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Legal Hop Independent Sequences in Graphs

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Abstract. Let G be any graph. A sequence $L = (w_1, \dots, w_k)$ of distinct vertices of G is called a legal hop independent sequence if $k = 1$ or L is a hop independent and $N_G[w_i] \setminus \bigcup_{j=1}^{i-1} N_G[w_j] \neq \varnothing$ for every $i \in \{2, \cdots, k\}$. The maximum length of a legal hop independent sequence in G, denoted by $\alpha_{\ell h}(G)$, is called the legal hop independence number of G. In this paper, we investigate its relationships with the hop independence and grundy domination parameter of a graph, respectively. In fact, the legal hop independence parameter is at most equal to the grundy domination (resp. hop independence) parameter on any graph G. Moreover, we derive some formulas and bounds of this parameter on some families of graphs, join, and corona of two graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: legal sequence, hop independent set, legal hop independent sequence, legal hop independence number

1. Introduction

Independent sets are used to model relationships in networks, such as social networks, computer networks, and biological networks. For example, in a social network, an independent set could represent a group of individuals who are not directly connected or acquainted with each other. Independent sets in graphs had studied on different kinds of graphs (see [2, 11, 17, 18]).

In 2022, hop independent set in a graph and its parameter was introduced by J. Hassan et al. [8]. They defined a set $S \subseteq V(G)$ is a hop independet set of G if any two distinct vertices in S are not at a distance two from each other, that is, $d_G(u, w) \neq 2$ for any distinct vertices $u, w \in S$. The maximum cardinality of a hop independent set of G,

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denoted by $\alpha_h(G)$, is called the hop independence number of G. They have shown that any maximum hop independent set S of G is always a hop dominating, that is, the hop indpendence number of a graph is always greater than or equal to the hop domination parameter. Moreover, they derived some bounds and formulas for some special graphs and graphs under some binary operations. Some studies related to hop independent sets, its variations, and other hop-related concepts can be found in [3, 9, 10, 13, 14].

Recently, J. Hassan and S. Canoy [5], introduced another variant of hop independence in a graph called hop independent hop domination. They have shown that the hop independent hop domination number of a graph G lies between the hop domination number and the hop independence number of graph G. They have characterized hop independent hop dominating sets in the shadow graph, join, corona, and lexicographic product of two graphs. Moreover, they have obtained exact values or bounds of the hop independent hop domination numbers of these graphs. Furthermore, researchers had studied variants of hop independent hop domination and other related studies on different types of graphs (see [1, 4, 6, 7, 12, 15, 16]).

In this paper, new variant of hop independence called legal hop independent sequences in a graph is introduced and investigated. The main focus of this concept is on the sequence of vertices of a graph wherein it must satisfy a certain condition aside from being a hop independent. The authors believe that this study may provide interesting results that would positively contribute to the independence theory and could lead to another interesting studies and application of the parameter.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by *$diam(G)$*, is called the *diameter* of G.

A subset D of $V(G)$ is called a *dominating* of G if for every $x \in V(G) \setminus D$, there exists $y \in D$ such that $xy \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.

Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of a graph G, and let $\hat{S} = \{v_1, v_2, \cdots, v_k\}$ be a corresponding set of a sequence S. Then S is called a *legal closed neighborhood sequence*(legal sequence) if $N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \varnothing$ for every $i \in \{2, \cdots, k\}$. If, in addition, $\hat{S}$ is a dominating set of G, then S is called a *Grundy dominating sequence.* The maximum length of a Grundy dominating sequence in a graph G is called the *Grundy domination number* of G, and is denoted by $\gamma_{gr}(G)$. Any Grundy dominating sequence with length equal to $\gamma_{gr}(G)$ is called a γ_{gr}-sequence of G.

Let $S_1 = (v_1, \ldots, v_n)$ and $S_2 = (u_1, \ldots, u_m)$ be two sequences of distinct vertices of G. The *concatenation* of S_1 and S_2, denoted by $S_1 \oplus S_2$, is the sequence given by $S_1 \oplus S_2 = (v_1, \ldots, v_n, u_1, \ldots, u_m)$.

