**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 17, No. 1, 2024, 362-371
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# On Reverse Derivations in $d$-Algebras

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**Abstract.** In the present paper, we apply the concept of reverse derivation in rings on the concept of $d-algebra$ to obtain the concept called a left-right (resp. right-left) reverse derivations of $d-$algebra $X$ (briefly, $(l,r)$ $resp.$ $(r,l)-$ reverse derivation of $d-algebra$ ), we will also, define some concepts such as regular map, composition two maps and study the related properties. Moreover, the notions of partial ordered edge $d-algebra$ as well as $d-subalgebra$ and their relation to our current study are obtained. In addition, some illustrative examples and counterexamples are discussed.

**2020 Mathematics Subject Classifications**: 03G25, 06F35
**Key Words and Phrases**: $d-$algebras, $BCI-$algebras, reverse derivations

## 1. Introduction

In the theory of rings, the study of derivation plays an important role in the properties of algebraic systems, analysis and algebraic geometry. It is known that Boolean algebra was developed from Boolean logic and similarly, $BCI-algebra$ was developed from $BCI-logic$.

An algebric structures $BCK-algebras$ and $BCI-algebras$ introduced by Imai.Y and Iseki. K ( see [12], [11] ) and have been extensively investigated by many researchers. It is shown that the notion of $BCK-algebra$ is a generalization of $BCK-algebra$. That is, every $BCK-algebra$ is a $BCI-algebra$, but the converse is not true.

The concept of $d-algebra$ introduced in [20], [19] which is one of the generalization of $BCK-$algebras. Then they investigated some interesting relations between $BCK-algebras$ and $d-$algebras, they also studied ideal theory in $d-$algebras and introduced the notion of $d-$ideal and investigated some relations among them.

The concept of derivation on a ring $R$ is defined as an additive map $d : R \longrightarrow R$ satisfying the condition $d(ab) = d(a)b + ad(b)$ $\forall$ $a, b \in R$. The notion of reverse derivations on a ring $R$ introduced in a paper [8] of Herstein as a map $d : R \longrightarrow R$ satisfying the condition $d(ab) = d(b)a + bd(a)$ $\forall$ $a, b \in R$ ( and in the case of Lie algebras book of Jacobson

DOI: https://doi.org/10.29020/nybg.ejpam.v17i1.5025

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[13]). The case of Lie algebras is very important because the definition of reverse derivations coincides with the notion of antiderivations (about reverse derivations of algebras and superalgebras see, [7], [9] and [15]), also we can see that the reverse derivations are a particular case of Jordan derivations.

In the same way, many researchers have studied the reverse derivation on different types of algebraic structures, such as the prime ring, semiprime rings and associative algebras .( For more details, see [1], [10], [5], [21] and [16])

In [14], the notion of derivation in rings and near rings theory applied to $BCI-$ algebras also the notion called a regular derivation in $BCI-$ algebras introduced by them, and discussed some of its properties, defined a $d-invariant$ ideal, also they gave conditions for an ideal to be $d-$invariant. The concept of derivations in non-commutative rings extended to left derivations, central derivations and $d-$derivations.

Several authors, ( For example, you can refer to [6], [18] and [17]) have studied derivations in $d$ and $BCI-$algebras. Recently, Al-omary RM ([2]) introduced the notion of $(\alpha, \beta)-$derivations of $d-$algebras and obtained some properties. Very recently, Aslıhan S, Damla Y ([4]) discussed the concept of generalized $(\alpha, \beta)-$derivations of $d-$algebras and studied some of it is properties.

Motivated by the previous results, it is natural to ask whether it is possible to define a reverse derivation on $d-$algebra $\mathcal{X}$. The aim of this paper is to introduce the concept of left-right (resp. right-left) reverse derivations of $d-$algebra $\mathcal{X}$ ( briefly, $(l, r)$ *resp.* $(r, l)-$ reverse derivation of $d-algebra$) and investigate some of it is properties. We discuss some properties regarding the regular, composition of two maps, edge $d-algebra$ and reverse derivation on $d-algebra$. Furthermore, some illustrative examples of what we studied were given.

