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2-Distance Zero Forcing Sets in Graphs

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Abstract. In this paper, we introduce new concept in graph theory called 2-distance zero forcing. We give some properties of this new parameter and investigate its connections with other parameters such as zero forcing and hop domination. We show that 2-distance zero forcing and hop domination(respectively, zero forcing parameter) are incomparable. Moreover, we characterize 2-distance zero forcing sets in some special graphs, and finally derive the exact values or bounds of the parameter using these results.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Zero forcing, 2-distance zero forcing, 2-distance zero forcing number

1. Introduction

Zero forcing is a propagation process in a graph that increases the number of blue vertices given on initial set of blue vertices, with all other vertices white, and a color-change rule. The color-change rule states that a blue vertex adjacent to a single white neighbor can force its neighbor to blue. Formally, if u is a blue vertex and w is the only white vertex in $N(u)$, then $u \longrightarrow w$ will be used to denote that u forces w to blue. Given a graph G, a zero forcing set B of G is a subset of vertices of $V(G)$ such that B is initially colored blue, and the remaining vertices in G are white, then iteratively applying applying the color-change rule given B results in every vertex in G becoming blue. Zero forcing sets have applications in control theory, network coding, and determining structural properties of graphs. Some studies related to zero forcing sets and its variants can be found in [1–6].

Recently, J. Manditong et al.[16], introduced new variant of zero forcing in a graph called zero forcing hop domination. They have established some properties of this parameter and determined its connections with other known parameters in graph theory. Moreover, they have obtained some exact values or bounds of the parameter on the generalized graph, some families of graphs, and graphs under some operations via characterizations. Some interesting studies related to zero forcing hop domination can be found in [7–15].

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In this paper, we introduce the concept of 2-distance zero forcing sets in a graph. Let G be a graph and let $x, y \in V(G)$. Then the 2-distance color change rule is if x is colored (active) vertex and exactly one hop neighbor y of x is uncolored (inactive) , then y will become colored (active). A 2-distance zero forcing set N of G is a subset of vertices of G such that when the vertices in N are colored (active) and the remaining vertices are uncolored(inactive) initially, repeated application of the 2-distance color change rule all vertices of G will become colored (active). The minimum cardinality of a 2-distance zero forcing set of G, denoted by $Z^2(G)$, is called the 2-distance zero forcing number of G. We study its connections with the standard zero forcing and hop domination parameter, respectively. Moreover, we investigate this parameter on some families of graphs such as complete, path, cycle, star, and complete bipartite graph. We believe, this new parameter and its results would serve as reference to future researchers who will study on variants of zero forcing, and would lead to an interesting topics of research in the future.

2. Terminology and Notation

A *path graph* is a non-empty graph with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$, where the $x_i^{'}s$ are all distinct. The path of order n is denoted by P_n. If G is a graph and u and v are vertices of G, then a path from vertex u to vertex v is sometimes called a *u-v path*. The *cycle graph* C_n is the graph of order $n \geq 3$ with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$.

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

Two vertices x, y of G are *adjacent*, or *neighbors*, if xy is an edge of G. The *open neighborhood* of x in G is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A vertex of a in G is a *hop neighbor* of a vertex b in G if $d_G(a, b) = 2$. The set $N_G^2(a) = \{b \in V(G) : d_G(a, b) = 2\}$ is called the *open hop neighborhood* of a. The *closed hop neighborhood* of a in G is given by $N_G^2[a] = N_G^2(a) \cup \{a\}$. The *open hop neighborhood* of $S \subseteq V(G)$ is the set $N_G^2(S) = \bigcup_{a \in S} N_G^2(a)$. The *closed hop neighborhood* of S in G is the set $N_G^2[S] = N_G^2(S) \cup S$.

A subset S of $V(G)$ is a *hop dominating* of G if for every $a \in V(G) \backslash S$, there exists $b \in S$ such that $d_G(a, b) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G.

Let G and H be any two graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The color change rule states that a blue vertex adjacent to a single white neighbor can force its neighbor to blue. Formally, if u is a blue vertex and w is the only white vertex in $N_G(u)$, then $u \longrightarrow w$ will be used to denote that u forces w blue. A *zero forcing set for a graph* G is a subset of vertices in Z such that if initially the vertices in Z are colored blue and the remaining vertices are colored white, the entire graph G May be colored blue by repeatedly applying the color-change rule. Furthermore, the *zero forcing number*, $Z(G)$, of a graph G is the minimum cardinality of a set of blue vertices(whereas vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ is turned blue after finitely many applications of "the color change rule": a white vertex is converted to a blue vertices if it is the only white neighbor of a blue vertex.

