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Second Order Transmuted Kumaraswamy Distribution and Its Related Results

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Abstract. The expansion of the application domains is greatly aided by the generalization of distributions. To achieve this, an extension of the two–parameter Kumaraswamy distribution to a three–parameter second order transmuted Kumaraswamy distribution has been presented in this study utilizing the cubic transmutation map. Along with the order statistics and parameter estimates, a number of statistical characteristics are shown. The performance of the estimated parameters has been demonstrated through simulation study. A few applications additionally serve to demonstrate the suggested distribution's dominating applicability.

2020 Mathematics Subject Classifications: 60E05, 62E15, 62G30, 62F10, 62H99

Key Words and Phrases: Cubic Transmutation, Probability Distribution, Kumaraswamy Distribution, Order Statistics, Maximum Likelihood Estimation

1. Introduction

Several distributions as a tool for data modeling have already been introduced by many researchers. In other circumstances, these tools are less well with complicated real-world data. Generalization can be an existing process to boost the potency of these tools. According to this idea, many researchers significantly contribute to the Kumaraswamy (Kw) distribution (Kumaraswamy [16]) to extend the flexibility of this distribution, which has the following *pdf*

$$g\left(x;a,b\right) = abx^{a-1}\left(1-x^{a}\right)^{b-1}, \quad x \in \left(0,1\right), \tag{1}$$

where $a, b \in \mathcal{R}^{+}$ are the two shape parameters. This distribution is the most widely applied statistical distribution in hydrological problems and it is similar to Beta distribution.

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Jones [17] introduced the two-parameter family of distributions, which share many features with the beta distribution and have both benefits and drawbacks in terms of tractability. Due to its tractability, the distribution may consequently play a special role in statistical modeling that adopts a quantile-based method. A generalization of the Kumaraswamy distribution (Kw-G) was first proposed by Cordeiro and de Castro [5]. With the different choices of G, some researchers developed Kumaraswamy-Weibull, Kumaraswamy-Gumbel, Kumaraswamy-generalized gamma, Kumaraswamy-half cauchy distributions, (Cordeiro et al. [7], [6], De Pascoa et al. [9], and Ghosh [12]) and so on. Wang et al. [23] described the inference on Kumaraswamy distribution. Wang [24] discussed the statistical characteristics, modeling, and inference of increasingly censored competing risks on Kumaraswamy distribution.

Recently some researchers contribute their research work on Kumaraswamy distribution including a general modified Kumaraswamy distribution for data modeling (Alshkaki [2]), Marshall–Olkin Kumaraswamy exponential distribution (Almarashi et al.[1]), arsine Kumaraswamy–Generalized family (Emam and Tashkandy [10]), inverse Kumaraswamy family (Daghistani [8]) etc.

The most common first order transmuted family of distributions was first introduced by Shaw and Buckley [20] from which a number of extended distributions have already been generated and have the following *cdf*

$$F\left(x\right)=\left(1+\lambda\right)G\left(x\right)-\lambda G\left(x\right)^{2},\quad x\in\mathcal{R}, \tag{2}$$

where $\lambda \in [-1, 1]$ is the transmutation parameter and $G(x)$ is the base distribution function of any standard probability model. Transmuted Kumaraswamy (TKw) distribution and transmuted inverted Kumaraswamy distribution are the commonly generated distributions from this family of distributions and these generated distributions were proposed by Khan et al. [15] and Sherwania et al. [21].

Saraçoğlu and Tanış [19] proposed the cubic rank transmuted Kumaraswamy distribution using the cubic rank transmutation map introduced by Granzotto et al. [14] by extending the transmutation map of equation (2). After that, Rahman et al. [18] extended the first order transmutation map (see equation (2)) to second order transmutation map, which has the following *cdf*

$$F\left(x\right)=\left(1-\lambda\right)G\left(x\right)+3\lambda G\left(x\right)^{2}-2\lambda G\left(x\right)^{3},\quad x\in\mathcal{R}, \tag{3}$$

where λ ranges between -1 to 1. Several flexible models have already been proposed using this transmutation map.