## 2. Elementaries

Here, we will repeat some basic properties and lemmas in $d-algebra$ which are usefull for developing the proof of our results. Definitions 1, 2 and the proofs of Lemmas 1, 2 can be seen in [20].

**Definition 1.** *A set $\emptyset \neq \mathcal{X}$ with a constant $0$ and a binary operation $*$ is called a $d-$algebra if $*$ satisfying the following axioms: $\forall\ x, y \in \mathcal{X}$,*
*(I) $x * x = 0$,*
*(II) $0 * x = 0$,*
*(III) If $x * y = 0$, $y * x = 0$, then $x = y$.*

**Definition 2.** *Let $(\mathcal{X}, *, 0)$ be a $d-$algebra, define $x * \mathcal{X} = \{x * a \mid a \in \mathcal{X}\}$. Then $\mathcal{X}$ is said to be edge $d-$algebra if $\forall x \in \mathcal{X}$ , $x * \mathcal{X} = \{x,\ 0\}$.*

**Lemma 1.** *In edge $d-$algebra $(\mathcal{X}, *, 0)$, the identity $x * 0 = x$ hold $\forall\ x \in \mathcal{X}$.*

**Lemma 2.** *If $\mathcal{X}$ is an edge $d-$algebra, then the identity $(x*(x*y))*y = 0$ hold $\forall\ x, y \in \mathcal{X}$.*

## 3. Main Results

In the present section, we introduce the concept of left-right (resp. right-left) reverse derivations of $d-$algebra $\mathcal{X}$ (briefly, $(l,r)$ *resp.* $(r,l)-$ reverse derivation of $d-algebra$ ) and will discuss some consequenes, also we will give some illustrative examples and counterexamples. Throughout this paper unless we mention otherwise, $\mathcal{X}$ denotes a $d-$algebra $(\mathcal{X}, *, 0)$ and $\forall\ x, y \in \mathcal{X}$ we write $x * y = xy$, also $x \wedge y = y(yx)$.

We will begin our study with the following definition, which can be found in [3].

**Definition 3.** *Suppose that $\mathcal{X}$ be a $d-$algebra, then $\mathcal{X}$ is called a super commutative if $x \neq y$, $xy = yx \neq 0$ for any non-zero $x, y \in \mathcal{X}$. Remark that the commutativity of $d-$algebras $\mathcal{X}$, defined as $x(xy) = y(yx)$ $\forall\ x, y \in \mathcal{X}$, that is $x \wedge y = y \wedge x$.*

In the next example, $d-$algebra $\mathcal{X}$ is a commutative but not super commutative:

**Example 1.** *Define a binary operation $*$ on $\mathcal{X} = \{0,\ a,\ b\}$ as follows:*

| $*$ | 0 | a | b |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| a | a | 0 | a |
| b | b | b | 0 |

*Then, it can be cheked that $\mathcal{X}$ is a commutative $d-$algebra, but not super commutative, (clearly, $for$ two elements $a, b \in \mathcal{X}$ we can see that $b\ * a = b$, while $a\ * b = a$, therefore, $b\ * a \neq a * b$).*

**Definition 4.** *A mapping $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ is called a $(l,r)-$ reverse derivation on a $d-$algebra $\mathcal{X}$, if $\forall\ x, y \in \mathcal{X}$ the identity $\zeta(xy) = \zeta(y)x \wedge y\zeta(x)$ holds.*

*Similarly, the $(r,l)-$ reverse derivation $\zeta$ on $\mathcal{X}$ can be defined as $\zeta(xy) = y\zeta(x) \wedge \zeta(y)x$ $\forall\ x, y \in \mathcal{X}$. Furthermore, $\zeta$ is called a reverse derivation of $\mathcal{X}$, if it is $(l,r)-$ and $(r,l)-$ reverse derivation at the same time.*

The existence of the ($(l,r)$ *resp.* $(r,l)-$ reverse derivation) of $d-algebra\ \mathcal{X}$, and thus the existence of the reverse derivation in $d-$algebra $\mathcal{X}$, is illustrated by the following example:

