n=1}^{\infty} |\alpha_n a(x_n)| : \ a \in M, \ \alpha \in B_{\lambda^*}\right\}, \text{ for all } x = (x_n)_n \in \lambda(E).$$

These seminorms turn out to be defined also on the space

$$\lambda[E] = \left\{x = (x_n)_n \subset E : (a(x_n))_n \in \lambda^*, \text{ for all } a \in E'\right\}.$$

Following [2] and [7], a sequence $(x_n)_n \subset E$ is said to be strongly λ-summable if, for every $M \in \mathcal{M}$ and $(a_n)_n \in \lambda^*[E'_M]$, the series $\Sigma|a_n(x_n)|$ converges. We mean by E'_M the linear subspace of E' spanned by M and equipped with the gauge $\|\cdot\|_M$ of M. Denote by $\lambda\langle E\rangle$ the space of all strongly $\lambda-$summable sequences in E. We will endow $\lambda\langle E\rangle$ with the locally convex topology introduced in [8] and defined by the family of seminorms $(\sigma_M)_{M\in\mathcal{M}}$, where

$$\sigma_M(x) = \sup\left\{\sum_{n=1}^{\infty}|a_n(x_n)| : a = (a_n)_n \in B_{\lambda^*(E'_M)}\right\}, \text{ for all } x = (x_n)_n \in \lambda\langle E\rangle.$$

Notice that, since λ is perfect, we have

$$\lambda\langle E\rangle \subset \lambda(E) \subset \lambda[E].$$

The spaces $\lambda\langle E\rangle$, $\lambda(E)$ and $\lambda[E]$ are sequentially complete, in particular, Banach spaces whenever λ and E are.

On the other hand, since λ' coincides with λ^*, one deduces from [5, Theorem 1] that $\lambda(E) = \lambda[E]$.

For any sequence $x = (x_n)_n$ in E and $p \in \mathbb{N}$, denote by $x^{(p)} = (x_1, x_2, \ldots, x_p, 0, 0, \ldots)$ the pth finite section of x. Let

$$x^{<p>} = x - x^{(p)} = (0, 0, \ldots, 0, x_{p+1}, x_{p+2}, \ldots).$$

If e_n is the nth unit coordinate vector of $\mathbb{C}^{\mathbb{N}}$, then $x^{(p)} = \sum_{n=1}^{p} x_n e_n$.

We will denote by $\lambda(E)_r$ (resp. $\lambda\langle E\rangle_r$), the subspace of $\lambda(E)$ (rep. $\lambda\langle E\rangle$) consisting of all the sequences $x = (x_n)_n$ which are limits of their finite sections $x^{(p)}$. The reader is referred to [6, 14] for notations and concepts related to the Köthe theory of sequence spaces and the general theory of locally convex spaces.

2. Bounded sets of $\lambda(E)$

If B is a closed, absolutely convex and bounded subset of E, and $S = B_{\lambda^*}$, let

$$\tilde{B} = \left\{(x_n)_n \in \lambda(E) : \ \forall\alpha = (\alpha_n)_n \in S, \ \sum_n \alpha_n x_n \in B\right\}. \quad (1)$$

We have the following result.

Proposition 1. *The collection $\{\tilde{B} : B$ bounded in $E\}$ constitutes a fundamental system of bounded sets for $\lambda(E)$.*

Proof. If B is a bounded set in E, then the corresponding set $\tilde{B}$ in (1) is bounded in $\lambda(E)$, by [8, Proposition 1]. Now, let $\mathbb{B}$ be a bounded set of $\lambda(E)$ and S the unit ball of λ^*. Consider the subset B of E defined by

$$B = \left\{y \in \lambda(E) : y = \sum_{n=1}^{\infty}\alpha_n x_n, \text{ for some } \alpha \in S \text{ and } x = (x_n)_n \in \mathbb{B}\right\}.$$

Let $a \in E'$, $M \in \mathcal{M}$ with $a \in M$ and $x = (x_n)_n \in \mathbb{B}$. Then,

$$\left|\left\langle a, \sum_{n=1}^{\infty} \alpha_n x_n \right\rangle\right| = \left|\sum_{n=1}^{\infty} \alpha_n a(x_n)\right| \leq \sum_{n=1}^{\infty} |\alpha_n a(x_n)| \leq \varepsilon_M(x).$$

Since S is a normal disk in λ^* and $\mathbb{B}$ is bounded in $\lambda(E)$, then B is a bounded disk in E. Moreover, by the definition of B, we see that $\mathbb{B} \subset \tilde{B}$. ■