3. Results

We begin this section by introducing the concepts of 2-distance zero forcing set and 2-distance zero forcing number of a graph.

Definition 1. Let G be a graph and let $x, y \in V(G)$. Then the 2-distance color change rule is if x is colored (active) vertex and exactly one hop neighbor y of x is uncolored (inactive), then y will become colored (active). Formally, if x is a colored(active) vertex and y is the only uncolored (inactive) vertex in $N_G^2(x)$, then $x \longrightarrow y$ will be used to denote that x 2-forces y to be colored (active). A 2-distance zero forcing set N of G is a subset of vertices of G such that when the vertices in N are colored (active) and the remaining vertices are uncolored(inactive) initially, repeated application of the 2-distance color change rule all vertices of G will become colored (active). The minimum cardinality of a 2-distance zero forcing set of G, denoted by $Z^2(G)$, is called the 2-distance zero forcing number of G.

Example 1. Consider the graph G in Figure 1 and let $N = \{a, b, d\}$. Then vertex c is 2-forced by vertex a and vertex e is 2-forced by either vertex d or b. Thus, N is a 2-distance zero forcing set of G. Moreover, $Z^2(G) = 3$.

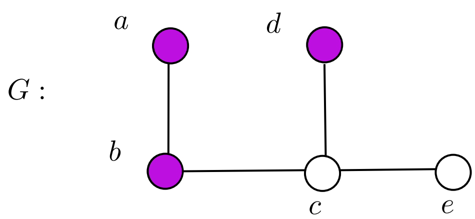


Figure 1: Graph G with $Z^2(G) = 3$

Proposition 1. *Let n be a positive integer. Then S is a 2-distance zero forcing set of K_n if and only if $S = V(K_n)$.*

Proof. Let S be a 2-distance zero forcing set of K_n. Suppose that $S \neq V(K_n)$. Then there exists $x \in V(K_n)$ such that $x \notin S$. However, $d_{K_n}(x, y) = 1$ for all $y \in S$. It follows that S cannot 2-forced x, a contradiction. Therefore, $S = V(K_n)$.

The converse is clear. □

Corollary 1. *Let n be a positive integer. Then $Z^2(K_n) = n$.*

Theorem 1. *$Z^2(G) = |V(G)|$ if and only if $diam(H) \leq 1$ for each component H of G.*

Proof. Suppose that $Z^2(G) = |V(G)|$. Suppose further that $diam(H) \geq 2$ for some component H of G. Then there exist $a, b \in V(H)$ such that $d_H(a,b) = 2 = d_G(a,b)$. Let $N = V(G) \setminus \{b\}$. Then N is a 2-distance zero forcing set of G. Thus, $Z^2(G) \leq |V(G)| - 1$, a contradiction. Therefore, $diam(H) \leq 1$ for each component H of G.

Conversely, suppose that $diam(H) \leq 1$ for each component H of G. If G is connected, then $G = K_n$. Thus, $Z^2(G) = |V(G)| = n$ by Corollary 1. Suppose that G is disconnected. Let $H_1, \dots, H_k, k \geq 2$ be components of G. Since $diam(H_i) \leq 1$, $Z_2(H_i) = |V(H_i)|$ for each $i \in \{1, \dots, k\}$. Thus,

$$Z^2(G) = Z^2(H_1) + \cdots + Z^2(H_k) = |V(H_1)| + \cdots + |V(H_k)| = |V(G)|.$$

□

Corollary 2. *Let n be a positive integer. Then, $Z^2(\overline{K}_n) = n$.*

Proposition 2. *Let G be a graph and let N be a 2-distance zero forcing set of G. Then every dominating vertex $v \in V(G)$, $v \in N$.*

Proof. Let $v \in V(G)$ be a dominating vertex of G. Then $N_G[v] = V(G)$, that is, v is adjacent to every vertex $u \in V(G) \setminus \{v\}$. Suppose that $v \notin N$. Then $d_G(v,w) = 2$ for some $w \in N$, a contradiction. Therefore, $v \in N$. □