**Example 2.** *Define a binary operation $*$ on a set $\mathcal{X} = \{0,\ a,\ b,\ c\}$ as follows:*

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $a$ | $0$ |
| $b$ | $b$ | $b$ | $0$ | $0$ |
| $c$ | $b$ | $b$ | $b$ | $0$ |

*Let $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ be a map defined as:*

$$\zeta(x) = \begin{cases} 0 & \text{if } x = 0,\ a,\ b \\ a & \text{if } x = c \end{cases}$$

*Then it is easily checked that $\mathcal{X}$ is a $d-$algebra, $\zeta$ is both a $(l,r)-$ and $(r,l)-$ reverse derivation on $\mathcal{X}$. Hence $\zeta$ is a reverse derivation on $\mathcal{X}$.*

**Remark 1.** *In Example 2, we can remark that $\mathcal{X}$ is a $d-$algebra but not edge $d-$algebra (because $c * 0 = b \neq c$). Also, we can remark that $\mathcal{X}$ neither super commutative ( because $for\ a,\ c \in \mathcal{X},\ we\ have\ c * a = b \neq a * c = 0$), nor commutative (note that $for\ a, c \in \mathcal{X}, we\ have\ a \wedge c = c\,(c\,a) = c * b = b$. On the other hand, $c \wedge a = a\,(a\,c) = a * 0 = a$, so that $a \wedge c \neq c \wedge a$).*

**Remark 2.** *Some generalizations of $(l,r)-$ derivations on $\mathcal{X}$ have relations with the concept of $(r,l)-$ reverse derivations on $\mathcal{X}$. Also we can observe that, both of $(r,l)-$reverse derivations and $(l,r)-$ derivations are the same on $\mathcal{X}$, if $\mathcal{X}$ is super commutative, but in general the converse may not be true as illustrated in the following example.*

**Example 3.** *Consider $\mathcal{X}$ and the $(r,l)-$ reverse derivation $\zeta(x)$ as in Example 2. Hence, it is not difficult to see that $\zeta(x)$ is also $(l,r)-$derivation of $\mathcal{X}$, but $\mathcal{X}$ not super commutative. Therefore, in Remark 2 the condition of super commutativity cannot be omitted.*

**Definition 5.** *Let $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ be a self map of a $d-algebra\ \mathcal{X}$. If $\zeta(0) = 0$, then $\zeta$ is called a regular.*

**Example 4.** *Assume that $\zeta$ is a $(r,l)-$ reverse derivation on the $d-algebra\ \mathcal{X}$ as in Example 2. It is obvious from the definition of $\zeta$ that $\zeta(0) = 0$, therefore $\zeta(x)$ is regular.*

**Theorem 1.** *If $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ is a $(r,l)-reverse$ derivation on an edge $d-algebra\ \mathcal{X}$, then $\zeta$ is regular.*

*Proof.* By assumption $\zeta$ is a $(r,l)-reverse$ derivation of $\mathcal{X}$, then we have $\zeta(xy) = y\zeta(x) \wedge \zeta(y)x\ \forall\ x, y \in \mathcal{X}$. Replace $y$ by $x$ in the previous equation and use the axiom $(I)$ in Definition 1, to get $\zeta(0) = \zeta(xx) = x\zeta(x) \wedge \zeta(x)x$ for any $x, y \in \mathcal{X}$.

Now, put $x = 0$ in the last equation, to get

$$\begin{aligned} \zeta(0) &= 0\zeta(0) \wedge \zeta(0)0 \\ &= 0 \wedge \zeta(0)0 \;\; [\textit{ By axiom (II) in Definition } 1] \\ &= \zeta(0)(\zeta(0)0) \;\; [\textit{ By } x \wedge y = y(yx)] \\ &= \zeta(0)\zeta(0) \;\; [\textit{By Lemma } 1] \\ &= 0. \;\; [\textit{ By axiom (I) in Definition } 1] \end{aligned}$$

Hence $\zeta$ is regular.

Now, replace the condition $\mathcal{X}$ is an edge $d-$algebra by $\mathcal{X}$ is a $d-$algebra in Theorem 1, to obtain the same results for $(l,r) - reverse$ derivation as in the next theorem.