The precise objectives of this study are to generate a flexible extended second order formation of Kumaraswamy distribution can be called cubic transmuted Kumaraswamy distribution (CTKw) (differs from the cubic rank transmuted Kumaraswamy (CRTKw) distribution) using the transmutation map mentioned in equation (3). To display several analytical shapes and more improved properties of the proposed distribution than CRTKw distribution. The performance of the estimated parameters (obtained via Maximum Likelihood Estimation (MLE)) of the CTKw distribution will be examined through simulation study and the suggested distribution's adaptability in real-world data sets can be obtained with the aid of goodness of fit as well.

The following is an outline of the article. Section 2 talks about the suggested CTIW distribution. The statistical aspects of the CTKw distribution, such as moments, generating functions, quantile functions, random number generation, and reliability function presented in Section 3, along with the order statistics of the distribution are explored in Section 4. The CTKw distribution parameter estimate is found in Section 5. The performance of MLEs is assessed via simulation in Section 6. Section 7 uses two actual applications from the real world to prove the dominant flexibility of CTKw distribution over comparable distributions. Section 8 concludes by listing a few last thoughts.

2. Cubic Transmuted Kumaraswamy Distribution

Kumaraswamy [16] first introduced the Kumaraswamy distribution which has the following distribution form as

$$F\left(x;a,b\right)=1-\left(1-x^{a}\right)^{b},\quad x\in\left(0,1\right), \tag{4}$$

where a and b are the shape parameters in the positive real number. The *cdf* of the transmuted Kumaraswamy distribution proposed by Khan et al. [15] using the transmutation map as mentioned in (2), can be given as

$$F_{TKw}\left(x;a,b,\lambda\right)=\left[1-\left(1-x^{a}\right)^{b}\right]\left[1+\lambda\left(1-x^{a}\right)^{b}\right],\quad x\in\left(0,1\right), \tag{5}$$

where λ is the same as (3).

Now, by substituting the *cdf* of the Kumaraswamy distribution from equation (4) to the *cdf* of the second order transmutation map in equation (3), a continuous random variable X is said to have CTKw distribution if it has the following distributional form

$$F_{CTKw}\left(x;a,b,\lambda\right)=\left[\left(1-x^{a}\right)^{b}-1\right]\left[\left(1-x^{a}\right)^{b}\left\{2\left(1-x^{a}\right)^{b}-1\right\}\lambda-1\right],\quad x\in\left(0,1\right), \tag{6}$$

where a and b are the shape parameters within the range of positive real number and $\lambda\in[-1,1]$. Then differentiating equation (6) with respect to x and hence the functional form of the corresponding *pdf* of the CTKw distribution is

$$f_{CTKw}\left(x;a,b,\lambda\right)=abx^{a-1}\left(1-x^{a}\right)^{b-1}\left[1-\lambda+6\lambda\left(1-x^{a}\right)^{b}-6\lambda\left(1-x^{a}\right)^{2b}\right],\quad x\in\left(0,1\right), \tag{7}$$

where the shape parameters $a,b\in\mathcal{R}^{+}$ and λ (transmutation parameter) lies between -1 to 1.

The following are some examples of special situations of the proposed CTKw distribution:

(i) The *cdf* of CTKw distribution corresponds with the uniform distribution for $a = b = 1$ and $\lambda = 0$.

(ii) For $a = b = 1$, the *cdf* of CTKw distribution reduces to cubic transmuted uniform distribution (see Rahman et al. [18]).

(iii) When $\lambda = 0$, the CTKw distribution reduces to the Kumaraswamy distribution.

(iv) The density function of the new CTKw distribution tends to zero when $x \to 0$.

In Figure 1, some potential *pdf* and *cdf* forms for the new CTKw distribution for various settings of the model parameters (a, b, and λ) are shown. The graphic demonstrates that the suggested CTKw may be utilized to model different possible behavior (both uni-modal and bi-modal) of data.