Lemma 2. *For every $t \in E$ and $\beta = (\beta_n)_n \in \lambda$, we have $(\beta_n t)_n \in \lambda\langle E\rangle$.*

Proof. For $t \in E$, let δ_t denote the evaluation defined on E' by $\delta_t(x') = x'(t)$. We have, if $M \in \mathcal{M}$, then $|\delta_t(x')| \leq P_M(t)\|x'\|_M$ for every $x' \in E'_M$. This means that $\delta_t \in (E'_M)'$ and that $\|\delta_t\| \leq P_M(t)$. Let $\beta = (\beta_n)_n \in \lambda$ and $(a_n)_n \in \lambda^*[E'_M]$. By the definition of $\lambda^*[E'_M]$, $(a_n(t))_n = (\delta_t(a_n)) \in \lambda^*$, and then

$$\sum_{n=1}^{\infty} |a_n(\beta_n t)| = \sum_{n=1}^{\infty} |a_n(t)\beta_n| < \infty.$$

Thus, $(\beta_n t)_n \in \lambda\langle E\rangle$. ■

Now, for $S = B_\lambda$ and a closed absolutely convex bounded subset B of E, define

$$\bar{B} = \left\{\sum_{k=1}^{\infty} \xi_k \beta^k x_k : \beta^k = (\beta^k_n)_n \in B_\lambda, x_k \in B, \text{ and } \sum_{k=1}^{\infty} |\xi_k| \leq 1\right\}. \tag{2}$$

Proposition 3. *The set $\bar{B}$ is a bounded subset of $\lambda\langle E\rangle$.*

Proof. Let $\{\beta^k = (\beta^k_n)_n\}_k^\infty$ and $\{x_k\}_k^\infty$ be sequences in B_λ and B respectively. Fix $k \in \mathbb{N}$, $M \in \mathcal{M}$ and $a = (a_n)_n \in \lambda^*[E'_M] = \lambda^*(E'_M)$, with $\|a\|_{\lambda^*(E'_M)} \leq 1$. As in the proof of the previous lemma, δ_{x_k} denotes the evaluation defined on E'_M. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|a_n(\beta^k_n x_k)\right| = \sum_{n=1}^{\infty} \left|\beta^k_n a_n(x_k)\right| &= \sum_{n=1}^{\infty} \left|\beta^k_n \delta_{x_k}(a_n)\right| \\
&= \|\beta^k\|_\lambda P_M(x_k) \sum_{n=1}^{\infty} \left|\frac{\beta^k_n}{\|\beta^k\|_\lambda} \frac{\delta_{x_k}}{P_M(x_k)}(a_n)\right| \\
&\leq \|\beta^k\|_\lambda P_M(x_k) \|a\|_{\lambda^*(E'_M)} \\
&\leq \|\beta^k\|_\lambda P_M(x_k) \leq P_M(x_k).
\end{aligned}$$

Since B is bounded in E, then there exists $\rho > 0$, so that $P_M(x_k) \leq \rho$ for every $k \in \mathbb{N}$; and, by the definition of σ_M, one has $\sigma_M(\beta^k x_k) \leq \rho$, for every $k \in \mathbb{N}$. Moreover, if $(\xi_k)_k$ satisfies $\sum_{k=1}^{\infty} |\xi_k| \leq 1$ then,

$$\sum_{k=1}^{\infty} \sigma_M(\xi_k \beta^k x_k) \leq \sum_{k=1}^{\infty} |\xi_k| \|\beta^k\|_\lambda P_M(x_k) \leq \rho \sum_{k=1}^{\infty} |\xi_k| \leq \rho. \tag{3}$$

By Lemma 2, the terms of the series $\sum_{k=1}^{\infty} \xi_k \beta^k x_k$ belong to $\lambda\langle E\rangle$. Since $\lambda\langle E\rangle$ is sequentially complete, we derive from (3) that this series is convergent in $\lambda\langle E\rangle$ and that the corresponding set $\bar{B}$ in (2) is well defined, contained and bounded in $\lambda\langle E\rangle$. ■

3. Köthe Duals of $\lambda(E)$ and $\lambda\langle E\rangle$

Following [4], if F is a linear subspace of $E^{\mathbb{N}}$, the generalized Köthe dual of F is defined by

$$F^* = \left\{(a_n)_n \subset E' : \sum |a_n(x_n)| \text{ converges for all } x = (x_n)_n \in F\right\}.$$

For every $x \in E$, denote by δ_x the evaluation defined, as in the proof of Lemma 2, by $\delta_x(x') = x'(x)$, for $x' \in E'$. Thanks to the linear and isometric map $\delta : x \to \delta_x$ from E to E'', we always have $F \subset F^{**}$. The sequence space F is said to be perfect if $F^{**} = F$.