**Theorem 2.** *If $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ is a $(l,r) - reverse$ derivation on a $d - algebra$ $\mathcal{X}$, then $\zeta$ is regular.*

*Proof.* Assume that $\zeta$ is a $(l,r) - reverse$ derivation of a $d - algebra$ $\mathcal{X}$. Then by definition of $\zeta$ we have, $\zeta(xy) = \zeta(y)x \wedge y\zeta(x)$ $\forall\ x, y \in \mathcal{X}$. Now, in the previous equation replace $y$ by $x$ and use the axiom $(I)$ in Definition 1, to get $\zeta(0) = \zeta(xx) = \zeta(x)x \wedge x\zeta(x)$, $\forall\ x \in \mathcal{X}$. Now, put $x = 0$ in the previous equation, to get

$$\begin{aligned} \zeta(0) &= \zeta(0)0 \wedge 0\zeta(0) \\ &= \zeta(0)0 \wedge 0 \;\; [\textit{Using axiom (II) in Definition } 1] \\ &= 0(0\zeta(0)) \;\; [\textit{By } x \wedge y = y(yx)] \\ &= 0. \;\; [\textit{Again using axiom (II) in Definition } 1] \end{aligned}$$

Hence $\zeta$ is regular.

**Theorem 3.** *Suppose that $\mathcal{X}$ be an edge $d-$algebra and $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ is a $(l,r)-$ reverse derivation of $\mathcal{X}$ such that $\zeta(x) = x$, then*

*(i)* *$\zeta$ is a reverse derivation on $\mathcal{X}$.*

*(ii)* $\zeta(xy) = \zeta(y)\zeta(x),\ \forall\ x, y \in \mathcal{X}$.

*Proof.* (1) Suppose that $\zeta$ be a $(l,r) - reverse$ derivation on edge $d-$algebra $\mathcal{X}$ where $\zeta(x) = x$ $\forall\ x \in \mathcal{X}$, therefore, to prove that $\zeta$ is a reverse derivation of $\mathcal{X}$ it is enough to verify that $\zeta$ is a $(r,l) - reverse$ derivation on $\mathcal{X}$ as follows: by assumption $\zeta$ is a $(l,r) - reverse$ derivation of $\mathcal{X}$, so $\forall\ x, y \in \mathcal{X}$ we get

$$\begin{aligned} \zeta(xy) &= \zeta(y)x \wedge y\zeta(x) \\ &= y\zeta(x) \wedge \zeta(y)x. \; [\textit{Using the assumption that } \zeta(y) = y,\ \zeta(x) = x] \end{aligned}$$

Thus, $\zeta$ is a $(r,l) - reverse$ derivation on $\mathcal{X}$, hence we conclude that $\zeta$ is a reverse derivation on $\mathcal{X}$.

(2) For all $x, y \in \mathcal{X}$, we have

$$\begin{aligned}
\zeta(xy) &= \zeta(y)x \wedge y\zeta(x) \ [By\ the\ Definition\ 4] \\
&= yx \wedge yx \ \ [By\ assumption\ that\ \zeta(x) = x] \\
&= yx[yx(yx)] \ \ [Using\ x \wedge y = y(yx)] \\
&= (yx)0 \ \ [Using\ axiom\ (\mathrm{I})\ in\ Definition\ 1] \\
&= yx \ \ [By\ Lemma\ 1] \\
&= \zeta(y)\zeta(x). \ \ [Again\ by\ assumption\ that\ \zeta(x) = x]
\end{aligned}$$

Hence, we get the required result.

By using the similar arguments as in Theorem 3 (2), it is easy to show that, if $\zeta$ is a $(r,l)-$reverse derivation of *edge* algebra $\mathcal{X}$, then also we get $\zeta(xy) = \zeta(y)\zeta(x)\ \forall\ x, y \in \mathcal{X}$.