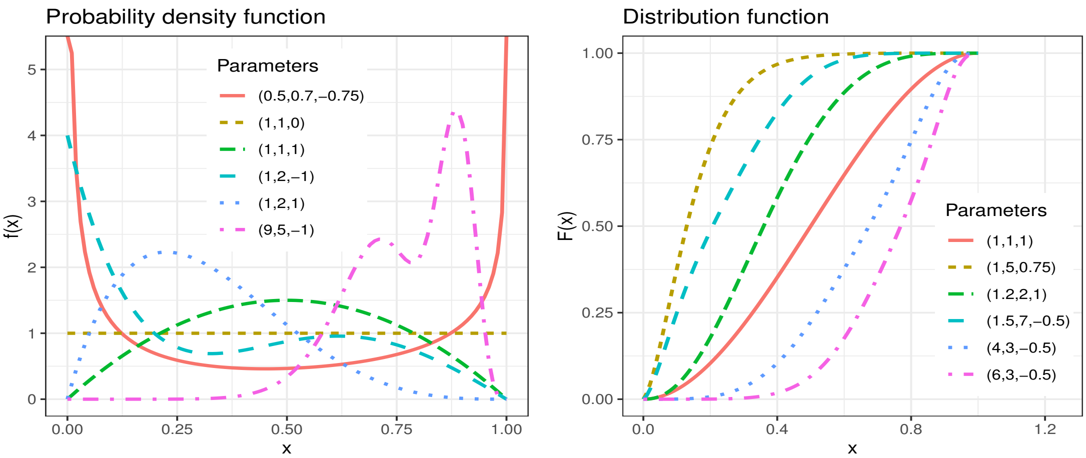

Figure 1: Plots of Density and Distribution functions of the proposed CTKw distribution for different combinations of the parameters

3. Properties of CTKw Distribution

The following subsections provide some crucial statistical characteristics of the proposed cubic transmuted Kumaraswamy distribution.

3.1. Moments

Expressions for various moments and shape characteristics of the proposed cubic transmuted Kumaraswamy distribution is provided in this subsection. The r^{th} moment of CTKw distribution can be given as

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r \, f(x) \, dx$$

$$
\begin{aligned}
&= \int_0^1 x^r\ abx^{a-1}\left(1-x^a\right)^{b-1}\left[1-\lambda+6\lambda\left(1-x^a\right)^b-6\lambda\left(1-x^a\right)^{2b}\right]dx \\
&= ab\left[(1-\lambda)\int_0^1 x^{a+r-1}\left(1-x^a\right)^{b-1}+6\lambda\int_0^1 x^{a+r-1}\left(1-x^a\right)^{2b-1}\right. \\
&\quad \left.-6\lambda\int_0^1 x^{a+r-1}\left(1-x^a\right)^{3b-1}\right] \\
&= b\Gamma\left(\frac{a+r}{a}\right)\left[\frac{(1-\lambda)\Gamma(b)}{\Gamma\left(b+\frac{r}{a}+1\right)}+\frac{6\lambda\Gamma(2b)}{\Gamma\left(2b+\frac{r}{a}+1\right)}-\frac{6\lambda\Gamma(3b)}{\Gamma\left(3b+\frac{r}{a}+1\right)}\right] \\
&= b\left[(1-\lambda)\beta\left(\frac{r}{a}+1,b\right)+6\lambda\beta\left(\frac{r}{a}+1,2b\right)-6\lambda\beta\left(\frac{r}{a}+1,3b\right)\right]
\end{aligned}
$$

$$\mu_r' = b\left[(1-\lambda)\Psi_{1,r}+6\lambda\Psi_{2,r}-6\lambda\Psi_{3,r}\right], \tag{8}$$

where $\Psi_{\kappa,r}$ is introduced for simplicity as

$$\Psi_{\kappa,r}=\beta\left(\frac{r}{a}+1,\kappa b\right),\quad \kappa=1,2,3.$$

The mean of the CTKw distribution can be given by substituting $r=1$ in equation (8) as

$$\mu=\ \mu_1'=b\left[(1-\lambda)\Psi_{1,1}+6\lambda\Psi_{2,1}-6\lambda\Psi_{3,1}\right]. \tag{9}$$

Again, the variance of the CTKw distribution can be given by substituting $r=2$ in equation (8) and using the equation $Var(X)=\mu_2=\mu_2'-\left(\mu_1'\right)^2$ as

$$
\begin{aligned}
Var(X) &= b\left[(1-\lambda)\Psi_{1,2}+6\lambda\Psi_{2,2}-6\lambda\Psi_{3,2}\right]-b^2\left[(1-\lambda)\Psi_{1,1}\right. \\
&\quad \left.+6\lambda\Psi_{2,1}-6\lambda\Psi_{3,1}\right]^2.
\end{aligned} \tag{10}
$$

For the greater value of r (>2), it is simple to obtain the other moments and shape characteristics of the proposed distribution. Now using the equations (9) and (10), it can be possible to obtain mean and variance charts with the different combinations of the parameters of the CTKw distribution that has been presented in the table 1 and 2 respectively. From the two tables, it can be visualized that with increasing the shape parameter a, mean values are rising gradually while variances are reducing slowly as demonstrated in Figure 2 and 3.