Proposition 3. *Let n be a positive integer. Then,*

$$Z^2(P_n) = \begin{cases} 1 & , n = 1 \\ 2 & , n \geq 2 \end{cases}$$

Proof. By Theorem 1, $Z^2(P_1) = 1$ and $Z^2(P_2) = 2$. Clearly, $Z^2(P_n) = 2$ for $n = 3, 4, 5$ Suppose that $n \geq 6$ Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $N = \{v_1, v_2\}$. If n is odd, then vertices $v_3, v_5, \dots, v_n$ are 2-forced by vertices $v_1, v_3, \dots, v_{n-2}$, respectively, and vertices $v_4, v_6, \dots, v_{n-1}$ are 2-forced by vertices $v_2, v_4, \dots, v_{n-3}$, respectively. If n is even, then vertices $v_3, v_5, \dots, v_{n-1}$ are 2-forced by vertices $v_1, v_3, \dots, v_{n-3}$, respectively, and vertices $v_4, v_6, \dots, v_n$ are 2-forced by vertices $v_2, v_4, \dots, v_{n-2}$, respectively. Therefore, N is a 2-distance zero forcing set of P_n. Since any singleton subset of $V(P_n)$ is not a 2-distance zero forcing set of P_n, it follows that N is a minimum 2-distance zero forcing set of P_n. Consequently, $Z^2(P_n) = 2$ for all $n \geq 2$. □

Proposition 4. *Let n be a positive integer. Then,*

(i)

$$Z^2(C_n) = \begin{cases} 3 \ , n = 3 \\ 2 \ , n = 4 \text{ or } n \geq 5 \text{ and odd} \\ 4 \ , n \geq 6 \text{ and even} \end{cases}$$

(ii)

$$Z^2(S_n) = \begin{cases} 2 \ , n = 1 \\ n \ , n \geq 2 \end{cases}$$

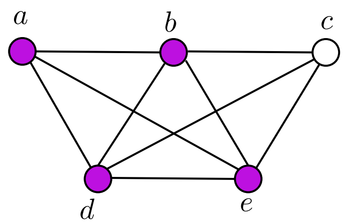
Proof. (i) By Theorem 1, $Z^2(C_3) = 3$. Clearly, $Z^2(C_4) = 2$. Suppose $n \geq 5$ and odd. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and let $N = \{v_1, v_3\}$. Then N is a minimum 2-distance zero forcing set of C_n. Thus, $Z^2(C_n) = 2$ for all $n \geq 5$ and odd. Next, suppose that $n \geq 6$ and even. Let $N' = \{v_1, v_2, v_3, v_4\}$. Then N' is a minimum 2-distance zero forcing set C_n. Thus, $Z^2(C_n) = 4$ for all $n \geq 6$ and even.

(ii) By Theorem 1, $Z^2(S_1) = 2$. Suppose that $n \geq 2$. Let $V(S_n) = \{d, v_1, ..., v_n\}$, where d is the dominating vertex of S_n. Consider $M = \{d, v_1, ..., v_{n-1}\}$. Then M is a minimum 2-distance zero forcing set of S_n. Therefore, $Z^2(S_n) = n$ for all $n \geq 2$. □

Theorem 2. *Let G be a graph. If H is a subgraph of G, then $Z^2(H) \leq Z^2(G)$ is not true in general.*

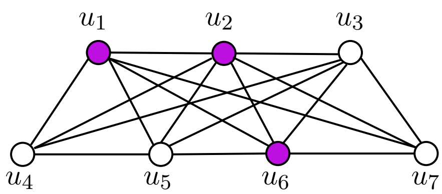
Proof. Consider the graph $K_2 + P_3$ below.

$K_2 + P_3$:



Let $S_1 = \{a, b, d, e\}$. Then S_1 is a minimum 2-distance zero forcing set of $K_2 + P_3$. Hence, $Z^2(K_2 + P_3) = 4$. Now, consider the graph $P_3 + P_4$ below.

