EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 2, 2024, 945-955
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Exact solutions for the modified Burgers equation with additional time-dependent variable coefficient

Bazar Babajanov[1,4], Fakhriddin Abdikarimov[2,*], Sarbinaz Bazarbaeva[3]

[1] *Department of Applied Mathematics and Mathematical Physics, Urgench State University, Urgench, Uzbekistan*
[2] *Khorezm Mamun Academy, Khiva, Uzbekistan*
[3] *Karakalpak State University, Nukus, Uzbekistan*
[4] *Khorezm Branch of Uzbekistan Academy of Sciences V. I. Romanovskiy Institute of Mathematics, Urgench, Uzbekistan*

Abstract. In this article, we investigated new travelling wave solutions for the modified Burgers equation with additional time-dependent variable coefficient via the functional variable method. The performance of this method is reliable and effective and gives the exact solitary wave solutions. All solutions of this equation have been examined and three dimensional graphics of the obtained solutions have been drawn by using the Matlab program. The exact solutions have its great importance to reveal the internal mechanism of the physical phenomena. This method presents a wider applicability for handling nonlinear wave equations.

2020 Mathematics Subject Classifications: 34A34, 34B15, 35Q51, 35J60, 35J66, 35L05

Key Words and Phrases: modified Burgers equation, solitary wave solutions, kinematics viscosity, variable coefficient, velocity, functional variable method, viscosity parameter, ordinary differential equation, continuous differentiable functions, internal mechanism, distinct variables

1. Introduction

Burgers equation was first given by Bateman and later was studied by Burgers as a mathematical model for turbulence[12, 15]. The Burgers equation has applications in various fields such as convection and diffusion, number theory, gas dynamics, heat conduction, elasticity, engineering and other scientific fields[29]. The Burgers equation is in the form

$$u_t + uu_x - \nu u_{xx} = 0,$$

where $u(x,t)$ denotes the velocity for space x and time t and $\nu > 0$ is a constant representing the kinematics viscosity of the fluid.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i2.5064

Email addresses: a.murod@mail.ru (B. Babajanov), goodluck_0714@mail.ru (F. Abdikarimov), sarbinazbazarbaeva6@gmail.com (S. Bazarbaeva)

The one-dimensional modified Burgers equation is in the form

$$u_t + u^2 u_x - \nu u_{xx} = 0,$$

where $u(x,t)$ is the dependent variable, ν is the viscosity parameter, t and x are the independent parameters. This equation describes in several areas of applied mathematics such as various practical transport problems, nonlinear waves in a medium with low-frequency pumping or absorption, ion reflection at quasi-perpendicular shocks, turbulence transport, the transport and dispersion of pollutants in rivers[14].

In the literature many numerical method was applied to approximate the solution of the modified Burgers equation by several authors. The collocation method with quintic splines[14], the colocation method with septic splines[31], the sextic B-spline collocation method[24], a non-polynomial spline based method[22], an explicit numerical scheme[13], Petrov-Galerkin method[33] and explicit exponential finite difference schemes have been used to obtain numerical solution of the modified Burgers equation by several authors[16].

Many direct methods of nonlinear evolutions equations have been developed to find solutions, such as tanh-function method[25], functional variable method[4, 5, 7, 9, 10], Hirota method[23], Backlund transform method[32], exp-function method[28], G/G' expansion method[6, 8] and extended tanh-method[19] are used for searching the exact solutions[2, 3, 11, 20, 26, 27].

In[17], the arteries were considered as thin-wall prestressed elastic tubes of variable radius, and the long-wavelength approximation was used. The propagation of weakly nonlinear waves in such an elastic tube filled with a liquid was investigated using the modified Korteweg-de Vries equation with a variable coefficient

$$u_t + 6u^2 u_x - u_{xxx} = h(t) u_x,$$

where t is the scale coordinate along the vessel axis after a static deformation (this coordinate characterizes the axi symmetric stenosis on the surface of the arterial wall), x is a variable depending on time and the coordinate along the vessel axis, $h(t)$ is the shape of the stenosis, and the function $u(x,t)$ characterizes the average axial velocity of the liquid.