3.2. Moment Generating Function

Another method of obtaining moments of any distribution is the moment generating function (mgf). The following theorem states and proves the mgf of the CTKw distribution.

Theorem 1. *The moment generating function of the continuous random variable X which follows CTKw distribution can be given as*

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}b\left[(1-\lambda)\beta\left(\frac{r}{a}+1,b\right)+6\lambda\beta\left(\frac{r}{a}+1,2b\right)-6\lambda\beta\left(\frac{r}{a}+1,3b\right)\right], \tag{11}$$

Table 1: Mean chart for the CTKw distribution

		$b=2$	$b=4$	$b=6$	$b=8$	$b=10$
	$a=1$	0.352	0.221	0.160	0.126	0.103
	$a=5$	0.745	0.668	0.623	0.591	0.568
$\lambda=-1$	$a=10$	0.856	0.810	0.781	0.761	0.745
	$a=15$	0.900	0.867	0.846	0.832	0.820
	$a=20$	0.923	0.897	0.881	0.870	0.861
	$a=1$	0.343	0.210	0.152	0.118	0.097
	$a=5$	0.751	0.672	0.626	0.594	0.570
$\lambda=-0.5$	$a=10$	0.861	0.813	0.785	0.764	0.749
	$a=15$	0.903	0.870	0.849	0.834	0.823
	$a=20$	0.926	0.900	0.884	0.872	0.863
	$a=1$	0.324	0.190	0.134	0.104	0.085
	$a=5$	0.764	0.681	0.633	0.600	0.575
$\lambda=0.5$	$a=10$	0.871	0.821	0.792	0.771	0.755
	$a=15$	0.911	0.876	0.855	0.840	0.828
	$a=20$	0.932	0.905	0.889	0.877	0.868
	$a=1$	0.314	0.179	0.126	0.096	0.078
	$a=5$	0.770	0.685	0.636	0.603	0.578
$\lambda=1$	$a=10$	0.876	0.825	0.795	0.774	0.758
	$a=15$	0.915	0.879	0.858	0.843	0.831
	$a=20$	0.935	0.908	0.891	0.879	0.870

where $t \in \mathcal{R}$.

Proof. *We know from the definition of* mgf *that*

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\ f(x)\ dx,$$

where the density function $f(x)$ *of the proposed CTKw distribution has already been given in equation (7). With the assist of series expansion provided by Gradshteyn and Ryzhik [13] we have,*

$$M_X(t) = \int_0^1 \sum_{r=0}^{\infty} \frac{t^r}{r!}\ x^r\ f(x)\ dx = \sum_{r=0}^{\infty} \frac{t^r}{r!}\ E\left(X^r\right). \tag{12}$$

Now by substituting the raw moment from equation (8) to (12), the mgf of the proposed CTKw distribution (11) is then proved.

3.3. Quantile Function

The quantile function of any distribution function can be obtained by solving the equation