Proposition 4. *Let λ be a perfect normed sequence space with dual space λ^* and E a locally convex space. Then $(\lambda(E)_r)^* = (\lambda(E))^*$ and $(\lambda\langle E\rangle_r)^* = (\lambda\langle E\rangle)^*$.*

Proof. We prove that $(\lambda(E)_r)^* = (\lambda(E))^*$, the same argument applies for the second equality. It is clear that $(\lambda(E))^* \subset (\lambda(E)_r)^*$. Let $a = (a_n)_n \in (\lambda(E)_r)^*$ and $x = (x_n)_n \in \lambda(E)$. To prove that $\sum_{n=1}^{\infty} |a_n(x_n)|$ converges, it is enough to prove that, for every $(\gamma_n)_n \in c_0$, the series $\sum_{n=1}^{\infty} |\gamma_n a_n(x_n)|$ converges. Set $y = (y_n)_n$ where $y_n = \gamma_n x_n$, for all $n \in \mathbb{N}$. We see that $y \in \lambda(E)$. In the other hand, for $M \in \mathcal{M}$, $a \in M$, $\alpha = (\alpha_n)_n \in B_{\lambda^*}$ and $p \in \mathbb{N}$, one has

$$\sum_{n=p+1}^{\infty} |\alpha_n a(\gamma_n x_n)| \leq \sup_{n\geq p+1} |\gamma_n| \sum_{n=p+1}^{\infty} |\alpha_n a(x_n)| \leq \|\gamma^{<p>}\|_{c_0} \epsilon_M(x).$$

This shows that $\epsilon_M(y^{<p>}) \leq \|\gamma^{<p>}\|_{c_0} \epsilon_M(x)$, and then $y \in \lambda(E)_r$ since $(\gamma^{<p>})_p$ converges to 0. Now, we have

$$\sum_{n=1}^{\infty} |\gamma_n a_n(x_n)| = \sum_{n=1}^{\infty} |a_n(\gamma_n x_n)| = \sum_{n=1}^{\infty} |a_n(y_n)| < \infty.$$

This completes the proof. ■

According to [7, Theorem 7], the continuous dual $\lambda(E)_r$ of $(\lambda(E)_r)'$ is given by $(\lambda(E)_r)' = \bigcup_{M\in\mathcal{M}} \lambda^*\langle E'_M\rangle$. In particular, if λ and E are Banach spaces then

$$(\lambda(E)_r)' = \lambda^*\langle E'\rangle. \tag{4}$$

The last equality is actually topological, by ([6, 15.12(2)]).

Proposition 5. *For every Banach space E, we have*

(a) the Köthe dual of $(\lambda(E))^*$ satisfies $(\lambda(E))^* = \lambda^*\langle E'\rangle = (\lambda(E)_r)'$,

(b) the Köthe dual of $(\lambda(E))^*$ satisfies $(\lambda(E))^{**} = \lambda(E'')$. In particular, if E is reflexive then $(\lambda(E))^{**} = \lambda(E)$.

Proof. By the definition of the spaces $\lambda^*\langle E'\rangle$ and $(\lambda(E))^*$, we have $\lambda^*\langle E'\rangle \subset (\lambda(E))^*$. Let $a = (a_n)_n \in (\lambda(E))^*$. Using the closed graph theorem ([6, 15.12(3)]), we can prove that the mapping $f_a : \lambda(E)_r \to \ell_1$ defined by $f_a(x) = (a_n(x_n))_n$ is continuous, and then $a \in (\lambda(E)_r)'$. So, $(\lambda(E))^* \subset (\lambda(E)_r)'$. The part (a) follows from (4). For (b), we have

$$\begin{aligned}(\lambda(E))^* &= (\lambda(E)_r)^*, && \text{(by Proposition 4)}\\ &= (\lambda(E)_r)', && \text{(by (a))}\\ &= (\lambda\widetilde{\otimes}_\varepsilon E)', && \text{(by [3, Prop. 2])}\\ &= \lambda^*\widetilde{\otimes}_\pi E', && \text{(by [6, 45.6(5)]).}\end{aligned}$$

On the other hand, since

$$\begin{aligned}(\lambda^*\langle E'\rangle)' = (\lambda\widetilde{\otimes}_\varepsilon E)'' = (\lambda^*\widetilde{\otimes}_\pi E')' &= L(\lambda^*, E''), && \text{(by [6, 41.3(6)])}\\ &= \lambda(E''), && \text{(by [10, Propoition 2])}\end{aligned}$$

then, $(\lambda(E))^{**} = \lambda(E'')$. ■

Now, by [8, Theorem 1], the continuous dual $(\lambda\langle E\rangle_r)'$ of $\lambda\langle E\rangle_r$ is given by the algebraic equality $(\lambda\langle E\rangle_r)' = \bigcup_{M\in\mathcal{M}} \lambda^*(E'_M)$. If λ and E are Banach spaces then

$$(\lambda\langle E\rangle_r)' = \lambda^*(E'). \tag{5}$$

This equality is topological, by ([6, 15.12(2)]).