A subset S of $V(G)$ is called a *hop independent* if for every pair of distinct vertices $x, y \in S$, $d_G(x, y) \neq 2$. The maximum cardinality of a hop independent set in G, denoted by $\alpha_h(G)$, is called the *hop independence* number of G. Any hop independent set S with cardinality equal to $\alpha_h(G)$ is called an α_h-set of G.

A graph is *complete* if every pair of distinct vertices are adjacent. A complete graph of order n is denoted by K_n.

A set $S \subseteq V(G)$ is called a *clique* in G if the subgraph $\langle S \rangle$ induced by S is a complete graph. The maximum size or cardinality of a clique of G, denoted by $\omega(G)$, is called the *clique number* of G. Any clique in G with cardinality $\omega(G)$ is called an ω-set in G. The *complement* of a graph G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u, v \in V(G) \text{ and } uv \notin E(G)\}$.

Let G and H be any two graphs. The *join* of G and H, denoted by $G+H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$$

.

The *corona* G and H, denoted by $G \circ H$, the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} + H^v \rangle$.

3. Results

We begin this section by introducing the concept of a legal hop independence in a graph.

Definition 1. Let G be any graph. A sequence $L = (w_1, \ldots, w_k)$ of distinct vertices of G is called a legal hop independent sequence if $k = 1$ or L is a hop independent and $N_G[w_i] \setminus \bigcup_{j=1}^{i-1} N_G[w_j] \neq \varnothing$ for every $i \in \{2, \cdots, k\}$. The maximum length of a legal hop independent sequence in G, denoted by $\alpha_{\ell h}(G)$, is called the legal hop independence number of G. Any legal hop independent sequence L of G with $|\hat{L}| = \alpha_{lh}(G)$, where $\hat{L} = \{w_1, \ldots, w_k\}$, is called an α_{lh}-sequence or a maximum legal hop independent sequence of G. Moreover, we call $\hat{L}$ an α_{lh}-set of G.

Example 1. Consider the graph $G = C_4$ in Figure 1. Let $L = (w_1, w_2)$. Then

$$N_G[w_1] = \{w_1, w_2, w_4\} \text{ and } N_G[w_2] = \{w_1, w_2, w_3\}.$$

From the equations above, we have

$$N_G[w_2] \backslash N_G[w_1] = \{w_1, w_2, w_3\} \backslash \{w_1, w_2, w_4\} = \{w_3\} \neq \emptyset.$$

It follows that L is a legal sequence in G. Since $d_G(w_1, w_2) = 1 \neq 2$, this means that $\hat{L} = \{w_1, w_2\}$ is a hop independent set of G. Thus, $L = (w_1, w_2)$ is a legal hop independent sequence of G. Moreover, since $d_G(w_1, w_3) = 2$ and $d_G(w_2, w_4) = 2$, it follows that L is a maximum legal hop independent sequence of G. Therefore, $\alpha_{\ell h}(G) = 2$.

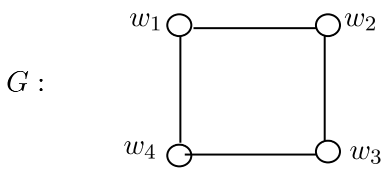

Figure 1: Graph G with $\alpha_{lh}(G) = 2$

Remark 1. *Let G be a graph. Then*

(i) *a legal sequence of G may not form a hop independent set of G;*

(ii) *a hop independent set of G may not form a legal sequence of G;*

(iii) *if $\hat{S}$ is an α_h-set of G and form a legal sequence, then S is an $\alpha_{\ell h}$-sequence of G and $\alpha_{\ell h}(G) = |\hat{S}|$; and*

(iv) *if L is a maximum legal sequence of G and $\hat{L}$ is a hop independent set of G, then L is an $\alpha_{\ell h}$-sequence of G and $\alpha_{\ell h}(G) = |\hat{L}|$.*

Theorem 1. *Let G be any graph. Then*

(i) $\alpha_{\ell h}(G) \leq \alpha_h(G)$;

(ii) $1 \leq \alpha_{\ell h}(G) \leq |V(G)|$; *and*

(iii) $\alpha_{\ell h}(G) \leq \gamma_{gr}(G)$.