$$F(x) = q, \tag{13}$$

Table 2: Variance chart for the CTKw distribution

		$b=2$	$b=4$	$b=6$	$b=8$	$b=10$
	$a=1$	0.081	0.040	0.023	0.015	0.011
	$a=5$	0.032	0.030	0.028	0.026	0.024
$\lambda=-1$	$a=10$	0.012	0.013	0.012	0.012	0.012
	$a=15$	0.006	0.007	0.007	0.007	0.007
	$a=20$	0.004	0.004	0.004	0.004	0.004
	$a=1$	0.068	0.033	0.019	0.013	0.009
	$a=5$	0.027	0.025	0.023	0.022	0.020
$\lambda=-0.5$	$a=10$	0.010	0.010	0.010	0.010	0.010
	$a=15$	0.005	0.006	0.006	0.006	0.005
	$a=20$	0.003	0.003	0.003	0.003	0.003
	$a=1$	0.043	0.020	0.011	0.007	0.005
	$a=5$	0.016	0.015	0.013	0.012	0.012
$\lambda=0.5$	$a=10$	0.006	0.006	0.006	0.006	0.005
	$a=15$	0.003	0.003	0.003	0.003	0.003
	$a=20$	0.002	0.002	0.002	0.002	0.002
	$a=1$	0.030	0.012	0.007	0.004	0.003
	$a=5$	0.010	0.009	0.009	0.008	0.007
$\lambda=1$	$a=10$	0.003	0.004	0.004	0.003	0.003
	$a=15$	0.002	0.002	0.002	0.002	0.002
	$a=20$	0.001	0.001	0.001	0.001	0.001

for x. Now , if $F(x)$ is the distribution function of the CTKw distribution as given in equation (6), then the corresponding quantile x_q of the CTKw distribution will be

$$x_q = \left[1 - \left\{ \frac{\sqrt[3]{\vartheta + 9\lambda^2(2q-1)}}{2\ 3^{2/3}\lambda} + \frac{\lambda+2}{2\sqrt[3]{3\vartheta + 27\lambda^2(2q-1)}} + \frac{1}{2} \right\}^{1/b} \right]^{1/a}, \quad 0 < x, q < 1, \tag{14}$$

where

$$\vartheta = \sqrt{3}\sqrt{-\lambda^3(\lambda(\lambda(\lambda+6) - 108(q-1)q - 15) + 8)}.$$

One can easily obtain all the three quartiles with the substitution of $q = \frac{25}{100}, \frac{50}{100}$, and $\frac{75}{100}$ respectively.

3.4. Random Sample Simulatioin

The process of selecting a sample from a larger population in a way that represents unpredictability and variability is modeled using a statistical approach called a random sample simulation. When it is neither practicable or practical to collect data from the full population, it is especially beneficial for performing experiments or studies. For simulating the random sample we have to solve the equation (13) in terms of u ($u \in U(0,1)$) instead

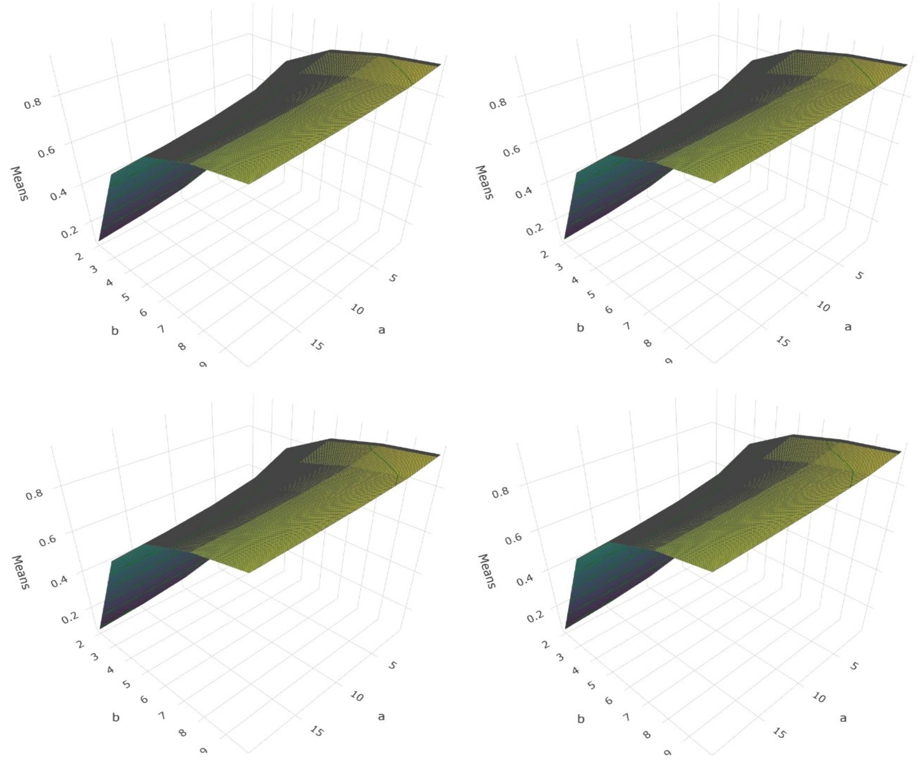

Figure 2: Mean plots of the proposed CTKw distribution with different combinations of the parameters a, b along with fixed values of $\lambda = -1$ (upperleft), $\lambda = -0.5$ (upperright), $\lambda = 0.5$ (lowerleft), and $\lambda = 1$ (lowerright)

of q i.e.

