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Asymptotic Behavior of Global Solutions of an Anomalous Gray-Scott Model

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Abstract. The object of this paper is to prove that existence global and asymptotic behavior of solutions for anomalous coupled reaction diffusion system (Gray-Scott model) with homogeneous Neumann boundary conditions. The existence and uniqueness of the local solution are given by the Banach fixed point theorem. Further, the asymptotic behavior is investigated by technique semi group estimates and the Sobolev embedding theorem .

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1. Introduction

The Gray-Scott system is a reaction-diffusion system. This means that it models a process that consists of a reaction and diffusion. In the case of the Gray-Scott model that reaction is a chemical reaction between two substances W and Z , both of which diffuse over time. During the reaction W gets used up, while Z is produced.

The system is characterised by two parameters: f is the rate at which is replenished, and k controls the rate at which Z is removed from the system. Varying these parameters leads to a wide range of interesting patterns, some of which look quite familiar.

The Gray-Scott system models the chemical reaction $W + 2Z \to 3Z$. This reaction consumes$W$ and produces Z. Consequently, the amount of both substances needs to be controlled to maintain the reaction. This is done by adding W at the "feed rate" f and removing Z at the "kill rate" k The removal of Z can also be described by another chemical reaction:

$Z \to P$. For this reaction P is an inert product, meaning it doesn't react. In this case the Parameter k controls the rate of the second reaction. Both substances diffuse over time at the diffusion rates d_1 and d_2. The Gray-Scott system is defined by two equations that describe the behavior of two reacting substances:

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$$\begin{cases} \frac{\partial w}{\partial t} = -d_1(-\Delta)w - w^2z + f(1-w) \\ \\ \frac{\partial z}{\partial t} = -d_2(-\Delta)z + w^2z - (f+k)z \end{cases}$$

The variables in these equations w and z are the concentrations of the two reacting substances W and Z. On the left-hand side of each equation is the time derivative of one of these concentrations, describing the rate at which it changes. The right-hand sides of the equations both contain three separate terms. The first describes the reaction between the two substances. Since one W and two Z react, the corresponding term includes w to the power of one and z to the power of two: w^2z as W gets consumed by the reaction, the term has a negative sign in the first equation. In the second equation it has a positive sign, as is produced in the reaction.

The second term of the first equation describes the rate at which W is replenished externally. This is necessary, as W would otherwise simply be used up. The feed rate is given by the parameter f. f is multiplied by $1-w$ to ensure that w is replenished at a rate dependent on the current concentration, which never exceeds Z does not need to replenished, since it is produced in the reaction. Booth, it needs to be removed in order maintain the reaction. The rate of removal, the kill rate, is controlled by the parameter k. To remove Z faster than W is added,k is added to f and multiplied by z, since the removal of Z is also supposed to be dependent on its concentration. The last term in both equations describe the diffusion of and, respectively.

In this current manuscript, we are interested in the fractional Gray Scott model which arises in the modelling of autocatalytic reactions. We study the global existence and asymptotic behavior of solutions to the system:

$$\begin{cases} \frac{\partial w}{\partial t} = -d_1(-\Delta)^{\delta}w - w^2z + f(1-w) & \text{in } \Omega\times\mathbb{R}^+, \\ \\ \frac{\partial z}{\partial t} = -d_2(-\Delta)^{\epsilon}z + w^2z - (f+k)z & \text{in } \Omega\times\mathbb{R}^+, \end{cases} \tag{1}$$

subjected with the boundary and initial conditions

$$\begin{cases} \frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 & \text{in } \partial\Omega\times\mathbb{R}^+, \\ \\ w\left(.,0\right) = w_0\left(.\right), z\left(.,0\right) = z_0\left(.\right) & \text{in } \Omega, \end{cases}$$

here Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, $w\left(t,x\right)$ and $z(t,x)$, $t\geq 0$, $x\in\Omega$ are real valued functions. $w_0\left(.\right)$ and $z_0\left(.\right)$ are non negatives, the constants d_1, d_2, k are positive and $0<\epsilon<1, 0<\delta<1$ and $f\geq 0$.

The system, obtained by replacing the fractional Laplacian by the classical one, is known as the chemical diffusion Gray Scott. This model was proposed by Gray and Scott in 1983. Later on , the Gray Scott model has attracted significant attention. It has been subject of a number of papers , for example Kirane [1], Hollis [4], Roth[9], Kouachi [10], Lin[6] ,..., etc.

Our paper is organized as follows. In section 2, we present some preliminaries and definitions which used in the following sections. In section 3, the definition of mild solution of the system (1) and the theorem of local existence are obtained. The main results of global existence and large time behavior for the solution are presented in section 4.

2. Notations and preliminary

In this section, we introduce some notations, definitions and lemmas which will be used in the sequel. Where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, we denote$(-\Delta_N)^\delta$ the fractional power of the Laplacian in Ω with homogenous Neumann boundary condition .

