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Matrix mixed inequalities

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Abstract. In this paper, we prove that all the eigenvalues of arbitrarily complex matrix are located in one closed disk, which is a refinement of some existing inequalities.

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1. Introduction

We denote by M_n the vector space of all complex $n \times n$ matrices. The notation $A \geq 0$ is used to mean that A is positive semidefinite. For $A \in M_n$, the conjugate transpose of A is denoted by A^*. Denote by $\lambda_j(A)$ $(1 \leq j \leq n)$ the class of all eigenvalues of $A \in M_n$ and $\|A\|_F = \sqrt{tr\,(AA^*)}$, $[A, B] = AB - BA$. The singular values of A are enumerated as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. These are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$.

The estimation and location of eigenvalues are always hot topics of matrix analysis [1], [2]. It plays an important role in many fields of applied science. Let $M \in M_n$ be an $n \times n$ complex matrix partitioned as

$$M = \begin{bmatrix} A_k & B_{k,n-k} \\ C_{n-k,k} & D_{n-k} \end{bmatrix},$$

where $1 \leq k \leq n-1$. The following estimation

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \|M\|_F^2 - \max_{1\leq k\leq n-1} \left(\|B_{k,n-k}\|_F - \|C_{n-k,k}\|_F\right)^2$$

is an elegant result on eigenvalues due to Tu [3].

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In [4], Gu proposed a new idea which uses only one single closed disk to locate eigenvalues of a given $n \times n$ complex matrix. He proved that all the eigenvalues of any complex matrix A are located in the following disk:

$$\left|\lambda_j - \frac{trA}{n}\right| \leq \left(\frac{n-1}{n}\left(\|A\|_F^2 - \frac{|trA|^2}{n}\right)\right)^{\frac{1}{2}} \tag{1}$$

for $j = 1, 2, \cdots, n$.

Zou et al. [5] showed that all eigenvalues of M are located in the following disk:

$$\left\{z \in C : \left|z - \frac{trM}{n}\right| \leq \sqrt{\|M\|_F^2 - \frac{|trM|^2}{n} - \max_{1\leq k\leq n-1}\left(\|B_{k,n-k}\|_F - \|C_{n-k,k}\|_F\right)^2}\right\}. \tag{2}$$

Let $M(x) = \begin{bmatrix} A_k & xB_{k,n-k} \\ x^{-1}C_{n-k,k} & D_{n-k} \end{bmatrix}$, where A_k is a $k \times k$ principal submatrix of M $(1 \leq k \leq n-1)$ and x is any non-zero real number.

For convenience, we write, respectively.

$$\triangle_M(k,x) = \|M\|_F^2 - \left[\left(1-x^2\right)\|B_{k,n-k}\|_F^2 + \left(1-x^{-2}\right)\|C_{n-k,k}\|_F^2\right] - \frac{|trM|^2}{n}$$

and

$$f_M(k,x) = \left((\triangle_M(k,x))^2 - \frac{1}{2}\|[M(x), M(x)^*]\|_F^2\right)^{\frac{1}{2}} + \frac{|trM|^2}{n}.$$

In [6], Wu et al. proved that

$$\left|\lambda_j(M) - \frac{trM}{n}\right| \leq \min_{x\neq 0}\min_{1\leq k\leq n-1}\sqrt{\frac{n-1}{n}}\left(f_M(k,x) - \frac{|trM|^2}{n}\right)^{\frac{1}{2}}, \tag{3}$$

which is a refinement of inequality (2).

It is natural to ask whether stronger inequality of (2) might be proved. This is a part of the motivation for our study.

2. Main result

We let the symbol S_l denote the set $\{1, \cdots n\}\backslash\{l\}$ for $l = 1, 2, \cdots, n$. In this section, a sharper estimation of the eigenvalues is presented. In order to obtain our result, we need the following lemmas.

Lemma 1. [7] *Let $A \in M_n$ with $n \geq 3$, then*

$$\left|\lambda_l(A) - \frac{trA}{n}\right|^2 \leq \frac{n-1}{n}\left(\sum_{j=1}^{n}|\lambda_j(A)|^2 - \frac{|trA|^2}{n} - \frac{1}{2}s^2(A)\right)$$

for $l = 1, 2, \cdots, n$ and $s(A) = \min_{1\leq l\leq n}\max_{j,k\in S_l}|\lambda_j(A) - \lambda_k(A)|$.