Proof. (i) Let G be any graph and let L be an $\alpha_{\ell h}$-sequence of G. Then its corresponding set $\hat{L}$ is a hop independent set of G. Since $\alpha_h(G)$ is the maximum cardinality of a hop independent set in G, it follows that $\alpha_h(G) \geq \left|\hat{L}\right| = \alpha_{\ell h}(G)$.

(ii) Let G be any graph and let $x \in V(G)$. Then (x) is a legal hop independent sequence of G. Hence, $\alpha_{\ell h}(G) \geq 1$. Since $\alpha_h(G) \leq |V(G)|$ for any graph G, it follows that $\alpha_{\ell h}(G) \leq |V(G)|$ by (i). Consequently, $1 \leq \alpha_{\ell h}(G) \leq |V(G)|$.

(iii) Let G be any graph and let L be an $\alpha_{\ell h}$-sequence of G. Then L is a legal sequence of G. Since any γ_{gr}-sequence is a maximum legal sequence, we have

$$\alpha_{\ell h}(G) = |\hat{L}| \leq \gamma_{gr}(G).$$

□

Theorem 2. $\alpha_{\ell h}(G) = 1$ *if and only if G is complete graph.*

Proof. Let $\alpha_{\ell h}(G) = 1$. Suppose G is non-complete. If G is connected, there exist $u, w \in V(G)$ such that $d_G(u, w) = 2$. Let $x \in N_G(u) \cap N_G(w)$. Then $w \in N_G[x] \backslash N_G[u]$, and so $N_G[x]\backslash N_G[u] \neq \emptyset$. Thus, $L = (u, x)$ is a legal sequence of G. Since $d_G(u, x) = 1$, it follows that $L = (u, x)$ is a legal hop independent sequence of G. Therefore, $\alpha_{\ell h}(G) \geq 2$, which is a contradiction.

Now, Suppose that G is disconnected graph. Let $G_1, ..., G_m$, where $m \geq 2$ be components of G. Then $\alpha_{\ell h}(G_i) \geq 1$ for each $i \in \{1, ..., m\}$. Thus,

$$\alpha_{\ell h}(G) = \alpha_{\ell h}(G_1) + ... + \alpha_{\ell h}(G_m) \geq 1 + ... + 1 \geq 2 \text{ since } m \geq 2.$$

However, this a contradiction to our assumption. Consequently, G must be complete graph.

For the converse, suppose that G is complete. Then $\gamma_{gr}(G) = 1$. Therefore, $\alpha_{\ell h}(G) = 1$ by Theorem 3 (ii) and (iii). □

Theorem 3. *Let G be any graph. Then $\alpha_{\ell h}(G) = |V(G)|$ if and only if every component of G is trivial.*

Proof. Suppose that $\alpha_{\ell h}(G) = |V(G)|$. Then $V(G)$ is the maximum $\alpha_{\ell h}$-set of G. Suppose there is a component K of G which is non-trivial. If K is complete, then $\alpha_{\ell h}(K) = 1$. Thus, $\alpha_{\ell h}(G) \leq |V(G)| - |V(K)| \leq |V(G)| - 1$, a contradiction. If K is non-complete, then $\alpha_h(K) \leq |V(K)|-1$. Hence, $\alpha_h(G) \leq |V(G)|-1$, and so $\alpha_{\ell h}(G) \leq |V(G)|-1$ by Theorem 3(i), a contradiction. Therefore, every component of G is trivial.