Let $\alpha_m\{m = 0, 1, ..., +\infty\}$ be the eigenvalues of the Laplacian operator in $L^2(\Omega)$ with homogenous Neumann boundary condition and let Ψ_m be the corresponding eigenfunction i.e. $\begin{cases} (-\Delta_N)^\delta \Psi_m = \alpha_m^\delta \Psi_m & \text{in } \Omega, \\ \frac{\partial \Psi_m}{\partial t} = 0 & \text{in } \partial\Omega, \end{cases}$

and

$$D((-\Delta_N)^\delta) = \{w \in L^2(\Omega)\,; \frac{\partial w}{\partial \eta} =, 0 \left\|(-\Delta_N)^\delta w\right\|_{L^2(\Omega)} \prec +\infty\}$$

$$\left\|(-\Delta_N)^\delta w\right\|_{L^2(\Omega)} = \sum_{m=1}^{+\infty} \left|\alpha_m^\delta \langle w, \Psi_m\rangle\right|^2$$

so for $w \in D((-\Delta_N)^\delta)$ we get

$$(-\Delta_N)^\delta w = \sum_{m=1}^{+\infty} \alpha_m^\delta \langle w, \Psi_m\rangle \Psi_m$$

We obtain the following integration by parts formula

$$\int_\Omega w(x)(-\Delta_N)^\delta z(x)dx = \int_\Omega z(x)(-\Delta_N)^\delta w(x)dx, \text{ for } w, z \in D((-\Delta_N)^\delta) \tag{2}$$

We will employ the following important inequalities of Strook and Varopoulos see([8], Theorem1)

$$\int_\Omega w(x)(-\Delta_N)^\delta w(x)dx \geq 0, \text{ for } w \in D((-\Delta_N)^\delta) \tag{3}$$

$$\int_\Omega w^{p-1}(x)(-\Delta_N)^\delta w(x)dx \geq \frac{4(p-1)}{p^2}\int_\Omega \left|(-\Delta_N)^{\frac{\delta}{2}} w(x)^{\frac{p}{2}}\right|^2 dx \geq 0, p \succ 1 \tag{4}$$

for all $w \in L^p(\Omega)$ such that $(-\Delta_N)^{\frac{\delta}{2}} w \in L^p(\Omega)$

Definition 1. *For $p \in (1, +\infty)$, we denote S_p (resp. T_p) the realisation of $(-\Delta)^{\delta}$ (resp. $(-\Delta)^{\epsilon}$) with a homogeneous Neumann boundary condition in $L^p(\Omega)$.*

It is well known that $-S_p$ (resp. $-T_p$) is a sectorial operator (see [4]); hence $-S_p$ (resp. $-T_p$) generates an analytic semigroup $\{e^{-tS_p}\}_{t\geq 0}$ $\left(\text{resp. } \{e^{-tT_p}\}_{t\geq 0}\right)$

Lemma 1. *For $\lambda \in [0,1]$ and $\mu \in \mathbb{R}$, there exists a constant $M(\lambda, \mu)$ such that, for all $t \succ 0$,*

$$\int_0^t c(s)^{-\lambda} e^{\mu s} ds \leq \begin{cases} M(\lambda,\mu)\, e^{\mu t}, if\ \mu \succ 0, \\ M(\lambda,\mu)\,(t+1), if\ \mu = 0, \\ M(\lambda,\mu)\,, if\ \mu \prec 0, \end{cases}$$

here

$$c(t) = \min\{t, 1\}.$$

Proof. see [5]

Lemma 2. *Let $p, q, r \in [0,1]$, $r \leq p \leq q$ and $\lambda \in [0,1]$ be such that $\frac{1}{p} = \frac{\lambda}{r} + \frac{1-\lambda}{q}$, we gain*

$$\left\| e^{-tS_p} w \right\|_p \leq e^{-\alpha_1^{\delta} \lambda t} c(t)^{\frac{-N}{2\delta}(\frac{1}{r}-\frac{1}{p})} \left\| w \right\|_r$$

Proof. see [3] By the interpolation inequality, we obtain

$$\left\| e^{-tS_p} w \right\|_p \leq \left\| e^{-tS_p} w \right\|_r^{\lambda} \left\| e^{-tS_p} w \right\|_q^{1-\lambda} \tag{5}$$