Similarly, we have

Proposition 6. *For every Banach space E, the following equalities hold*

(a) $(\lambda\langle E\rangle)^* = \lambda^*(E') = (\lambda\langle E\rangle_r)'$,

(b) $(\lambda\langle E\rangle)^{**} = \lambda\langle E''\rangle$. *In particular, if E is reflexive then* $(\lambda\langle E\rangle)^{**} = \lambda\langle E\rangle$.

Proof. The proof is similar to that of Proposition 5, but we present it for the sake of completeness.

By [5, Theorem 1], $\lambda^*(E') = \lambda^*[E']$, and then, by the definition of the space $\lambda\langle E\rangle$, we have $\lambda^*(E') \subset (\lambda\langle E\rangle)^*$. On the other hand, in view of the closed graph theorem ([6, 15.12(3)]), we deduce that every $a = (a_n)_n \in (\lambda\langle E\rangle)^*$ corresponds to a continuous linear form on $\lambda\langle E\rangle_r$ by setting $f_a(x) = \sum_{n=1}^{\infty} a_n(x_n)$. Thus, $(\lambda\langle E\rangle)^* \subset (\lambda\langle E\rangle_r)'$. The part (a) follows from (5). Regarding (b), we have

$$\begin{aligned}(\lambda\langle E\rangle)^* &= (\lambda\langle E\rangle_r)^*, && \text{(by Proposition 4)}\\ &= (\lambda\langle E\rangle_r)', && \text{(by (a))}\\ &= \lambda^*(E'), && \text{(by [8, Theorem 1]).}\end{aligned}$$

This leads to

$$(\lambda\langle E\rangle)^{**} = (\lambda^*(E'))^* = \lambda^{**}\langle E''\rangle = \lambda\langle E''\rangle, \qquad \text{(by (a) of Proposition 5).}$$

This ends the proof. ■

Proposition 7. *Suppose that E and λ are Banach spaces with λ reflexive. Then,*

$$\lambda^*\langle E'\rangle_r = \lambda^*\langle E'\rangle.$$

Proof. Let $a = (a_n)_n \in \lambda^*\langle E'\rangle$, and consider

$$\varphi_a : \lambda(E'') \to \ell_1,\ \varphi_a((x''_n)_n) = (x''_n(a_n))_n.$$

The linear mapping φ_a is well defined, since $\lambda(E'') = \lambda[E'']$ by [5, Theorem 1]. Let us show that φ_a is weak to weak continuous. If $(\alpha_n)_n \in \ell_\infty$ then $(\alpha_n a_n)_n \in \lambda^*\langle E'\rangle$ and for all $(x''_n)_n \in \lambda(E'')$, we have

$$\big\langle(\alpha_n)_n,\ (x''_n(a_n))_n\big\rangle = \big\langle(\alpha_n a_n)_n, (x''_n)_n\big\rangle.$$

Let $\mathcal{B}$ be a bounded set in $\lambda(E)_r$. The Alaoglu-Bourbaki Theorem ([6, 20.9(4)]) asserts that $\mathcal{B}$ is relatively weak* compact. We derive from [6, 22.4(3)] that $\big\{(a_n(x_n))_n :\ (x_n)_n \in \mathcal{B}\big\}$ is relatively compact in ℓ_1 and then

$$\lim_{p\to\infty} \sup\left\{\sum_{n=p+1}^{\infty} |a_n(x_n)| :\ (x_n)_n \in \mathcal{B}\right\} = 0.$$

This means that $(a^{<p>})_p$ is a null sequence in $\lambda^*\langle E'\rangle$. This completes the proof. ■

Proposition 8. *Let E be a Banach and λ a reflexive Banach sequence space. Then, the elements of $\lambda^*\langle E'\rangle$ are the sequences $a = (a_n)_n \subset E'$ that have the form*

$$a = \sum_{k=1}^{\infty} \lambda_k \beta^k x'_k,$$

where $(\lambda_k)_k \in \ell_1$ and, for every $k \in \mathbb{N}$, $\beta^k = (\beta^k_n)_n \in B_\lambda$ and $x'_k \in B_{E'}$.