The modified KdV-Burgers equation with variable coefficients is defined as

$$u_t + u_{xxx} + 3\alpha u^2 u_x + \beta u_{xx} = 0,$$

where α and β are constant coefficients, and they incorporate the effects of nonlinearity $(\alpha u^2 u_x)$ and dissipation (βu_{xx}) into the equation; β is the coefficient of the kinematic viscosity of a fluid $(\beta < 0)$. When the dispersion term $u_{xxx} = 0$, then this equation was was formulated from the modified Burgers equation[30]. When $\beta = 0$, this equation is just the so called mKdV equation, which originates from nonlin ear optics[1] and the propagation of long internal waves in a fluid when the coefficient of the ordinary nonlinear term in the KdV equation. The higher order nonlinear term $u^2 u_x$ dominates over higher or dispersive terms[21].

In this article, we consider the modified Burgers equation with additional time-dependent variable coefficient

$$u_t + h_1(t) u^2 u_x - h_2(t) u_{xx} + \omega(t) u_x = 0, \tag{1}$$

where $u(x,t)$ is an unknown function, $x \in R$, $t \geq 0$, $h_1(t) \neq 0$, $h_2(t) \neq 0$, $\omega(t) \neq 0$ are given continuous differentiable functions and $h_2(t) > 0$ is a variable representing the kinematics viscosity of the fluid.

The equation (1) arises in many physical problems including the motions of waves in nonlinear optics, plasma or fluids, water waves, ion-acoustic waves in a collision less plasma. The first element u_t designates the evolution term and the second one shows the term of dispersion.

The main aim of this paper is to find the exact soliton solutions of the equation (1) via functional variable method. The main advantage of the proposed method over other methods is that it provides more new exact traveling wave solutions. All solutions of this equation have been examined and three dimensional graphics of the obtained solutions have been drawn by using the Matlab program. The exact solutions have its great importance to reveal the internal mechanism of the physical phenomena.

2. Description of the method

The basic idea of the functional variable method proposed in[18]. Let us consider the nonlinear differential equation with independent variables x, y, z, t and a dependent variable u

$$P(u,\, u_t,\, u_x,\, u_y,\, u_z,\, u_{xy},\, u_{yz},\, u_{xz},\, ...) = 0, \tag{2}$$

where P is a polynomial in $u(t,\, x,\, y,\, z,\, ...)$ and its partial derivatives. The equation (2) is a nonlinear partial differential equation that is not integrable, in general. Sometime it is difficult to find a complete set of solutions.

Step 1. The following transformation is used for the new wave variable as

$$\xi = \sum_{i=0}^{p} \alpha_i \chi_i + \delta, \tag{3}$$

where χ_i are distinct variables, when $p = 1$, $\xi = \alpha_0\chi_0 + \alpha_1\chi_1 + \delta$. If the quantities α_0, α_1 are constants, then, they are called the wave pulsation and χ_0, χ_1 are the variables t and x, respectively.

We can introduce the following transformation for a travelling wave solution of equation (2)

$$u(\chi_0,\ \chi_1,\, ...) = u(\xi), \tag{4}$$

and the chain rule

$$\frac{\partial u}{\partial \chi_i} = \alpha_i \frac{du}{d\xi}, \quad \frac{\partial^2 u}{\partial \chi_i \partial \chi_j} = \alpha_i \alpha_j \frac{d^2 u}{d\xi^2}, ... \tag{5}$$

Using equation (3) and equation (5), the nonlinear partial differential equation (2) can be transformed into an ordinary differential equation of the form

$$Q(u,\, u', u'', u''', ...) = 0, \tag{6}$$

where Q is a polynomial in $u(\xi)$ and its total derivatives, while $u' = \frac{du}{d\xi}$.