Practising the following inequalities

$$\left\| e^{-tS_p} w \right\|_r \leq e^{-\alpha_1^{\delta} t} \left\| w \right\|_r \tag{6}$$

and

$$\left\| e^{-tS_p} w \right\|_q \leq t^{\frac{-N}{2\delta}(\frac{1}{r}-\frac{1}{q})} \left\| w \right\|_r, \tag{7}$$

we get

$$\left\| e^{-tS_p} w \right\|_p \leq e^{-\alpha_1^{\delta} \lambda t} t^{\frac{-N}{2\delta}(1-\lambda)(\frac{1}{r}-\frac{1}{q})} \left\| w \right\|_r, \tag{8}$$

here $(1-\lambda)(\frac{1}{r} - \frac{1}{q}) = (\frac{1}{r} - \frac{1}{p})$.
Consequently, we gain

$$\begin{aligned} \left\| e^{-tS_p} w \right\|_p &\leq e^{-\alpha_1^{\delta} \lambda t} t^{\frac{-N}{2\delta}(\frac{1}{r}-\frac{1}{p})} \left\| w \right\|_r \\ &\leq e^{-\alpha_1^{\delta} \lambda t} c(t)^{\frac{-N}{2\delta}(\frac{1}{r}-\frac{1}{p})} \left\| w \right\| r \end{aligned} \tag{9}$$

where

$$c(t) = \min\{t, 1\}.$$

Remark : *An main result in [8] comes in our case $\forall \xi \succ 0, \exists M(\xi) \in \mathbb{R}^+$ such that*

$$\left\| e^{-tT_p} z \right\|_{\infty} \leq M(\xi) t^{\frac{-N}{(N+\xi)^{\epsilon})}} \left\| z \right\|_{\frac{N}{2}+\xi'}, for\ all\ z \in L^{\infty}(\Omega), t \succ 0. \tag{10}$$

The relation (10) with $\epsilon = 1$ injected with a successive iterations method have been used in ([8] proposition 3.3) to prove that the solutions are bounded in $C\left(\overline{\Omega}\right)$.

3. Local existence

In this section, we investigate the local existence of mild solutions to the problem (1)-(2)

Lemma 3 (definition). *(Mild solution)*

Let $w_0, z_0 \in L^\infty(\Omega)$ *and* $T \succ 0$. *We say that* $(w,z) \in C([0,T]; L^\infty(\Omega) \times L^\infty(\Omega))$ *is a mild solution of (1)-(2) if* w, z *satisfy the followings integral equations for* $t \in [0,T]$:

$$\begin{cases} w(t) = e^{-d_1 t S_p} w_0 + \int_0^t e^{-d_1(t-s)S_p}(-w^2 z + f(1-w))ds, \\ z(t) = e^{-d_2 t T_p} z_0 + \int_0^t e^{-d_2(t-s)T_p}(w^2 z - (f+k)z)ds. \end{cases} \tag{11}$$

Theorem 1. *local existence*

Let $w_0, z_0 \in C(\overline{\Omega})$, *then there exist a maximal time* $T_{\max} \succ 0$ *and a unique mild solution* $(w,z) \in C([0,T_{\max}); C(\overline{\Omega}) \times C(\overline{\Omega}))$ *to the system (1)-(2) , with the alternative:*

-either $T_{\max} = +\infty$;

-or $T_{\max} \prec +\infty$ *and* $\lim_{t\to T_{\max}}(\|w(t)\|_\infty + \|z(t)\|_\infty) = +\infty$.

Proof. $\forall T \succ 0$, *we define the Banach space:*

$$B_T := \{(w,z) \in C\left([0,T]; C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right)\right); \|(w,z)\| \le 2\|(w_0,z_0)\| = L\},$$

where $\|.\|_\infty := \|.\|_{L^\infty(\Omega)}$ *and*$\|.\|$ *is the norm of* B_T *defined by:*

$$\|(w,z)\| := \|w\|_{L^\infty([0,T];L^\infty(\Omega))} + \|z\|_{L^\infty([0,T];L^\infty(\Omega))}.$$

Next, $\forall (w,z) \in B_T$,, *we define* $\Phi(w,z); = (\Phi_1(w,z), \Phi_2(w,z))$ *where for* $t \in [0,T]$

$$\Phi_1(w,z) = e^{-d_1 t S_p} w_0 + \int_0^t e^{-d_1(t-s)S_p}(-w^2 z + f(1-w))ds$$

and

$$\Phi_2(w,z) = e^{-d_2 t T_p} z_0 + \int_0^t e^{-d_2(t-s)T_p}(w^2 z - (f+k)z)ds.$$

We will show the local existence by the Banach fixed point theorem.

- $\Phi : B_T \to B_T$:*Let* $(w,z) \in B_T$. *Practising the estimate (8) (with* $r = p = +\infty$*),we attain*

$$\begin{aligned} \|\Phi_1(w,z)\|_\infty &\le \|w_0\|_\infty + \int_0^t \left\|-w^2 z(s)\right\|_\infty ds + f\int_0^t \|(1-w)(s)\|_\infty ds \\ &\le \|w_0\|_\infty + TL^3 + fT + fTL. \end{aligned}$$

  Similarly, we obtain

$$\|\Phi_2(w,z)\|_\infty \le \|z_0\|_\infty + TL^3 + (f+k)TL.$$

Therefore we get,

$$\begin{aligned} \|\Phi(w,z)\|_\infty &\le (\|w_0\|_\infty + \|z_0\|_\infty) + 2TL^3 + (2f+k)TL + fT \\ &\le 2(\|w_0\|_\infty + \|z_0\|_\infty) \end{aligned}$$

by choosing T such that $T \le \frac{1}{2(6L^2+(2f+k))}$.