Conversely, suppose that every component of G is trivial. Let $V(G) = \{v_1, v_2, ..., v_m\}$. Then $\langle\{x_1\}\rangle, \langle\{x_2\}\rangle, ..., \langle\{x_m\}\rangle$ are the components of G. Thus, $N_G[x_i] = \{x_i\}$ for each $i \in \{1, 2, ..., m\}$, and so $x_j \in N_G[x_j]\backslash \bigcup_{k=1}^{j-1} N_G[x_k]$ $\forall$ $j \in \{2, ..., m\}$. That is, $S = (v_1, v_2, ..., v_m)$ is a legal sequence of G. Notice that $d_G(x_s, x_t) \neq 2$ $\forall$ $s \neq t$, where $s, t \in \{1, 2, ..., m\}$. Therefore, S is a legal hop independent sequence of G. Consequently, $\alpha_{\ell h}(G) = |V(G)|$. □

The following result follows from the above Theorem.

Corollary 1. *Let $m \geq 1$ be any positive integer. Then $\alpha_{\ell h}(\overline{K}_m) = m$.*

Theorem 4. *Let s and t be positive integers that satisfy $2 \leq s \leq t$. Then there exists a connected graph K such that $\alpha_{\ell h}(K) = s$ and $\alpha_h(K) = t$.*

Proof. Supose that $s < t$. Let $q = t - s$ and consider the graph K in Figure 2, where $\langle\{x, w_s, u_1, u_2, ...u_q\}\rangle$ and $\langle\{y, w_s, u_1, u_2, ...u_q\}\rangle$ induced a complete graph, respectively. Let $L = (w_1, w_2, ..., w_s)$ and $\hat{L}_0 = \{w_1, w_2, ..., w_s, u_1, u_2, ..., u_q\}$. Then L and $\hat{L}_0$ are maximum legal hop independent sequence and maximum hop independent set of K, respectively. Therefore, $\alpha_{\ell h}(K) = s$ and $\alpha_h(K) = s+q = t$, and so $\alpha_{\ell h}(K) = s < t = \alpha_h(K)$.

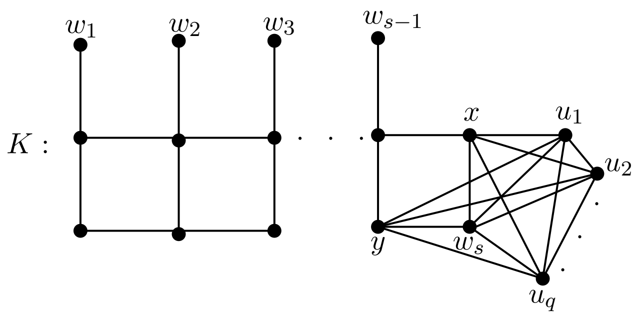

Figure 2: A graph K' with $\alpha_{lh}(K') < \alpha_h(K')$

For $s = t$, consider the graph K' in Figure 3. Let $Q = (a_1, a_2, ..., a_t)$ and $\hat{Q} = \{a_1, a_2, ...a_t\}$. Then Q and $\hat{Q}$ are maximum legal hop independent sequence and maximum hop independent set of K', respectively. Thus, $\alpha_{\ell h}(K') = t = s = \alpha_h(K')$. □

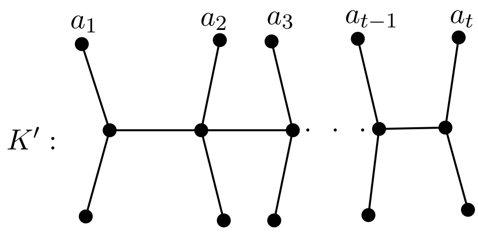

Figure 3: A graph K' with $\alpha_{lh}(K') = \alpha_h(K')$

Definition 2. Let G be any graph. A sequence $L = (a_1, \ldots, a_n)$ is called a clique legal sequence if $n = 1$ or L is a legal sequence and its corresponding set $\hat{L}$ induces a complete graph. The maximum length of a clique legal sequence in G, denoted by $\alpha_{c\ell h}(G)$, is called the clique legal number of G. Moreover, we call $\hat{L}$ a clique legal set of G.