**Definition 6.** *If* $\zeta,\ \zeta^{'}$ *are two self maps on a* $d-algebra$ $\mathcal{X}$*, then the map* $\zeta \circ \zeta^{'} : \mathcal{X} \longrightarrow \mathcal{X}$ *defined as* $\zeta \circ \zeta^{'}(x) = \zeta(\zeta^{'}(x)),\ \forall\ x \in \mathcal{X}$*.*

**Theorem 4.** *Suppose that* $\zeta$ *and* $\zeta^{'}$ *are two* $(r,l) - reverse\ derivations$ *on an edge* $d-algebra$ $\mathcal{X}$*, then the map* $\zeta \circ \zeta^{'}$ *is regular.*

*Proof.* Let $\zeta,\ \zeta^{'}$ are two $(r,l) - reverse$ derivations on $\mathcal{X}$. Then by definition, we have $\zeta \circ \zeta^{'}(xy) = y(\zeta \circ \zeta^{'})(x) \wedge (\zeta \circ \zeta^{'})(y)x,\ \forall\ x, y \in \mathcal{X}$. Replacing $y$ by $x$ in the last equation and using axiom $(I)$ in the Definition 1, we get $(\zeta \circ \zeta^{'})(0) = (\zeta \circ \zeta^{'})(xx) = x(\zeta \circ \zeta^{'})(x) \wedge (\zeta \circ \zeta^{'})(x)x,\ \forall\ x \in \mathcal{X}$. Now, put $x = 0$ in the last equation, to get

$$\begin{aligned}
(\zeta \circ \zeta^{'})(0) &= 0(\zeta \circ \zeta^{'})(0) \wedge (\zeta \circ \zeta^{'})(0)0 \\
&= 0 \wedge (\zeta \circ \zeta^{'})(0)0 \qquad [By\ axiom\ (\mathrm{II})\ in\ Definition\ 1] \\
&= 0 \wedge (\zeta \circ \zeta^{'})(0) \quad [By\ Lemma\ 1] \\
&= (\zeta \circ \zeta^{'})(0)((\zeta \circ \zeta^{'})(0)0) \quad [By\ x \wedge y = y(yx)] \\
&= (\zeta \circ \zeta^{'})(0)(\zeta \circ \zeta^{'})(0) \qquad [Again\ by\ Lemma\ 1] \\
&= 0. \qquad [By\ axiom\ (\mathrm{I})\ in\ Definition\ 1]
\end{aligned}$$

Hence $\zeta \circ \zeta^{'}(0) = 0$, and so $\zeta \circ \zeta^{'}$ is regular.

In the following theorem, the condition $\mathcal{X}$ is an *edge* algebra (i.e., $x\ 0 = 0$) omitted and attempt to get the same results in Theorem 4, for $(l,r) - reverse$ derivation of $d - algebra$ $\mathcal{X}$.

**Theorem 5.** *Suppose that* $\zeta$ *and* $\zeta^{'}$ *are two* $(l,r) - reverse\ derivations$ *of a* $d - algebra$ $\mathcal{X}$*, then* $\zeta \circ \zeta^{'}$ *is regular.*

*Proof.* By assumption that $\zeta$ *and* $\zeta^{'}$ are two $(l,r)-$reverse derivations on $\mathcal{X}$, then $\forall$ $x, y \in \mathcal{X}$, we have $\zeta \circ \zeta^{'}(xy) = \zeta \circ \zeta^{'}(y)x \wedge y\zeta \circ \zeta^{'}(x)$. Replace $y$ by $x$ in the previous equation, to get $(\zeta \circ \zeta^{'})(0) = (\zeta \circ \zeta^{'})(xx) = (\zeta \circ \zeta^{'})(x)x \wedge x(\zeta \circ \zeta^{'})(x)$, $\forall\ x \in \mathcal{X}$. Now, put $x = 0$ in the last equation, to get

$$\begin{aligned}(\zeta \circ \zeta^{'})(0) &= (\zeta \circ \zeta^{'})(0)0 \wedge 0(\zeta \circ \zeta^{'})(0)\\ &= (\zeta \circ \zeta^{'})(0)0 \wedge 0 \ \ [Using\ axiom\ (\mathrm{II})\ in\ Definition\ 1]\\ &= 0(0(\zeta \circ \zeta^{'})(0)0) \ \ [By\ x \wedge y = y(yx)]\\ &= 0. \ \ [By\ axiom\ (\mathrm{II})\ in\ Definition\ 1]\end{aligned}$$

Hence $\zeta \circ \zeta^{'}(0) = 0$, and so $\zeta \circ \zeta^{'}$ is regular.