Hence $\Phi(w,z) \in B_T$ for $T \le \frac{1}{2(6L^2+(2f+k))}$.

- *$\Phi(w,z)$ is contraction map : for $(w,z), (w'', z") \in B_T$, we obtain*

$$\begin{aligned} \|\Phi_1(w,z) - \Phi_1(w",z")\|_\infty &\le \int_0^t \left\|-w^2 z(s) + w"^2 z"(s)\right\|_\infty ds + f\int_0^t \|(w-w")(s)\|_\infty ds \\ &\le \int_0^t \left\|w^2\right\| \|z-z"\| + \|z"\| (\|w\| + \|w"\|)(\|w-w"\| ds + \\ &\quad f\int_0^t \|w-w"\| ds \\ &\le 3TL^2 + fT \|(w,z)-(w",z")\| \end{aligned}$$

by the same way,

$$\|\Phi_1(w,z) - \Phi_1(w",z")\|_\infty \le 3TL^2 + (f+k)T \|(w,z)-(w",z")\|$$

So

$$\begin{aligned} \|\Phi(w,z) - \Phi(w",z")\| &\le 6TL^2 + 2fT + kT \\ &\le 1/2 \,|\|(w,z)-(w",z")\||\,. \end{aligned}$$

for $T \le \frac{1}{2(6L^2+(2f+k))}$.

Consequently, in view of the Banach fixed point theorem ç, Φ admits a fixed point on B. Thus the system (1)-(2) has a mild solution.

The solution can be extended on a maximal interval $[0, T_{\max})$ where $T_{\max} := \sup\{T \succ 0; (w,z)\}$. is a solution to (1)-(2)

4. Global existence and asymptotic behavior

In this section, we state and prove the main result using the ideas of [2] or [1].

Theorem 2. *Let $(w_0, z_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ be such that $w_0 \ge 0, z_0 \ge 0$. Then there exists a unique global solution (w,z) of (1)-(2) which satisfy :*

$\cdot w(x,t) \geq 0,\ z(x,t) \geq 0;\ x \in \Omega,\ t \geq 0.$
$\cdot w \in C\left(\mathbb{R}^{+}, C\left(\overline{\Omega}\right)\right),\ \ z \in C\left(\mathbb{R}^{+}, C\left(\overline{\Omega}\right)\right).$
$\cdot \lim_{t\to+\infty} \|z(t)\|_{\infty} = 0;\ \ \exists w_{\infty} \geq 0.$*such that* $\lim_{t\to+\infty} \|w(x,t) - w_{\infty}\|_{\infty} = 0.$

Proof. .

$\cdot$***Step1.***

We define $w^{+} = \max(0, w)$ *and* $w^{-} = \max(0, -w)$. *We write* $w = w^{+} - w^{-}$, *multiply the first equation of system (1) by* $(-w^{-})$ *and integrate over* Ω; *we get*

$$\int_{\Omega} \frac{\partial w^{-}}{\partial t} w^{-} dx = -d_1 \int_{\Omega} -S_p w^{-} w^{-} dx - \int_{\Omega} (w^{-2} z) w^{-} dx + \int_{\Omega} f(1 - w^{-}) w^{-} dx.$$

By estimate (4) , we have

$$\frac{d}{dt} \int_{\Omega} (w^{-})^2 dx \leq 2[\int_{\Omega} (w^{-2})(z w^{-}) dx + \int_{\Omega} f w^{-} dx + \int_{\Omega} f (w^{-})^2 dx.$$

Since (w, z) *is a local solution on* $[0, T_{\max})$, *then* w *and* z *are bounded on* $[0, T]$ *for* $T \prec T_{\max}$; *Furthermore, there exist continuous functions* $m(t)$ *and* $h(t)$ *such that* $\|z(t)\|_{\infty} \leq m(t)$ *and* $\|w(t)\|_{\infty} \leq h(t)$.

It holds that

$$\frac{d}{dt} \int_{\Omega} (w^{-})^2 dx \leq 2[(m(t)h(t) + f) \int_{\Omega} (w^{-2}) dx + f h(t)]. \tag{12}$$

As $\int_{\Omega} (w^{-2})(0) dx = 0$ *and* $f \geq 0$, *Gronwall's inequality [3] allows us to attain* $\int_{\Omega} (w^{-2}) dx = 0$; *Consequently,* $w(x,t) \geq 0$.

By same manner, we gain

$$\int_{\Omega} \frac{\partial z^{-}}{\partial t} z^{-} dx = -d_2 \int_{\Omega} -T_p z^{-} z^{-} dx + \int_{\Omega} w^2 (z^{-})^2 dx - \int_{\Omega} (f + k)(z^{-})^2 dx.$$

Employing inequality (4) , we obtain

$$\frac{d}{dt} \int_{\Omega} (w^{-})^2 dx \leq 2[\int_{\Omega} (w^2 - (f + k))(z^{-})^2 dx.$$

It follows that

$$2[h^2(t) - (f + k)] \int_{\Omega} (z^{-})^2 dx.$$

By integration we have $z^{-} = 0$ *which gives* $z(x,t) \geq 0$.