Definition 3. Let G be any graph. A clique legal sequence L is called a clique legal dominating sequence or a clique Grundy dominating sequence if its corresponding set $\hat{L}$ is a dominating set of G. The maximum length of a clique Grundy dominating sequence in G, denoted by $\gamma_{c\ell gr}(G)$, is called the clique Grundy domination number of G. Moreover, a clique legal sequence L of G is called a clique legal non-dominating sequence if $\hat{L}$ is not a dominating set of G.

Theorem 5. *[8] Let G and H be graphs. Then S is a non-empty hop independent set of $G+H$ if and only if one of the following statements holds:*

(i) $S \cap V(H) = \varnothing$ and $S \cap V(G)$ is a clique in G.

(ii) $S \cap V(G) = \varnothing$ and $S \cap V(H)$ is a clique in H.

(iii) $S \cap V(G)$ and $S \cap V(G)$ are clique in G and H, repectively.

Theorem 6. *[6] Let G and H be two non-complete graphs. A sequence D of distinct verices of $G+H$ is a Grundy dominating sequence in $G+H$ if and only if one of the following conditions holds:*

(i) D is a Grundy dominating sequence of G.

(ii) D is a Grundy dominating sequence of H.

(iii) $D = D_G \oplus (w)$ for some non-dominating legal closed neighborhood sequence D_G of G and $w \in V(H)$.

(iv) $D = D_H \oplus (v)$ for some non-dominating legal closed neighborhood sequence D_H of H and $v \in V(G)$.

Theorem 7. *Let H and K be two non-complete graphs. A sequence L of distinct vertices of $H+K$ is a legal hop independent sequence in $H+K$ if and only if one of the following conditions holds:*

(i) L is a clique legal sequence in H

(ii) L is a clique legal sequence in K

(iii) $L = L_H \oplus (a)$, where L_H is a clique legal non-dominating sequence in H and $a \in V(K)$.

(iv) $L = L_K \oplus (b)$, where L_K is a clique legal non-dominating sequence in K and $b \in V(H)$.

Proof. Suppose that L is a legal hop independent sequence of $H+K$. Assume that $\hat{L} \subseteq V(H)$. Then $\hat{L}$ is a clique in H by Theorem 5. By Theorem 6, L is a legal sequence in H. Thus, L is a clique legal sequence in H and so (i) holds. Similarly, if $\hat{L} \subseteq V(K)$, then L is a clique legal sequence in K. That is, (ii) holds.

Now, let L_H and L_K be subsequences of L such that $\hat{L}_H = \hat{L} \cap V(H)$ and $\hat{L}_K = \hat{L} \cap V(K)$. Suppose that $\hat{L}_H \neq \varnothing$ and $\hat{L}_K \neq \varnothing$. Then $L = L_H \oplus (a)$ for some non-dominating legal sequence L_H in H and $a \in V(K)$ by Theorem 6. By Theorem 5, L_H is clique in H. Thus, L_H is a clique legal non-dominating sequence in H, and so (iii) holds. Similarly, by Theorem 5 and Theorem 6, (iv) holds.

The converse is clear. □

Corollary 2. *Let H and K be two non-complete graphs. Then*

$$\alpha_{\ell h}(H+K)=\begin{cases} max\{\gamma_{c\ell gr}(H),\gamma_{c\ell gr}(K)\}, & if\ both\ H\ and\ K\ admit\ a\ clique\\ & grundy\ domination.\\ max\{\alpha_{c\ell h}(H)+1,\alpha_{clh}(K)+1\}, & if\ both\ H\ and\ K\ does\ not\ admit\\ & a\ clique\ grundy\ domination.\\ max\{\alpha_{c\ell h}(H)+1,\gamma_{c\ell gr}(K)\}, & if\ K\ admits\ a\ clique\ grundy\\ & domination\ and\ H\ does\ not.\\ max\{\alpha_{c\ell h}(K)+1,\gamma_{c\ell gr}(H)\}, & if\ H\ admits\ a\ clique\ grundy\\ & domination\ and\ K\ does\ not.\end{cases}$$