**Theorem 6.** *If* $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ *is a* $(l, r) - reverse$ *derivation on a* $d - algebra$ $\mathcal{X}$*, then* $\forall\ x \in \mathcal{X}\ \zeta(x\zeta(x)) = 0$.

*Proof.* By assumption, $\mathcal{X}$ is a $d - algebra$ and $\zeta$ is a $(l, r) - reverse$ derivation such that $\zeta(x\zeta(x)) = 0\ \forall\ x \in \mathcal{X}$, therefore we have

$$\begin{aligned}\zeta(x\zeta(x)) &= (\zeta \circ \zeta)(x)x \wedge \zeta(x)\zeta(x) \quad [By\ the\ Definitions\ 4, 6, respectively]\\ &= (\zeta \circ \zeta)(x)x \wedge 0 \quad [By\ axiom\ (\mathrm{I})\ in\ Definition\ 1]\\ &= 0(0(\zeta \circ \zeta)(x)) \quad [By\ x \wedge y = y(yx)]\\ &= 0. \quad [By\ axiom\ (\mathrm{II})\ in\ Definition\ 1]\end{aligned}$$

Thus, $\forall\ x \in \mathcal{X}\ \zeta(x\zeta(x)) = 0$ as required.

**Theorem 7.** *If* $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ *is a* $(l, r) - reverse$ *derivation on an edge* $d - algebra$ $\mathcal{X}$*, then* $\forall\ x \in \mathcal{X}\ \zeta(\zeta(x)x) = 0$.

*Proof.* It is given that, $\zeta$ is a $(l, r) - reverse$ derivation of an edge $d - algebra$ $\mathcal{X}$, then for any $x \in \mathcal{X}$, we have

$$\begin{aligned}\zeta(\zeta(x)x) &= \zeta(x)\zeta(x) \wedge x(\zeta \circ \zeta)(x) \quad [By\ the\ Definitions\ 4, 6, respectively]\\ &= 0 \wedge x(\zeta \circ \zeta)(x) \quad [By\ axiom\ (\mathrm{I})\ in\ Definition\ 1]\\ &= x(\zeta \circ \zeta)(x)[(x(\zeta \circ \zeta)(x))0] \quad [By\ x \wedge y = y(yx)]\\ &= x(\zeta \circ \zeta)(x)(x(\zeta \circ \zeta)(x)) \quad [By\ Lemma\ 1]\\ &= 0. \quad [By\ axiom\ (\mathrm{II})\ in\ Definition\ 1]\end{aligned}$$

The proof is completed as required.

**Definition 7.** *Define a relation "* $\leq$ *" on a* $d - algebra$ $\mathcal{X}$ *by* $x \leq y$*,* $iff\ xy = 0$ *for any* $x, y \in \mathcal{X}$*. Thus,* $\mathcal{X}$ *becomes a partially ordered by the relation* $x \leq y$*, denoted it by* $(\mathcal{X}, \leq)$*.*

**Definition 8.** *[20] A* $d - subalgebra$ $S$ *of a* $d - algebra$ $\mathcal{X}$ *is a non-empty subset of* $\mathcal{X}$ *satisfing the condition* $x * y \in S$*, whenever* $x, y \in S$*.*

**Example 5.** *Let $S = \{0,\ a,\ b\}$ and $H = \{0,\ c\}$ be two non-empty sets of the $d-algebra$ $\mathcal{X}$ shown in the Example 2. Clearly, we can verify that $S = \{0,\ a,\ b\}$ is a $d-subalgebra$ in $\mathcal{X}$. But $H = \{0,\ c\}$ is not a $d-subalgebra$ in $\mathcal{X}$, because $c * 0 = b$ not in $H$.*

**Proposition 1.** *Assume that $\zeta : \mathcal{X} \longrightarrow \mathcal{X}$ is a $(l,r)-reverse\ derivation$ such that $\mathcal{X}$ is an edge $d-algebra$ with partial order $\leq$. Then*