$\cdot$ ***Step 2.***

We will derive a uniform bound of $\|w(t)\|_{\infty}$.

Multiplying the first equation of (1) by w^{p-1} *and integrating over* Ω, *we get* $\frac{d}{dt} \int_{\Omega} w^p dx \leq 0$ *thanks to relation (5) and* $f = 0$. *Hence, we get*

$$\|w(t)\|_{\infty} \leq \|w_0\|_{\infty}, \forall t \in [0, T_{\max}). \tag{13}$$

Now, we integrate the first equation of (1) over Ω and practise the integration by parts formula (3) which haven $\int\limits_{\Omega}(-\Delta_N)^{\delta} w(x)dx = 0$ and $f = 0$; we attain

$$\int_{\Omega} \frac{\partial w}{\partial t} dx = -\int\limits_{\Omega} w^2 z dx \leq 0$$

Therefore, the function $t \mapsto \int_{\Omega} w(x,t)dx$ is nonincreasing. Since $w \geq 0$. Then it admits a limit as $t \to +\infty$:

$$\lim_{t\to+\infty} \frac{1}{|\Omega|} \int_{\Omega} w(x,t)dx = w_{\infty} \geq 0.$$

We add the equations of system(1) and we integrate over Ω and putting $f = 0$ to obtain

$$\frac{d}{dt} \int_{\Omega} (w+z)dx = -k \int_{\Omega} z dx \leq 0. \tag{14}$$

We observe that the function $t \mapsto \int_{\Omega}(w+z)dx \geq 0$ is nonincreasing. Accordingly admits a limit; Also we obtain

$$\lim_{t\to+\infty} \int_{\Omega} z(x,t)dx = l \geq 0.$$

Thence, $z \in L^{\infty}\left(\mathbb{R}^+; L^1(\Omega)\right)$. Thanks to estimate (11) we can proceed analogously to the proof of of([8],proposition 3.3). Consequently, we get $z \in C\left(\mathbb{R}^+, C\left(\overline{\Omega}\right)\right)$.

· ***Step 3.***

Integrating the equation (15) over $[0,T]$, we have

$$\begin{aligned} -k \int_0^t \int_{\Omega} z(x,s)dxds &= \int_{\Omega} (w_0 + z_0)dx - \int_{\Omega} (w+z)(x,t)dx \\ &\leq \int_{\Omega} (w_0 + z_0)dx. \end{aligned}$$

Also, $\int_0^t \int_{\Omega} z(x,s)dxds$ is finite. Moreover, $z(x,t)$ is uniformly continuous in t.

In fact, let $h \succ 0$ and $0 \leq t \prec t+h \prec T_{\max}$, it follows that,

$$\begin{aligned} \|z(t+h) - z(t)\|_{\infty} &\leq \left\|(e^{-d_2 h T_p} - I)e^{-d_2 t T_p} z_0\right\|_{\infty} + \int_0^t \left\|(e^{-d_2 h T_p} - I)e^{-d_2(t-h)T_p} w^2 z(s)\right\|_{\infty} ds \\ &\quad -(f+k)\int_0^t \left\|(e^{-d_2 h T_p} - I)e^{-d_2(t-h)T_p} z(s)\right\|_{\infty} ds + \int_t^{t+h} \left\|e^{-d_2(t+h-s)T_p} w^2 z(s)\right\|_{\infty} ds \\ &\quad -(f+k)\int_t^{t+h} \left\|e^{-d_2(t+h-s)T_p} z(s)\right\|_{\infty} ds \\ &= A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned}$$

Employing ([6],lemma2.1), we get that for every $\theta \in (0.1)$

$$\begin{aligned} A_1 &\leq M_2(\theta) h^\theta \left\| T_p^\theta e^{-d_2 t T_p} z_0 \right\|_\infty \\ &\leq M_2(\theta) M_1(\theta) h^\theta t^{-\theta} e^{-d_2 \alpha_1^\epsilon t} \|z_0\|_\infty \end{aligned}$$

We also find

$$A_2 \leq M_2(\theta) M_1(\theta) h^\theta \int_0^t (t-s)^{-\theta} e^{-d_2 \alpha_1^\epsilon (t-s)} \left\| w^2 z(s) \right\|_\infty ds.$$

Using lemma 1, we can observe that

$$\int_0^t (t-s)^{-\theta} e^{-d_2 \alpha_1^\epsilon (t-s)} \leq M_3(\theta, -d_2 \alpha_1^\epsilon).$$