$$F(x) = u.$$

Or,

$$\left[(1-x^a)^b - 1\right]\left[(1-x^a)^b\left\{2\,(1-x^a)^b - 1\right\}\lambda - 1\right] = u,$$

for x. After some simplification, the simulating random sample is then given as

$$X = \left[1 - \left\{\frac{\sqrt[3]{\vartheta + 9\lambda^2(2u-1)}}{2\;3^{2/3}\lambda} + \frac{\lambda+2}{2\sqrt[3]{3\vartheta + 27\lambda^2(2u-1)}} + \frac{1}{2}\right\}^{1/b}\right]^{1/a}, \tag{15}$$

where

$$\vartheta = \sqrt{3}\sqrt{-\lambda^3(\lambda(\lambda(\lambda+6) - 108(u-1)u - 15) + 8)}.$$

Here, this equation (15) will further use in the simulation study.

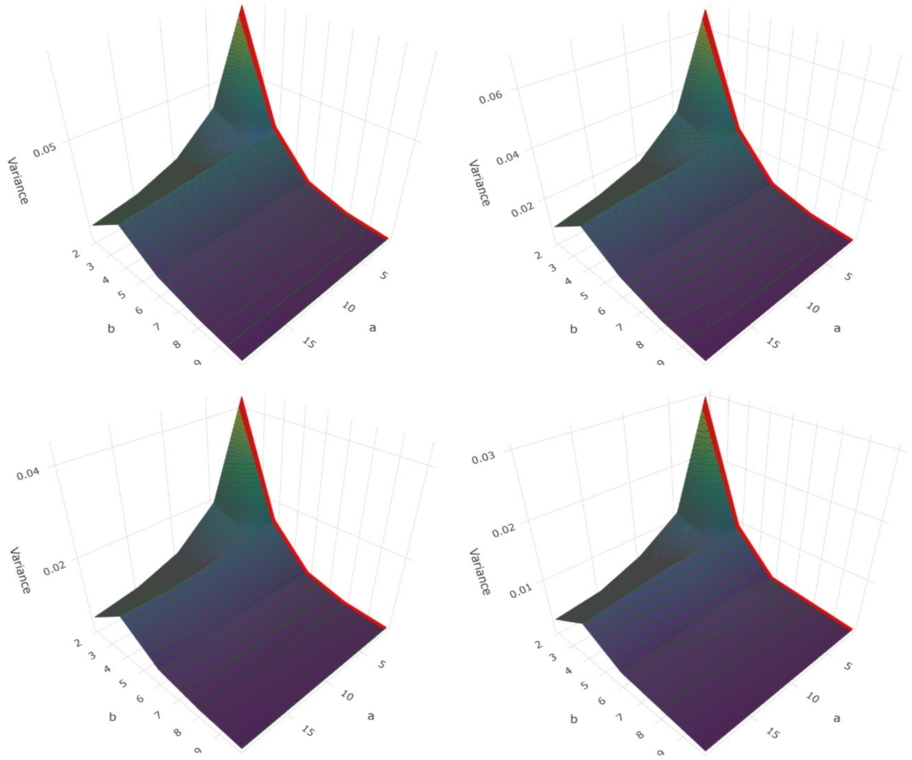

Figure 3: Variance plots of the proposed CTKw distribution with different combinations of the parameters a, b along with fixed values of $\lambda = -1$ (upperleft), $\lambda = -0.5$ (upperright), $\lambda = 0.5$ (lowerleft), and $\lambda = 1$ (lowerright)