Theorem 8. *[6] Let G be a complete graph and let H be a non-complete graph. A sequence D of distinct vertices of $G+H$ is a Grundy dominating sequence in $G+H$ if and only if one of the following condition holds:*

(i) *$D=(v)$ for some $v\in V(G)$.*

(ii) *D is a Grundy dominating sequence of H.*

(iii) *$D=D_H\oplus(v)$ for some non-dominating legal neighborhood sequence D_H of H and $v\in V(G)$.*

Theorem 9. *Let S and T be complete and non-complete graph, respectively. A sequence L' of distinct vertices of $S+T$ is a legal hop independent sequence if and only if one of the following conditions holds:*

(i) *$L'=(s)$ for some $s\in V(S)$.*

(ii) *L' is a clique legal sequence of T.*

(iii) *$L'=L_T\oplus(w)$, where L_T is a clique legal non-dominating sequence in T and $w\in V(S)$.*

Proof. Let L' be a legal hop independent sequence of $S+T$. Assume that $\hat{L}'\subseteq V(S)$. Since S is complete, $L'=(s)$ for some $s\in V(G)$. Hence, (i) holds.

Suppose that $\hat{L}'\subseteq V(T)$. Since $\hat{L}'$ is hop independent in $S+T$, $\hat{L}'$ is clique in T by Theorem 5. By Theorem 7, L' is legal sequence in T. Thus, L' is a clique legal sequence in T, and so (ii) holds.

Now, assume that $\hat{L}'=\hat{L}_S\cup\hat{L}_T$, where $\hat{L}_s=\hat{L}'\cap V(S)$ and $\hat{L}_T=\hat{L}'\cap V(T)$. Then $\hat{L}'$ is a dominating set of $S+T$. By Theorem 8, $L'=L_T\oplus(w)$ for some non-dominating legal sequence L_T of T and $w\in V(S)$. Since $\hat{L}'$ is a hop independent set in $S+T$, $\hat{L}_T$

must be clique in T. Hence, (iii) holds.

The converse is clear. □

Corollary 3. *Let S and T be complete and non-complete graphs, respectively. Then*

$$\alpha_{\ell h}(S+T) = \begin{cases} \gamma_{c\ell gr}(T), & if\ T\ admits\ a\ clique\ grundy\ domination. \\ \alpha_{c\ell h}(T)+1, & otherwise. \end{cases}$$

Theorem 10. *Let G be a connected graph and H be any graph. Then L is a legal hop independent sequence of $G \circ H$ if $\hat{L} = \bigcup_{v \in v} \hat{L}_v$, where $\hat{L}_v$ is a clique legal set in H^v for each $v \in V(G)$. Moreover, $\alpha_{\ell h}(G \circ H) \geq \alpha_{c\ell h}(H) \cdot |V(G)|$.*

Proof. Let $\hat{L} = \bigcup_{v \in v} \hat{L}_v$, where $\hat{L}_v$ is a clique legal set in H^v for some $v \in V(G)$. Then L is a legal sequence of $G \circ H$. Now, let $a, b \in \hat{L}$. If $a, b \in \hat{L}_v$, then $d_G(a,b) = 1$. Thus, $\hat{L}$ is a hop independent of $G \circ H$, and so we are done. Suppose that $a \in L_u$ and $b \in L_w$ for some $u, w \in V(G)$. Clearly, $d_{G \circ H}(a,b) \geq 3$. Since a and b are arbitrary, it follows that $\hat{L}$ is a hop independent set of $G \circ H$. Hence, $\hat{L}$ is legal hop independent set of $G \circ H$. Consequently, $\alpha_{\ell h}(G \circ H) \geq \alpha_{c\ell h}(H) \cdot |V(G)|$. □

4. Conclusion

The concept of legal hop independence in graphs has been introduced and investigated in this study. The parameter was defined on any simple and undirected graph. Moreover, it was found out that the legal hop independence number of a graph is always less than or equal to either Grundy domination number or hop independence number of a graph. The realization result in Theorem 4 says that the difference between the hop independence number and the legal hop independence number of a graph can be made arbitrarily large. Furthermore, some exact values and bounds of the parameter have been obtained on some special graphs, join and corona of two graphs.