Step 2. We make a transformation in which the unknown function u is considered as a functional variable in the form

$$u' = F(u), \tag{7}$$

then, the solution can be found by the relation

$$\int \frac{du}{F(u)} = \xi + C, \tag{8}$$

here C is a constant of integration which is set equal to zero for convenience. Some successive differentiations of u in terms of F are given as

$$\begin{aligned}
&u'' = \tfrac{dF(u)}{du}\tfrac{du}{d\xi} = \tfrac{dF(u)}{du}F(u) = \tfrac{1}{2}\tfrac{d\left(F^2(u)\right)}{du},\\
&u''' = \tfrac{1}{2}\tfrac{d^2\left(F^2(u)\right)}{du^2}\sqrt{F^2(u)},\\
&u^{(IV)} = \tfrac{1}{2}\left[\tfrac{d^3\left(F^2(u)\right)}{du^3}F^2(u) + \tfrac{d^2\left(F^2(u)\right)}{du^2}\tfrac{d\left(F^2(u)\right)}{du}\right],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{9}$$

Step 3. The ordinary differential equation (6) can be reduced in terms of u, F and its derivatives upon using the expressions of equation (7) and (9) into equation (6) gives

$$R(u, \frac{dF(u)}{du}, \frac{d^2F(u)}{du^2}, \frac{d^3F(u)}{du^3}, ...) = 0. \tag{10}$$

After integration, equation (10) provides the expression of $F(u)$ and this, together with equation (7), give appropriate solutions to the being considered problem.

3. Algorithm for finding solutions

We use the following algorithm to calculate the exact solution of the equation (1) by the functional variable method. Using the wave variable

$$u(x,t) = u(t,\xi), \xi = a(t) + b(t)x, \tag{11}$$

that will convert equation (1) to following form

$$u'_t + (a_t(t) + b_t(t))\, u'_\xi + h_1(t)b(t)u^2u'_\xi - h_2(t)b^2(t)u''_\xi + \omega(t)b(t)u'_\xi = 0, \tag{12}$$

where $a(t)$ and $b(t)$ are an unknown time-dependent functions, we will determine these functions later.

Let $a(t)$, $b(t)$, $h_1(t)$, $h_2(t)$ and $\omega(t)$ are constant functions. We use the following transformation

$$\xi = a + bx. \tag{13}$$

We put $a(t)$, $b(t)$, $h_1(t)$, $h_2(t)$ and $\omega(t)$ into (11) and (12), integration constants are considered zero. It is easy to show that after transformation, the equation (12) can be transformed into an ordinary differential equation of the form

$$h_1bu^2u'_\xi - h_2b^2u''_\xi + \omega bu'_\xi = 0. \tag{14}$$

Integrating once equation (14), we have

$$\frac{h_1}{3}u^3 - h_2 b u'_\xi + \omega u = 0. \tag{15}$$

It is easy to deduce from equation (15) an expression for the function u'_ξ

$$u'_\xi = ru + nu^3, \tag{16}$$

where $r = \frac{\omega}{h_2 b}$, $n = \frac{h_1}{3h_2 b}$.

We search the solution of equation (1) in the form:

$$u(t,\xi) = \sum_{k=0}^{m} q_k(t)\Phi^k(\xi) = q_0(t) + q_1(t)\Phi(\xi) + ... + q_m(t)\Phi(\xi)^m, \tag{17}$$

where Φ satisfies equation (16) as

$$\Phi' = \lambda\Phi + \mu\Phi^3, \tag{18}$$

where λ and μ are free parameters and m is an undetermined integer and $q_k(t)$ are coefficients to be determined later.

One of the most useful techniques for obtaining the parameter m in (17) is the homogeneous balance method. Substituting (17) into equation (12) and by making balance between the linear term u'' and the nonlinear term uu' to determine the value of m, and by simple calculation we have got that $3m + 2 = m + 4$, this in turn gives $m = 1$, and the solution (17) takes the form

$$u(t,\xi) = \sum_{k=0}^{1} q_k(t)\Phi^k(\xi) = q_0(t) + q_1(t)\Phi(\xi). \tag{19}$$

Now, we substitute (19) into (12) along with (18) and set each coefficient of $\Phi^k\,(\Phi')^p$ ($k = 0,\,1,\,2$ and $p = 0,\,1$) to zero to obtain a set of algebraic equations for $q_0(t)$, $q_1(t)$, $a(t)$ and $b(t)$:

$$\begin{cases} b_t(t) = 0, \\ q_{0t}(t) + h_1(t)q_0^2(t) = 0, \\ q_{1t}(t) + 2q_0(t)q_1(t)h_1(t)b(t) = 0, \\ a_t(t) - \lambda h_2(t)b^2(t) + \omega(t)b(t) = 0 \\ h_1(t)q_1^2(t) - 3\mu h_2(t)b(t) = 0. \end{cases} \tag{20}$$

Solving the system of algebraic equations, we can obtain $a(t)$, $b(t)$, $q_0(t)$ and $q_1(t)$. For this, we consider the following 2 cases in the system of equations (20).