3.5. Reliability Analysis

Reliability analysis is a systematic method for evaluating and quantifying the performance and dependability of systems, goods, services, or other elements across time. The reliability function of the proposed CTKw distribution is simply defined as

$$\begin{aligned} R(t) &= 1 - F(t) \\ &= (1-t^a)^b \left[\lambda \left\{3\,(1-t^a)^b - 2\,(1-t^a)^{2b} - 1\right\} + 1\right]. \end{aligned}$$

In reliability and survival analysis, the hazard function is a key idea. It calculates the instantaneous rate at which certain occurrences, such as failures, take place over a certain period, assuming that the event has not yet happened. The hazard function of the CTKw distribution is denoted as $h(t)$ and defined as

$$\begin{aligned} h(t) &= \frac{f(t)}{R(t)} \\ &= \frac{abt^{a-1}\left[\lambda \left\{6\,(1-t^a)^b - 6\,(1-t^a)^{2b} - 1\right\} + 1\right]}{(1-t^a)^b \left[\lambda \left\{3\,(1-t^a)^b - 2\,(1-t^a)^{2b} - 1\right\} + 1\right]}. \end{aligned}$$

Figure 4 now illustrates several potential reliability and hazard function forms, including monotonically declining reliability functions and a variety of hazard function types, including increasing, decreasing, both increasing and decreasing, and others for several combinations of the model parameters (a, b, and λ).

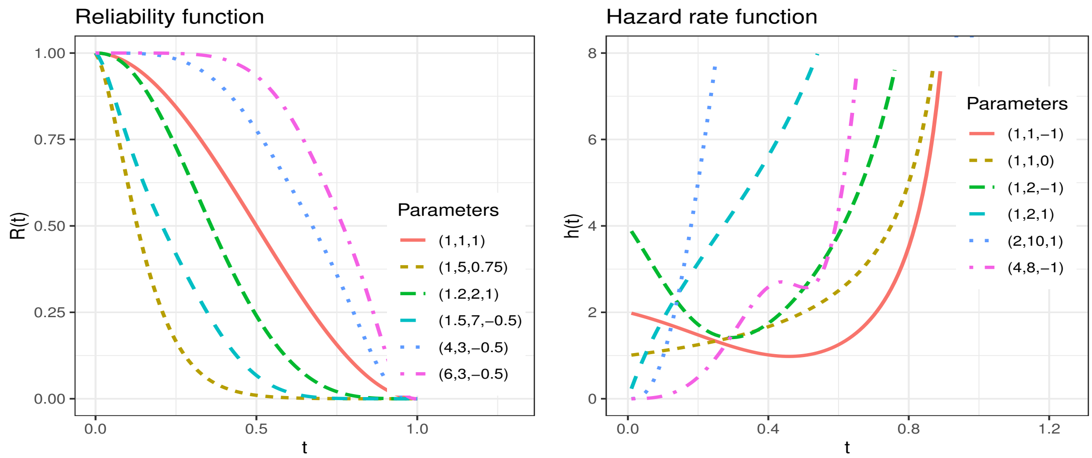

Figure 4: Plots of Reliability and Hazard functions of the proposed CTKw distribution for different combinations of the parameters

4. Order Statistics

Let, $X_1, X_2, ..., X_n$ be iid random variables which are arranged in ascending order according to order statistics. These statistics offer insightful information on the distribution and properties of the data. Then by the definition of the *pdf* of i^{th} order statistics, we know

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x). \tag{16}$$

With the substitution of the density and distribution function of the proposed CTKw distribution in equation (16), the obtaining *pdf* of i^{th} order statistics for the proposed distribution can be given as

$$\begin{aligned} f_{i:n}(x) \quad &= \quad \frac{n!}{(i-1)!(n-i)!} \left[abx^{a-1} (1-x^a)^{b-1} \left\{1 - \lambda + 6\lambda (1-x^a)^b - 6\lambda (1-x^a)^{2b}\right\}\right] \\ &\times \left[\left\{(1-x^a)^b - 1\right\} \left\{(1-x^a)^b \left(2 (1-x^a)^b - 1\right) \lambda - 1\right\}\right]^{i-1} \\ &\times \left[(1-x^a)^b \left\{\lambda \left(3 (1-x^a)^b - 2 (1-x^a)^{2b} - 1\right) + 1\right\}\right]^{n-i}, \end{aligned} \tag{17}$$

where $i = 1, 2, ..., n$. The *pdf* of the lowest order statistic for CTKw distribution can be given by substituting $i = 1$ in (17) as

$$f_{1:n}(x) \quad = \quad n \left[abx^{a-1} (1-x^a)^{b-1} \left\{1 - \lambda + 6\lambda (1-x^a)^b - 6\lambda (1-x^a)^{2b}\right\}\right]$$

$$\times \left[(1-x^a)^b \left\{ \lambda \left(3(1-x^a)^b - 2(1-x^a)^{2b} - 1 \right) + 1 \right\} \right]^{n-1}.$$