As $z \in C(\mathbb{R}^+, C(\overline{\Omega}))$ *we obtain* $\|z(t)\|_\infty \leq C, \forall t \succ 0$, *here* C *is positive constant.*
So, we have

$$A_2 \leq M_2(\theta) M_1(\theta) M_3(\theta, -d_2 \alpha_1^\epsilon) C h^\theta \|w_0\|^2.$$

Similarly, it follows that

$$A_3 \leq M_2(\theta) M_1(\theta) M_3(\theta, -d_2 \alpha_1^\epsilon) C h^\theta.$$

Using the relation see([7])

$$\left\| e^{-d_2 t T_p} w \right\|_\infty \leq R \|w\|_\infty,$$

we obtain

$$\begin{aligned} A_4 &= \int_0^h \left\| e^{-d_2 \tau T_p} w^2 z(t-h-\tau) \right\|_\infty d\tau \\ &\leq MRh \|w\|_\infty^2. \end{aligned}$$

Therefore, $\forall t \succeq \beta \succ 0$

$$\|z(t+h) - z(t)\|_\infty \leq M(\theta, \epsilon, \beta) h^\theta.$$

Consequently, $\lim_{t \to +\infty} \int_\Omega z(x,t) dx = 0$.

Then, for $\theta \in (0.1)$, *applying* T_p^θ *to both sides of the second equation of (12) and estimate, we have*

$$\left\| T_p^\theta z(t) \right\|_p \leq \left\| T_p^\theta e^{-d_2 t T_p} z_0 \right\|_p + \int_0^t \left\| T_p^\theta e^{-d_2 (t-s) T_p} w^2 z(s) \right\|_p ds.$$

Using ([3],theorem1.4.3), we obtain

$$\left\| T_p^\theta z(t) \right\|_p \leq Rc(t)^{-\theta} e^{-d_2 \alpha_1^\epsilon t} \|z_0\|_p + R |\Omega|^{\frac{1}{P}} \|w_0\|_\infty^2 \|z_0\|_\infty \int_0^t c(t-s)^{-\theta} e^{-d_2 \alpha_1^\epsilon (t-s)} ds.$$

Thus by Lemma (1) and $\forall t \geq \beta$, *we get* $\left\|T_p^{\theta} z(t)\right\|_p \leq R(\theta, p, \beta)$.

Tereupon $\{z(t)\}_{t \geq \beta}$ *is uniformly bounded in* $D(T_p^{\theta})$*; so by Sobolev's imbeding theorem, the compactness of* $\{z(t)\}_{t \geq \beta}$ *in* $C\left(\overline{\Omega}\right)$ *is assured.*

Therefore, there exists a sequence $\{t_j\}_{j \geq 0}, t_j \to +\infty$ *such that* $z(t_j) \to z^*$ *in* $C\left(\overline{\Omega}\right)$ *as* $j \to +\infty$.

Away we find $\lim_{t \to +\infty} \|z(t)\|_{\infty} = 0$.

Similarly, we can prove that $\{w(t)\}_{t \geq \beta}$ *is precompact in* $C\left(\overline{\Omega}\right)$*; So, there exists a sequence* $\{\tau_j\}_{j \geq 0}, \tau_j \to +\infty$ *such that* $w(\tau_j) \to w^*$ *in* $C\left(\overline{\Omega}\right)$ *as* $j \to +\infty$.

We have $(w+z)(t) \to w^*$ *in* $C\left(\overline{\Omega}\right)$, *then* $(w+z)(t) \to w^*$ *in* $L^1(\Omega)$ *as* $t \to +\infty$.

Using the fact that $\lim_{t \to +\infty} \int_{\Omega} z(x,t)dx = 0$ *and* $\lim_{t \to +\infty} \frac{1}{|\Omega|} \int_{\Omega} w(x,t)dx = w_{\infty}$, *we gain* $(w+z)(t) \to w_{\infty}$ *in* $L^1(\Omega)$ *as* $t \to +\infty$. *By uniqueness of the limit,* $w^* = w_{\infty}$.

Theorem 3. *Let* (w, z) *be the solution of (1)-(2). Therefore assume that* $k - w_{\infty} \succ 0$. *Then there exist positive constants* T *and* R *such that*

$$\|w(t) - w_{\infty}\|_{\infty} \leq \begin{cases} R \exp^{-\varkappa(t-T)} & \text{if } 2d_1\alpha_1^{\delta}\lambda \neq h(w_{\infty}) \\ R(t-T+1)\exp^{-\varkappa(t-T)} & \text{if } 2d_1\alpha_1^{\delta}\lambda = h(w_{\infty}), \end{cases}$$

$\|z(t)\|_{\infty} \leq R\exp(-h(w_{\infty})(t-T), t \geq T$

where $\varkappa = \min\left\{h(w_{\infty}), 2d_1\alpha_1^{\delta}\lambda\right\}, h(w_{\infty}) = (w_0 + \varepsilon)^2 - k \succ 0$.