Some graphs that were not considered in this study could be an interesting cases to consider for further investigation of this newly defined parameter. In addition, researchers may consider to study the complexity and algorithm of determining the legal hop independence number of any graph.

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References

[1] V. Bilar, M.A. Bonsocan, J. Hassan, and S. Dagondon. Vertex cover hop dominating sets in graphs. *Eur. J. Pure Appl. Math.*, 17(1):93–104, 2024.

[2] E. Davies, M. Jenssen, W. Perkins, and B. Roberts. independent sets, matchings, and occupancy fractions. *J. Lond. Math. Soc.*, 96:211–220, 2017.

[3] J. Hassan, AR. Bakkang, and ASS. Sappari. j^2-hop domination in graphs: Properties and connections with other parameters. *Eur. J. Pure Appl. Math.*, 16(4):2118–2131, 2023.

[4] J. Hassan and S. Canoy. Connected grundy hop dominating sequences in graphs.,. *Eur. J. Pure Appl. Math.*, 16(2):1212–1227, 2023.

[5] J. Hassan and S. Canoy. Hop independent hop domination in graphs.,. *Eur. J. Pure Appl. Math.*, 15(4):1783–1796, 2023.

[6] J. Hassan and S. Canoy Jr. Grundy dominating and grundy hop dominating sequences in graphs: Relationships and some structural properties. *Eur. J. Pure Appl. Math.*, 16(2):1154–1166, 2023.

[7] J. Hassan and S. Canoy Jr. Grundy total hop dominating sequences in graphs. *Eur. J. Pure Appl. Math.*, 16(4):2597–2612, 2023.

[8] J. Hassan, S. Canoy Jr., and A. Aradais. Hop independent sets in graphs. *Eur. J. Pure Appl. Math.*, 15(2):467–477, 2022.

[9] J. Hassan, A. Lintasan, and N.H. Mohammad. Some properties and realization problems involving connected outer-hop independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 16(3):1848–1861, 2023.

[10] J. Hassan, A. Tapeing, H. Copel, A.R Bakkang, and S.D. Aming. j^2-independence parameters of some graphs.,. *Eur. J. Pure Appl. Math.*, 17(1):124–134, 2024.

[11] Liu Jiuqiang. Maximal and maximum independent sets in graphs. *Dissertations.*, 1985.

[12] S. Canoy Jr. and J. Hassan. Weakly convex hop dominating sets in graphs. *Eur. J. Pure Appl. Math.*, 15(4):1783–1796, 2022.

[13] S. Kaida, K.J. Maharajul, J. Hassan, L. S. Laja, A.B. Lintasan, and A.A. Pablo. Certified hop independence: Properties and connections with other variants of independence. *Eur. J. Pure Appl. Math.*, 17(1):435–444, 2024.

[14] J. Manditong, J. Hassan, LS Laja, AA. Laja, NHM. Mohammad, and SU. Kamdon. Connected outer-hop independent dominating sets in graphs under some binary operations. *Eur. J. Pure Appl. Math.*, 16(3):1817–1829, 2023.

[15] J. Manditong, A. Tapeing, J. Hassan, A.R. Bakkang, N.H. Mohammad, and S.U. Kamdon. Some properties of zero forcing hop dominating sets in a graph. *Eur. J. Pure Appl. Math.*, 17(1):324–337, 2024.

[16] J. Mohamad and H. Rara. On resolving hop domination in graphs. *Eur. J. Pure Appl. Math.*, 14(1):324–337, 2021.

[17] H.S. Wilf. The number of maximal independent sets in a tree. *SIAM J. Alg. Disc. Meth.*, 7:125–130, 1986.

[18] J. Zito. The structure and maximum number of maximum independent sets in trees. *J. Graph Theory.*, 15(2):207–221, 1991.