$P_3 + P_4$:

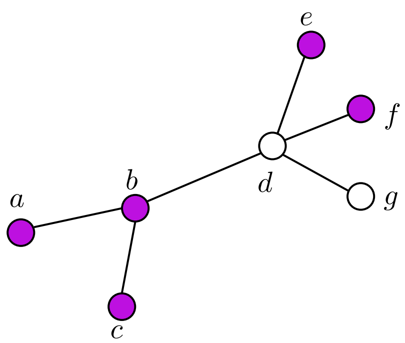


Let $S_2 = \{u_1, u_2, u_6\}$. Then u_3, u_4, u_7 and u_5 are 2-forced by u_1, u_6, u_4 and u_7, respectively. Thus, S_2 is a 2-distance zero forcing set of $P_3 + P_4$. It can be verified that $Z^2(P_3 + P_4) = 3$. Consequently, the assertion follows. □

Theorem 3. *Let H be a graph. If K is a subgraph of H, then $Z^2(K) \geq Z^2(H)$ is not true in general.*

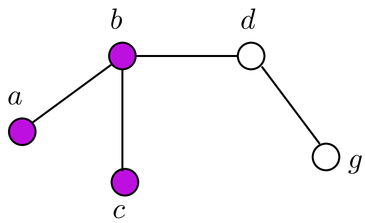
Proof. Consider the graph H below.

H :



Let $Q_1 = \{a, b, c, e, f\}$. Then Q_1 is a minimum 2-distance zero forcing set of H. Thus, $Z^2(H) = 5$. Now, consider the subgraph K of H below.

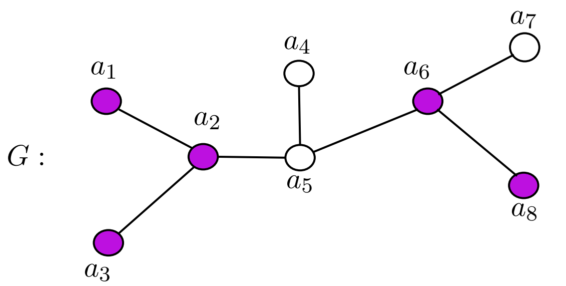
K :



Let $Q_2 = \{a, b, c\}$. Then Q_2 is a minimum 2-distance zero forcing set of K. Thus, $Z^2(K) = 3$. Therefore, the assertion follows. □

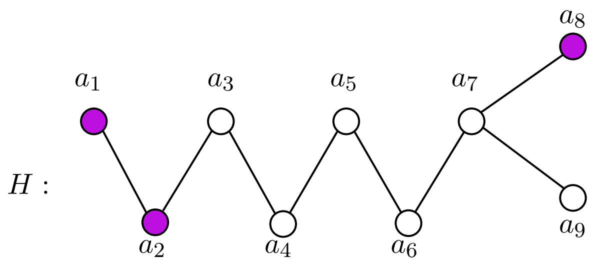
Proposition 5. *Let G be a graph. Then the 2-distance zero forcing $Z^2(G)$ and hop domination parameters $\gamma_h(G)$ of G are incomparable.*

Proof. Consider the graph G below.



Let $Q = \{a_5, a_6\}$. Then Q is a hop dominating set of G since $N_G^2[Q] = V(G)$. Since $N_G^2[a_i] \neq V(G)$ for every $i \in \{1, 2, \cdots, 8\}$, it follows that Q is a minimum hop dominating set of G. Thus, $\gamma_h(G) = 2$. Now, let $S = \{a_1, a_2, a_3, a_6, a_8\}$. Then S is a minimum 2-distance zero forcing set of G. Therefore, $Z^2(G) = 5$.

On the other hand, consider the graph H below.



Let $D = \{a_1, a_2, a_8\}$. Then vertices a_3, a_5 and a_7 are 2-forced by the vertices a_1, a_3 and a_5, respectively, and vertices a_4, a_6 and a_9 are 2-forced by the vertices a_2, a_4 and a_6, respectively. This follows that D is a 2-distance zero forcing set of H. Moreover, $Z^2(H) = 3$. Next, let $D' = \{a_3, a_4, a_5, a_6\}$. Then D' is a minimum hop dominating set of H. Consequently, $\gamma_h(H) = 4$. □

Theorem 4. *Let G be a graph. Then $Z^2(G) = \gamma_h(G) = |V(G)|$ if and only if every component of G is complete.*

Proof. Suppose that $Z^2(G) = |V(G)| = \gamma_h(G)$. Then $V(G)$ is both the minimum 2-distance zero forcing and minimum hop dominating set of G. Suppose there is a component of G which is non-complete. Then there exist $a, b \in V(Q)$ such that $d_G(a, b) = 2$. Let $S = V(G) \setminus \{a\}$. Then S is both a 2-distance zero forcing and a hop dominating set of G. a contradiction. Therefore, every component of G is complete.