*(i) $\zeta(xy) \leq \zeta(y)x$ for all $x, y \in \mathcal{X}$.*

*(ii) If $\zeta^{-1}(0) = \{x \in \mathcal{X} \mid \zeta(x) = 0\}\ \forall\ x \in \mathcal{X}$ such that $\zeta$ is regular, then $\zeta^{-1}(0)$ is a $d-subalgebra$ of $\mathcal{X}$.*

*(iii) If $x, y \in \zeta^{-1}(0)$, then $x \wedge y \in \zeta^{-1}(0)$.*

*Proof.* (1) We have, $\zeta$ is a $(l,r)-$ reverse derivation on edge $d-algebra$ $\mathcal{X}$, then for any $x, y \in \mathcal{X}$, we get

$$\begin{aligned} \zeta(xy) &= \zeta(y)x \wedge y\zeta(x) \quad [Using\ the\ Definition\ 4] \\ &= y\zeta(x)[y\zeta(x)(\zeta(y)x)] \quad [By\ x \wedge y = y(yx)]. \end{aligned}$$

Now, multiplying both sides by $\zeta(y)x$ from the right hand, we get

$$\begin{aligned} \zeta(xy)\zeta(y)x &= [y\zeta(x)(y\zeta(x)\zeta(y)x)]\zeta(y)x \\ &= 0 \quad [By\ Lemma\ 2]. \end{aligned}$$

That is, $\zeta(xy)\zeta(y)x = 0$ for all $x, y \in \mathcal{X}$. Now, using Definition 7, we get $\zeta(xy) \leq \zeta(y)x$ for all $x, y \in \mathcal{X}$, as required.

(2) It is given that $\zeta$ is regular. Hence, $\zeta^{-1}(0) \neq \emptyset$. Let $x, y \in \zeta^{-1}(0)$. Then by the part (1) of the present theorem, we get

$$\zeta(xy) \leq \zeta(y)x \ \text{ for all } x, y \in \mathcal{X}. \tag{1}$$

But, $y \in \zeta^{-1}(0)$ so that $\zeta(y) = 0$, now replacing $\zeta(y)$ by 0 in Equation (1) and using axiom $(II)$ in Definition 1 respectively, we get $\zeta(xy) \leq 0x = 0\ \forall\ x, y \in \mathcal{X}$. Therefore, $\zeta(xy) = 0\ \forall\, x, y \in \mathcal{X}$, hence $xy \in \zeta^{-1}(0)$. Which yields that $\zeta^{-1}(0)$ is a $d-subalgebra$ of $\mathcal{X}$.

(3) Let $\zeta$ be a $(l,r)-$ reverse derivation on $\mathcal{X}$. Then we get,

$$\begin{aligned} \zeta(x \wedge y) &= \zeta(y(yx)) \ \ [By\ x \wedge y = y(yx)] \\ &= \zeta(yx)y \wedge (yx)\zeta(y) \quad [By\ Definition\ 4] \\ &= [\zeta(x)y \wedge x\zeta(y)]y \wedge (yx)\zeta(y) \quad [Again\ using\ Definition\ 4] \\ &= (0y \wedge x0)y \wedge (yx)0 \ \ [By\ assumption\ x, y \in \zeta^{-1}(0),\ i.e., \zeta(x) = \zeta(y) = 0] \\ &= (0 \wedge x)y \wedge yx \quad [By\ \text{(II)}\ in\ Definition\ 1\ and\ Lemma\ 1] \\ &= (x(x0))y \wedge yx \ \ [By\ x \wedge y = y(yx)] \\ &= 0y \wedge yx \ \ [By\ \ Lemma\ 1\ and\ (I)\ in\ Definition\ 1, respectively] \\ &= 0 \wedge yx \ \ [By\ (II)\ in\ Definition\ 1] \\ &= yx((yx)0) \ \ [By\ x \wedge y = y(yx)] \\ &= yx(yx) \ \ [Again\ by\ \ Lemma\ 1] \\ &= 0. \ \ [By\ \ (I)\ in\ Definition\ 1] \end{aligned}$$

Hence, we conclude that, $\zeta(x \wedge y) = 0$ for any $x, y \in \zeta^{-1}(0)$ which means $x \wedge y \in \zeta^{-1}(0)$ as required.

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