Let $q_0(t) = 0$, then the system of algebraic equations (20) has the following solution

$$\begin{cases} a(t) = \int_0^t \left(\lambda S_2^2 h_2(\tau) - S_2\omega(\tau)\right) d\tau + S_1, \\ b(t) = const = S_2, \\ q_0(t) = 0, \\ q_1(t) = const = S_3. \end{cases} \tag{21}$$

$$h_2(t) = kh_1(t),\ k = const, \tag{22}$$

where S_1, S_2 and S_3 are the integration constants and are identified from initial data of the pulse. Notice that $h_1(t)$ and $h_2(t)$ serve as constraint relations between the coefficient functions and which indicate that (22) must be satisfied to assure the existence and the formation process of soliton structures.

Taking account of (11), (18), (19) and (21), we get the exact solutions for equation (1)

$$u_1(x,t) = S_3\sqrt{\frac{e^{2\left(\int_0^t\left(\lambda S_2^2h_2(\tau)-S_2\omega(\tau)\right)d\tau+S_1+S_2x\right)}}{1-e^{2\left(\int_0^t\left(\lambda S_2^2h_2(\tau)-S_2\omega(\tau)\right)d\tau+S_1+S_2x\right)}}}. \tag{23}$$

Let $q_0(t) \neq 0$, then the system of algebraic equations (20) has the following solution

$$\begin{cases} a(t) = \int_0^t \left(\lambda C_2^2 h_2(\tau) - C_2\omega(\tau)\right) d\tau + C_1, \\ b(t) = const = C_2, \\ q_0(t) = \frac{1}{\int_0^t h_1(\tau)d\tau + C_3}, \\ q_1(t) = \frac{C_4}{\left(\int_0^t h_1(\tau)d\tau+C_3\right)^{2C_2}}. \end{cases} \tag{24}$$

$$h_2(t) = \frac{C_4^2}{3\mu C_2}\frac{h_1(t)}{\left(\int_0^t h_1(\tau)d\tau + C_3\right)^{4C_2}}, \tag{25}$$

where C_1, C_2, C_3 and C_4 are the integration constants and are identified from initial data of the pulse. Notice that $h_1(t)$ and $h_2(t)$ serve as constraint relations between the coefficient functions and which indicate that (25) must be satisfied to assure the existence and the formation process of soliton structures.

Taking account of (11), (18), (19) and (24), we get the exact solutions for equation (1)

$$u_2(x,t) = \frac{1}{\int_0^t h_1(\tau)d\tau + C_3} + \frac{C_4}{\left(\int_0^t h_1(\tau)d\tau + C_3\right)^{2C_2}}\sqrt{\frac{e^{2\left(\int_0^t\left(\lambda C_2^2h_2(\tau)-C_2\omega(\tau)\right)d\tau+C_1+C_2x\right)}}{1-e^{2\left(\int_0^t\left(\lambda C_2^2h_2(\tau)-C_2\omega(\tau)\right)d\tau+C_1+C_2x\right)}}}. \tag{26}$$

4. Examples

Solitary wave solutions represent an important type of solutions for nonlinear partial differential equations as many nonlinear partial differential equations have been found to have a variety of solitary wave solutions. The solitary wave solutions were obtained in this article and could be helpful in analyzing long wave propagation on the surface of a fluid layer, iron sound waves in plasma, and vibrations in a nonlinear string. Also, solitary wave in the concept of mathematical physics is defined as a self-reinforcing wave package that retains its shape. It propagates at a constant amplitude and velocity.