Conversely, suppose that every component of G is complete. Then by Theorem 1, $Z^2(G) = |V(G)|$. Moreover, $\gamma_h(G) = |V(G)|$. Consequently,

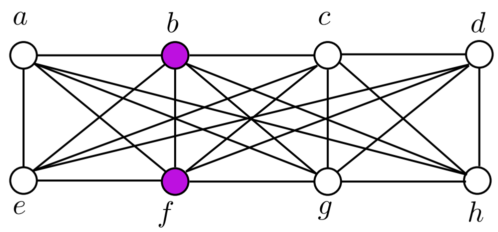
$$Z^2(G) = \gamma_h(G) = |V(G)|.$$

□

Proposition 6. *Let G be a graph. Then zero forcing $Z(G)$ and 2-distance zero forcing $Z^2(G)$ parameter of G are incomparable.*

Proof. Consider the graph $P_4 + P_4$ below.

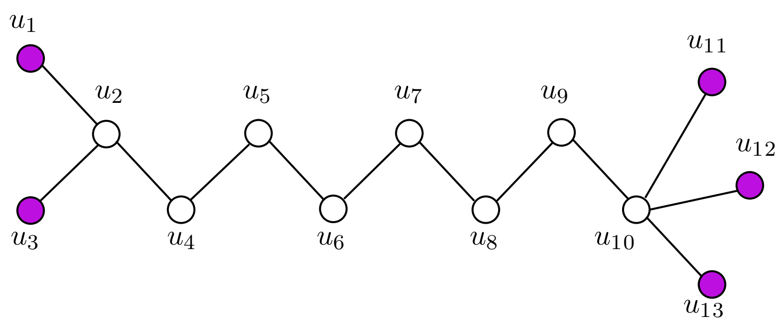
$P_4 + P_4$:



Let $S_1 = \{b, f\}$ and $S_2 = \{a, b, c, d, e\}$. Then S_1 and S_2 are minimum 2-distance zero forcing and zero forcing sets of $P_4 + P_4$, respectively. Thus, $Z^2(P_4 + P_4) = 2$ and $Z(P_4 + P_4) = 5$.

Next, consider the graph G below.

G :

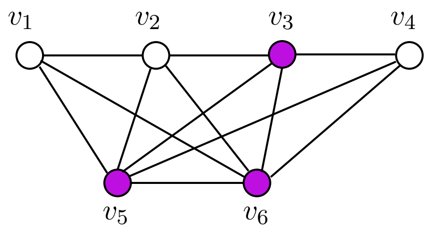


Let $D_1 = \{u_1, u_{11}, u_{12}, u_{13}\}$ and $D_2 = \{u_1, u_3, u_{11}, u_{12}, u_{13}\}$. Then D_1 and D_2 are minimum zero forcing and 2-distance zero forcing set of G, respectively. Therefore, $Z(G) = 4$ and $Z^2(G) = 5$. □

Theorem 5. *There exists a graph G such that $Z^2(G) = Z(G)$.*

Proof. Consider the graph $K_2 + P_4$ below.

$K_2 + P_4$:



Let $D' = \{v_1, v_5, v_6\}$ and $D'' = \{v_3, v_5, v_6\}$. Then D' and D'' are minimum zero forcing and 2-distance zero forcing sets of K_2+P_4, respectively. Hence, $Z(K_2+P_4) = 3 = Z^2(K_2+P_4)$. Let $G = K_2 + P_4$. Then the assertion follows. □

Theorem 6. *Let G be a graph. Then $Z^2(G) = Z(G) = |V(G)|$ if and only if every component of G is trivial.*

Proof. Suppose that $Z^2(G) = Z(G) = |V(G)|$. Suppose there is component K of G which is non-trivial. Let $V(K) = \{k_1, ..., k_n\}$, $k \geq 2$. Consider $Q = V(G) \setminus \{k_1\}$. Then Q is a zero forcing set of G. Thus, $Z(G) \leq |V(G)| - 1$, a contradiction. Therefore, every component of G is trivial.

The converse is clear. □

4. Conclusion

The concept of 2-distance zero forcing in a graph has been introduced and investigated in this paper. The 2-distance zero forcing numbers of some graphs are obtained. The connections of the 2-distance zero forcing parameter with the standard zero forcing and hop domination parameter have been presented. Interested researchers may study this concept on graphs that were not considered in this study. Interested researchers may also consider on providing an application of this parameter.

Acknowledgements

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