Proof. As in Proposition 7, we have

$$\begin{aligned}
\lambda^*\langle E'\rangle &= (\lambda(E)_r)', && \text{by [7, Theorem 7]}\\
&= (\lambda\widetilde{\otimes}_\varepsilon E)', && \text{by [3, Prop. 2]}\\
&= \mathcal{I}(\lambda \times E), && \text{integral bilinear forms on } \lambda \times E,\ \text{by [6, 45.1(2)]}\\
&= \mathcal{L}^I(\lambda, E'), && \text{integral mappings of } \lambda \text{ in } E',\ \text{by [6, 45.4(1)]}\\
&= \mathcal{N}(\lambda, E'). && \text{nuclear operators from } \lambda \text{ in } E' \text{ by [6, 45.6(1) and 45.6(4)]}
\end{aligned}$$

For $a = (a_n)_n \in \lambda^*\langle E'\rangle$, the corresponding $f_a \in \mathcal{L}^I(\lambda, E')$ is defined by $f_a(\alpha) \in E'$ such that $f_a(\alpha)(t) = B(\alpha, t) = \langle a, \alpha t\rangle$, for $\alpha = (\alpha_n)_n \in \lambda$ and $t \in E$, where $B \in \mathcal{I}(\lambda \times E)$ is the integral bilinear form on $\lambda \times E$ corresponding to f_a.
Now, since $f_a \in \mathcal{N}(\lambda, E')$, then by [6, 42.5(5)-(6)], there are $(\lambda_k)_k \in \ell_1$, a sequence

$\{\beta^k = (\beta^k_n)_n : \ k \in \mathbb{N}\}$ in B_λ and a sequence $(x'_k)_k$ in $B_{E'}$ such that, for every $\alpha \in \lambda$ and $t \in E$, we have

$$\langle a, \alpha t\rangle = f_a(\alpha)(t) = \sum_{k=1}^{\infty} \lambda_k \langle \beta^k, \alpha\rangle x'_k(t).$$

This implies that $a_n = \sum_{k=1}^{\infty} \lambda_k \beta^k_n x'_k$ for every $n \in \mathbb{N}$ and that $a = \sum_{k=1}^{\infty} \lambda_k \beta^k x'_k$.
By ascending the previous chain of equalities, we easily see that the inverse is true. ■

Proposition 9. $(\lambda(E))^* = (\lambda(E))'$ *if and only if* $\lambda(E)_r = \lambda(E)$.

Proof. By (a) of Proposition 5, if $\lambda(E)_r = \lambda(E)$ then $(\lambda(E))^* = (\lambda(E)_r)' = (\lambda(E))'$. Inversely, suppose that $(\lambda(E))^* = (\lambda(E))'$. As in the proof of Proposition 7, for every $x = (x_n)_n \in \lambda(E)$,

$$\varphi_x : (\lambda(E))' \to \ell_1, \ \varphi_x((a_n)_n) = (a_n(x))_n,$$

is well defined, linear and weak to weak continuous. Denote by $\mathcal{H}$ the closed unit ball of $(\lambda(E))^*$. Here also, the Alaoglu-Bourbaki Theorem ([6, 20.9(4)]) guarantees that $\mathcal{H}$ is relatively weak* compact. We derive from [6, 22.4(3)], that $\{(a_n(x_n))_n : \ (a_n)_n \in \mathcal{H}\}$ is relatively compact in ℓ_1 and then

$$\lim_{p\to\infty} \varepsilon_M(x^{<p>}) = \lim_{p\to\infty} \sup \left\{ \sum_{n=p+1}^{\infty} |a_n(x_n)| : \ (a_n)_n \in \mathcal{H} \right\} = 0.$$

Thus, $x \in \lambda(E)_r$. ■

Proposition 10. $(\lambda\langle E\rangle)^* = (\lambda\langle E\rangle)'$ *if and only if* $\lambda\langle E\rangle_r = \lambda\langle E\rangle$.

Proof. The proof is similar to that of Proposition 9 when interchanging the roles of $\lambda(E)$ and $(\lambda(E))'$ by those of $\lambda\langle E\rangle$ and $(\lambda\langle E\rangle)'$, respectively. ■

Conclusion

Let E be a Banach space and λ a perfect Banach sequence space which is reflexive. We prove that $\lambda(E)$ and $\lambda\langle E\rangle$ have the AK property if and only if the Köthe dual and the continuous dual are equal. If, moreover E is reflexive, these spaces become perfect.

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