We illustrate the application of algorithm to solving the equation (1). Exact soliton solution of the equation (1) can be defined explicitly for exact values of $h_1(t) = t$, $h_2(t) = t$, $\omega(t) = t$, $\lambda = 1$, $\mu = 1$. According to (21), we obtain $q_0(t)$, $q_1(t)$, $a(t)$ and $b(t)$:

$$q_0(t) = 0, q_1(t) = 3, a(t) = 3t^2, b(t) = 3. \tag{27}$$

In this case, the soliton solution of the equation (1) has the form

$$u_1(x,t) = \sqrt{\frac{9e^{6(t^2+x)}}{1 - e^{6(t^2+x)}}}. \tag{28}$$

This solution of the equation (1) have been checked and using mathematical software Matlab and three-dimensional graphics of the obtained solutions have been shown. Solitary wave solutions represent an important type of solutions for nonlinear partial differential equations as many nonlinear partial differential equations have been found to have a variety of solitary wave solutions.

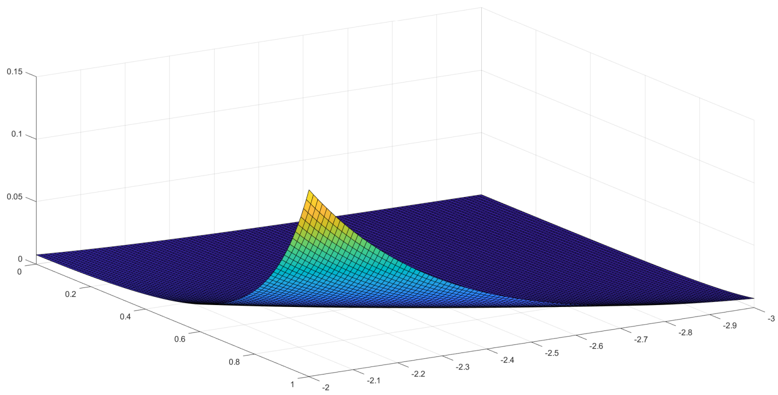

Figure 1: Soliton wave solution of the equation (1) for $h_1(t) = t$, $h_2(t) = t$, $\omega(t) = t$, $\lambda = 1$, $\mu = 1$.

We illustrate the application of algorithm to solving the equation (1). Exact soliton solution of the equation (1) can be defined explicitly for exact values of $h_1(t) = 2t$, $h_2(t) = \frac{t}{t^2+1}$, $\omega(t) = -8t$, $\lambda = 32$, $\mu = \frac{8}{3}$. According to (24), we obtain $q_0(t)$, $q_1(t)$, $a(t)$ and $b(t)$:

$$q_0(t) = \frac{1}{t^2+1}, q_1(t) = \frac{1}{\sqrt{t^2+1}}, a(t) = ln(t^2+1) + t^2, b(t) = \frac{1}{4}. \tag{29}$$

In this case, the soliton solution of the equation (1) has the form

$$u_2(x,t) = \frac{1}{t^2+1} + \sqrt{t^2+1}\sqrt{\frac{e^{2\left(t^2+\frac{1}{4}x\right)}}{1-(t^2+1)^2 e^{2\left(t^2+\frac{1}{4}x\right)}}}. \tag{30}$$

This solution of the equation (1) have been checked and using mathematical software Matlab and three-dimensional graphics of the obtained solutions have been shown. Solitary wave solutions represent an important type of solutions for nonlinear partial differential equations as many nonlinear partial differential equations have been found to have a variety of solitary wave solutions.