Lemma 2. [7] *Let* $A \in M_n$, *then*

$$\sum_{j=1}^{n} |\lambda_j(A)|^2 \leq \sqrt{\left(\|A\|_F^2 - \frac{|trA|^2}{n}\right)^2 - \frac{\|[A, A^*]\|_F^2}{2}} + \frac{|trA|^2}{n}.$$

Next we give a new proof of Lemma 2.2 in [6], which plays a key role in their discussion.

Lemma 3. *Let* $M = \begin{bmatrix} A_k & B_{k,n-k} \\ C_{n-k,k} & D_{n-k} \end{bmatrix}$ *with eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$, *then*

$$\sum_{j=1}^{n} |\lambda_j|^2 \leq \min_{x \neq 0} \min_{1 \leq k \leq n-1} f_M(k, x)$$

is valid for any non-zero number x.

Proof. Let $X = \begin{bmatrix} xI_k & 0 \\ 0 & I_{n-k} \end{bmatrix}$, then $M(x) = XMX^{-1}$, where I_k is a $k \times k$ unit matrix. Obviously, $M(x)$ is similar to M. By Lemma [2], we have

$$\begin{aligned} \sum_{j=1}^{n} |\lambda_j(M)|^2 &= \sum_{j=1}^{n} |\lambda_j(M(x))|^2 \qquad (4) \\ &\leq \sqrt{\left(\|M(x)\|_F^2 - \frac{|trM(x)|^2}{n}\right)^2 - \frac{\|[M(x), M(x)^*]\|_F^2}{2}} + \frac{|trM(x)|^2}{n} \\ &= \sqrt{\left(\|M(x)\|_F^2 - \frac{|trM|^2}{n}\right)^2 - \frac{\|[M(x), M(x)^*]\|_F^2}{2}} + \frac{|trM|^2}{n}, \end{aligned}$$

where

$$\|M(x)\|_F = \left(\|M\|_F^2 - \left[\left(1 - x^2\right)\|B_{k,n-k}\|_F^2 + \left(1 - x^{-2}\right)\|C_{n-k,k}\|_F^2\right]\right)^{\frac{1}{2}}. \qquad (5)$$

Combing inequality (4) and equality (5), we conclude Lemma 3.

We now focus on the location of the eigenvalues of complex matrices.

Theorem 1. *Let* $M = \begin{bmatrix} A_k & B_{k,n-k} \\ C_{n-k,k} & D_{n-k} \end{bmatrix}$ *with eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *(* $n \geq 3$*), then all of eigenvalues of* M *are included by the following disk:*

$$\left|\lambda_l(M) - \frac{trM}{n}\right| \leq \min_{x \neq 0} \min_{1 \leq k \leq n-1} \sqrt{\frac{n-1}{n}} \left(f_M(k, x) - \frac{|trM|^2}{n} - \frac{1}{2}s^2(M)\right)^{\frac{1}{2}}$$

for $l = 1, 2, \cdots, n$ *and* $s(M) = \min_{1 \leq l \leq n} \max_{j,k \in S_l} |\lambda_j(M) - \lambda_k(M)|$.

Proof. Combining Lemmas 2.1 and 2.3, we deduce that

$$\begin{aligned}\left|\lambda_l(M) - \frac{trM}{n}\right|^2 &\leq \frac{n-1}{n}\left(\sum_{j=1}^{n}|\lambda_j(M)|^2 - \frac{|trM|^2}{n} - \frac{1}{2}s^2(M)\right)\\ &\leq \frac{n-1}{n}\left(\min_{x\neq 0}\min_{1\leq k\leq n-1} f_M(k,x) - \frac{|trM|^2}{n} - \frac{1}{2}s^2(M)\right)\\ &\leq \min_{x\neq 0}\min_{1\leq k\leq n-1}\frac{n-1}{n}\left(f_M(k,x) - \frac{|trM|^2}{n} - \frac{1}{2}s^2(M)\right).\end{aligned}$$

Therefore,

$$\left|\lambda_l(M) - \frac{trM}{n}\right| \leq \min_{x\neq 0}\min_{1\leq k\leq n-1}\sqrt{\frac{n-1}{n}}\left(f_M(k,x) - \frac{|trM|^2}{n} - \frac{1}{2}s^2(M)\right)^{\frac{1}{2}}$$

for $s(M) = \min_{1\leq l\leq n}\max_{j,k\in S_l}|\lambda_j(M) - \lambda_k(M)|$.

This completed the proof.

For complex matrix with order $n(n > 2)$, then the computation of Theorem 2.4 requires approximately $\frac{n^3}{2}$ additional calculations compared to the computation of inequality(3). This indicates that its computational complexity is greater than the computational complexity in (3). But, in theory, Theorem 2.4 is a refinement of (3).

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