Proof. [***Proof of Theorem***]

For $\varepsilon \succ 0$, *there exists a constant* $T \succ 0$ *such that fort* $\geq T$ $w_{\infty} - \varepsilon \prec w(t) \prec w_{\infty} + \varepsilon$.

Putting $0 \prec \varepsilon \prec k - w_{\infty}$.

We multiply the second equation of (1) by z^{p-1} *and integrate over* Ω; *It follows that*

$$\frac{d}{dt}\int_{\Omega} z^p dx \leq p\int_{\Omega}(w^2 - (f+k))z^p dx, \forall t \geq T,$$

in the light of relation (5) .

So

$$\frac{d}{dt}\|z(t)\|_p^p \leq p((w_{\infty}+\varepsilon)^2 - (f+k))\|z(t)\|_p^p, \forall t \geq T.$$

Thus, for $1 \leq p \leq +\infty$, *we have*

$$\|z(t)\|_p \leq \|z(T)\|_p \exp((w_{\infty}+\varepsilon)^2 - (f+k))(t-T), t \geq T. \tag{15}$$

Which implies

$$\|z(t)\|_{\infty} \leq \|z(T)\|_{\infty} \exp((w_{\infty}+\varepsilon)^2 - (f+k))(t-T), t \geq T. \tag{16}$$

For the rate of convergence of $\|w(t) - w_{\infty}\|_{\infty}$ *to zero, we define as in [14] two bounded linear operators* I *and* G *by*

$Iu := \langle u\rangle, Gu := u - \langle u\rangle$, here $\langle u\rangle := \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$.
Adding the two equations of problem (1) and integrating over Ω, we obtain if $f = 0$

$$\langle w(t)\rangle = -\langle z(t)\rangle + \langle w_0\rangle + \langle z_0\rangle - k\int_0^t \langle z(s)\rangle\, ds, \text{ as } t \to \infty.$$

It holds that

$$w_\infty = \langle w_0\rangle + \langle z_0\rangle - k\int_0^t \langle z(s)\rangle\, ds.$$

Hence

$$|\langle w(t) - w_\infty\rangle| \leq\prec \langle z(t)\rangle + k\int_0^t \langle z(s)\rangle\, ds.$$

Using inequality (16) we get

$$\begin{aligned}\int_t^\infty \langle z(s)\rangle\, ds &\leq \langle z(T)\rangle \int_t^\infty \exp((w_\infty+\varepsilon)^2 - (k))(s-T)ds \\ &\leq R_1 \langle z(T)\rangle \exp((w_\infty+\varepsilon)^2 - (k))(t-T), t \geq T,\end{aligned}$$

where $R_1 = \frac{1}{k-(w_\infty+\varepsilon)^2}$. Out

$$|\langle Iw(t) - w_\infty\rangle| \leq R_2 \langle z(T)\rangle \exp((w_\infty+\varepsilon)^2 - (k))(t-T), t \geq T, \tag{17}$$

where $R_2 = \max\{1, kR_1\}$.
Accordingly $w(t)$ satisfies the integral equation for $t \geq T$,

$$w(t) = e^{-d_1 t S_p} w_0 + \int_0^t e^{-d_1(t-s)S_p}(-w^2 z + f(1-w))(s)ds,$$

as $f = 0$, we get

$$\begin{aligned}w(t) &= e^{-d_1 t S_p} w_0 - \int_0^t e^{-d_1(t-s)S_p}(w^2 z)(s)ds, \\ &= e^{-d_1(t-T)S_p} w(T) - \int_T^t e^{-d_1(t-s)S_p}(-w^2 z)(s)ds,\end{aligned}$$

Thus we have

$$Gw(t) = e^{-d_1(t-T)S_p} Gw(T) - \int_T^t e^{-d_1(t-s)S_p} G(w^2 z)(s)ds.$$

Using Lemma 2 , we get the estimate

$$\left\| e^{-d_1(t-T)S_p} Gw(T) \right\|_p \leq R e^{-d_1 \alpha_1^\delta (t-T)} \|w(T)\|_p \tag{18}$$

Now , to calculate $A(t) = \int_T^t \left\| e^{-d_1(t-s)S_p} G(w^2 z)(s)ds \right\|_p$.

$$\int_T^t \left\| e^{-d_1(t-s)S_p} G(w^2 z)(s) ds \right\|_p \leq R \int_T^t c(t-s)^{\frac{-N}{2\delta}(\frac{1}{q}-\frac{1}{p})} e^{-2d_1\alpha_1^\delta \lambda (t-s)} \|z(s)\|_q$$