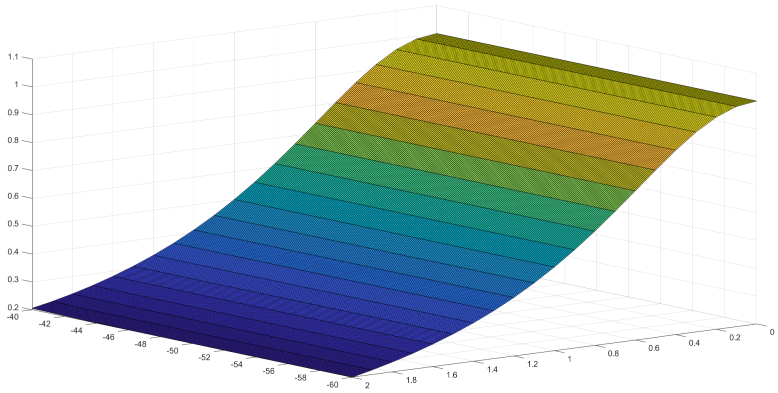

Figure 2: Soliton wave solution of the equation (1) for $h_1(t) = 2t$, $h_2(t) = \frac{t}{t^2+1}$, $\omega(t) = -8t$, $\lambda = 32$, $\mu = \frac{8}{3}$.

5. Conclusion

This paper discusses several traveling wave solutions of the modified Burgers equation with additional time-dependent variable coefficient by the functional variable method. The main advantage of the proposed method over other methods is that it provides more new exact traveling wave solutions. We have found soliton solutions of this equation and three dimensional graphics of the obtained solutions have been drawn by using the Matlab program. After visualizing the graphs of the soliton solutions wave solutions, the use of distinct values of random parameters is demonstrated to better understand their physical features. It is known that the parameters included in the solutions have a deep connection with the amplitudes and velocities. In this regard, we can explore some of the nonlinear phenomena that take place in physics, applied mathematics and technology. We conclude that the exact solutions have its great importance to reveal the internal mechanism of the physical phenomena.

Conflict of Interest

The author declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Author contributions

Bazar Babajanov and Fakhriddin Abdikarimov conceived of the presented idea. Bazar Babajanov developed the theory. Fakhriddin Abdikarimov performed the computations. Fakhriddin Abdikarimov and Sarbinaz Bazarbaeva verified the methods. All authors discussed the results and contributed to the final manuscript and contributed to the article and approved the submitted version.

References

[1] G. P. Agrawal. *Nonlinear Fiber Optics.* Academic Press, San Diego, California, 1989.

[2] O. Abu Arqub. Computational algorithm for solving singular fredholm time-fractional partial integrodifferential equations with error estimates. *Journal of Applied Mathematics and Computing*, 59:227–243, 2019.

[3] O. Abu Arqub and H. Rashaideh. The rkhs method for numerical treatment for integrodifferential algebraic systems of temporal two-point bvps. *Neural Computing and Applications*, 30:2595–2606, 2018.

[4] B. Babajanov and F. Abdikarimov. The application of the functional variable method for solving the loaded non-linear evaluation equations. *Frontiers in Applied Mathematics and Statistics*, 8:912674, 2022.

[5] B. Babajanov and F. Abdikarimov. Exact solutions of the nonlinear loaded benjamin-ono equation. *WSEAS Transactions on Mathematics*, 21:666–670, 2022.

[6] B. Babajanov and F. Abdikarimov. Expansion method for the loaded modified zakharov-kuznetsov equation. *Advanced Mathematical Models & Applications*, 7(2):168–177, 2022.

[7] B. Babajanov and F. Abdikarimov. Solitary and periodic wave solutions of the loaded modified benjamin-bona-mahony equation via the functional variable method. *Researches in Mathematics*, 30(1):10–20, 2022.

[8] B. Babajanov and F. Abdikarimov. Soliton solutions of the loaded modified calogero-degasperis equation. *International Journal of Applied Mathematics*, 35(3):381–392, 2022.

[9] B. Babajanov and F. Abdikarimov. New exact soliton and periodic wave solutions of the nonlinear fractional evolution equations with additional term. *Partial Differential Equations in Applied Mathematics*, 8:100567, 2023.

[10] B. Babajanov and F. Abdikarimov. Solitary and periodic wave solutions of the loaded boussinesq and the loaded modified boussinesq equation. *Journal of Mathematics and Computer Science*, 30(1):67–74, 2023.

[11] B. Babajanov and F. Abdikarimov. Soliton and periodic wave solutions of the nonlinear loaded (3+1)-dimensional version of the benjamin-ono equation by functional variable method. *Journal of Nonlinear Modeling and Analysis*, 5(4):782–789, 2023.

[12] H. Bateman. Some recent researches on the motion of fluids. *Monthly Weather Review*, 43:163–170, 1915.