Using the estimate (16), we find

$$\begin{aligned} A(t) &\leq R\|z(T)\|_q \int_T^t c(t-s)^{\frac{-N}{2\delta}(\frac{1}{q}-\frac{1}{p})} e^{-2d_1\alpha_1^\delta\lambda(t-s)} e^{((w_\infty+\varepsilon)^2-(k))(s-T)} ds \\ &\leq R\|z(T)\|_q \int_0^{t-T} c(t-T-\tau)^{\frac{-N}{2\delta}(\frac{1}{q}-\frac{1}{p})} e^{-2d_1\alpha_1^\delta\lambda(t-T-\tau)} e^{(w_\infty+\varepsilon)^2-(k))(\tau)} d\tau \\ & R\|z(T)\|_q e^{(w_\infty+\varepsilon)^2-(k))(t-T)} \times \int_0^{t-T} c(t-T-\tau)^{\frac{-N}{2\delta}(\frac{1}{q}-\frac{1}{p})} e^{(((w_0+\varepsilon)^2-k)-2d_1\alpha_1^\delta\lambda)(t-T-\tau)} d\tau \end{aligned}$$

We note $A_\varepsilon(t) = \int_0^{t-T} c(t-T-\tau)^{\frac{-N}{2\delta}(\frac{1}{q}-\frac{1}{p})} e^{(((w_0+\varepsilon)^2-k)-2d_1\alpha_1^\delta\lambda)(t-T-\tau)} d\tau.$

If we choose p and q satisfying $0 \leq^{\frac{-N}{2\delta}(\frac{1}{q}-\frac{1}{p})} \prec 1$ *and use Lemma 1, we obtain*

∘ *When* $2d_1\alpha_1^\delta\lambda \prec (w_0+\varepsilon)^2 - k = h(w_\infty)$, *we choose ε such that* $0 \prec \varepsilon \prec h(w_\infty) - 2d_1\alpha_1^\delta\lambda$, *so*

$$A_\varepsilon(t) \leq R(\frac{N}{2\delta}(\frac{1}{q}-\frac{1}{p}), h(w_\infty) - 2d_1\alpha_1^\delta\lambda - \varepsilon) e^{(h(w_\infty)-2d_1\alpha_1^\delta\lambda-\varepsilon)(t-T)}.$$

∘ *When* $2d_1\alpha_1^\delta\lambda \geq (w_0+\varepsilon)^2 - k = h(w_\infty)$,

$$A_\varepsilon(t) \leq R(\frac{N}{2\delta}(\frac{1}{q}-\frac{1}{p}), h(w_\infty) - 2d_1\alpha_1^\delta\lambda - \varepsilon).$$

Hence

$$A(t) \leq R\|z(T)\|_q e^{-\mu(t-T)} \tag{19}$$

where

$$\mu = \begin{cases} 2d_1\alpha_1^\delta\lambda & \text{if } 2d_1\alpha_1^\delta\lambda \prec h(w_\infty) \\ (w_0+\varepsilon)^2 - k & \text{if } 2d_1\alpha_1^\delta\lambda \geq h(w_\infty) \end{cases}$$

Combining relation (19) and (20), we obtain

$$\|Gw(T)\|_p \leq R\|(w,z)(T)\|_p e^{-\mu(t-T)}; t \geq T. \tag{20}$$

So, we get from (18) and (21)

$$\|w(t) - w_\infty\|_p \leq R\|(w,z)(T)\|_\infty e^{-\mu(t-T)}; t \geq T.$$

Now, we have

$$w_\infty - Re^{-\mu(t-T)} \prec w(t) \prec w_\infty + Re^{-\mu(t-T)}; t \geq T.$$

Therefore, we can assert that $\forall 1 \leq p \leq +\infty$, $\|z(t)\|_p \leq R\|z(T)\|_p e^{-h(w_\infty)(t-T)}, t \geq T.$

Hence, $Gw(t) - w_\infty$ can be estimated as

$$|Gw(t) - w_\infty| \leq R\,|Gw(T)|\, e^{-h(w_\infty)(t-T)}; t \geq T.$$

and

$$\|Gw(t)\|_\infty \leq \begin{cases} R\exp^{-\varkappa(t-T)} & \text{if } 2d_1\alpha_1^\delta\lambda \neq h(w_\infty) \\ R(t-T+1)\exp^{-\varkappa(t-T)} & \text{if } 2d_1\alpha_1^\delta\lambda = h(w_\infty), \end{cases}$$

where $\varkappa = \min\left\{h(w_\infty), 2d_1\alpha_1^\delta\lambda\right\}$.
Accordingly

$$\|w(t) - w_\infty\|_\infty \leq \begin{cases} R\exp^{-\varkappa(t-T)} & \text{if } 2d_1\alpha_1^\delta\lambda \neq h(w_\infty) \\ R(t-T+1)\exp^{-\varkappa(t-T)} & \text{if } 2d_1\alpha_1^\delta\lambda = h(w_\infty). \end{cases}$$

5. Conclusion

In this paper, we considered a Gray Scott system with anomalous diffusion described by a fractional Laplacian power which accounts for a sub-diffusive situation. In addition to the global existence of bounded solutions, we showed the convergence of the solution $(w_\infty, 0)$. (w_∞): the final state of susceptible individuals and 0: the final state of infected individuals. The exponent of the fractional Laplacian influences the asymptotic behavior in time from w to w_∞.

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