[13] A. G. Bratsos and L. A . Petrakis. An explicit numerical scheme for the modified burgers equation. *International Journal for Numerical Methods in Biomedical Engneering*, 27(2):232–237, 2011.

[14] A.G. Bratsos. A fourth-order numerical scheme for solving the modified burgers equation. *Computers & Mathematics with Applications*, 60(5):1393–1400, 2010.

[15] J. M. Burgers. A mathematical model illustrating the theory of turbulence. *Advances in Applied Mechanics*, 1:171–199, 1948.

[16] G. Celikten and E. N. Aksan. Explicit exponential finite difference methods for the numerical solution of modified burgers equation. *Eastern Anatolian Journal of Science*, 3(1):45–50, 2017.

[17] H. Demiray. Variable coefficient modified kdv equation in fluid-filled elastic tubes with stenosis: solitary waves. *Chaos, Solitons & Fractals*, 42:358–364, 2009.

[18] W. Djoudi and A. Zerarka. Exact solutions for the kdv-mkdv equation with time-dependent coefficients using the modified functional variable method. *Cogent Mathematics*, 3(1):1–9, 2016.

[19] S. A. El-Wakil and M. A. Abdou. New exact travelling wave solutions using modified extended tanh-function method. *Chaos, Solitons & Fractals*, 31(4):840–852, 2007.

[20] M. Farhan, Z. Omar, F. Mebarek-Oudina, J. Raza, Z. Shah, R. V Choudhari, and O. D. Makinde. Implementation of the one-step one-hybrid block method on the nonlinear equation of a circular sector oscillator. *Computational Mathematics and Modeling*, 31:116–132, 2020.

[21] J. A. Gear and R. Grimshaw. A second-order theory for solitary waves in shallow fluids. *Physics of Fluid*, 26(14):14–29, 1983.

[22] A. Griewank and T. S. El-Danaf. Efficient accurate numerical treatment of the modified burgers equation. *Applicable Analysis*, 88(1):75–87, 2009.

[23] R. Hirota. Exact solution of the kdv equation for multiple collisions of solutions. *Physical Review Letters*, 27:1192–1194, 1971.

[24] D. Irk. Sextic b-spline collocation method for the modified burgers equation. *Matmematics and Computers in Simulation*, 38(9):1599–1620, 2009.

[25] W. Malfliet. Solitary wave solutions of nonlinear wave equations. *American Journal of Physics*, 60(7):650–654, 1992.

[26] Sh. Momani, O. Abu Arqub, and B. Maayah. Piecewise optimal fractional reproducing kernel solution and convergence analysis for the atangana-baleanu-caputo model of the lienard's equation. *Fractals*, 28(8):2040007, 2020.

[27] Sh. Momani, B. Maayah, and O. Abu Arqub. The reproducing kernel algorithm for numerical solution of van der pol damping model in view of the atangana-baleanu fractional approach. *Fractals*, 28(8):2040010, 2020.

[28] H. Naher, F. A. Abdullah, and M. A. Akbar. The exp-function method for new exact solutions of the nonlinear partial differential equations. *International Journal of Physical Sciences*, 6(29):6706–6716, 2011.

[29] M. A. Ramadan and T. S. El-Danaf. Numerical treatment for the modified burgers equation. *Matmematics and Computers in Simulation*, 70(2):90–98, 2005.

[30] M. A. Ramadan and T. S. El-Danaf. Numerical treatment for the modified burgers equation. *Mathematics and Computers in Simulation*, 70(2):90–98, 2005.

[31] M. A. Ramadan, T. S. El-Danaf, and F. E. I. ABD Alaal. A numerical solution of the burgers equation using septic b-splines. *Chaos, Solitons & Fractals*, 26(3):795–804, 2005.

[32] C. Rogers and W. F. Shadwick. Backlund transformations and their applications. *Mathematics in Science and Engineering*, 161:334, 1982.

[33] T. Roshan and K. S. Bhamra. Numerical solutions of the modified burgers equation by petrov-galerkin method. *Applied Mathematics and Computation*, 218(7):3673–3679, 2011.