Again, the *pdf* of the highest order statistic for CTKw distribution can be given by substituting $i = n$ in (17) as

$$\begin{aligned} f_{n:n}(x) &= n \left[abx^{a-1} (1-x^a)^{b-1} \left\{ 1 - \lambda + 6\lambda (1-x^a)^b - 6\lambda (1-x^a)^{2b} \right\} \right] \\ &\times \left[\left\{ (1-x^a)^b - 1 \right\} \left\{ (1-x^a)^b \left(2(1-x^a)^b - 1 \right) \lambda - 1 \right\} \right]^{n-1}. \end{aligned}$$

Further, with $\lambda = 0$, one can easily obtain the *pdf* of the i^{th} order statistics for the base Kumaraswamy distribution as

$$g_{r:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} abx^{a-1} (1-x^a)^{b-1} \left[1 - (1-x^a)^b \right]^{i-1} \left[(1-x^a)^b \right]^{n-i}.$$

The density function of the k^{th} order statistic for the proposed CTKw distribution can be obtained by using the following formula

$$E(X_{i:n}^k) = \int_0^1 x_i^k \; f_{i:n}(x) \; dx.$$

5. Estimation of the Parameters and Inference

Using Maximum Likelihood Estimation (MLE), the CTKw distribution's parameters are estimated in this section. A frequent and effective strategy in statistics is MLE model parameter estimation. The goal of MLE is to determine the parameter values that maximize the likelihood of witnessing the provided data under the presumptive statistical model. Take into consideration a random sample of size n from the suggested CTKw distribution, with the likelihood function as

$$\mathcal{L} = \prod_{i=1}^{n} \left[abx_i^{a-1} (1-x_i^a)^{b-1} \left\{ 1 - \lambda + 6\lambda (1-x_i^a)^b - 6\lambda (1-x_i^a)^{2b} \right\} \right].$$

Then the corresponding log-likelihood (LL) function

$$l = ln(\mathcal{L}).$$

After some simplification, the LL function of the proposed CTKw distribution can be given as

$$\begin{aligned} l &= n\log(a) + n\log(b) + (a-1)\sum_{i=1}^{n} \log(x_i) + (b-1)\sum_{i=1}^{n} \log(1-x_i^a) \\ &+ \sum_{i=1}^{n} \log\left[1 - \lambda + 6\lambda (1-x_i^a)^{b} - 6\lambda (1-x_i^a)^{2b} \right]. \end{aligned} \tag{18}$$

The LL function must now maximize for MLE to estimate the parameters. In order to do this, consider the derivatives of (18) with respect to the unknowable parameters a, b, and λ, and proceed as follows.

$$\begin{aligned}\frac{\partial l}{\partial a} &= \frac{n}{a} + \sum_{i=1}^{n} \log(x_i) - (b-1)\sum_{i=1}^{n} \frac{x_i^a \log(x_i)}{1-x_i^a} \\ &+ \sum_{i=1}^{n} \frac{12b\lambda x_i^a \log(x_i)\left(1-x_i^a\right)^{2b-1} - 6b\lambda x_i^a \log(x_i)\left(1-x_i^a\right)^{b-1}}{1-\lambda+6\lambda\left(1-x_i^a\right)^{b} - 6\lambda\left(1-x_i^a\right)^{2b}},\end{aligned}$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log\left(1-x_i^a\right) + \sum_{i=1}^{n} \frac{6\lambda\left(1-x_i^a\right)^{b}\log\left(1-x_i^a\right) - 12\lambda\left(1-x_i^a\right)^{2b}\log\left(1-x_i^a\right)}{1-\lambda+6\lambda\left(1-x_i^a\right)^{b} - 6\lambda\left(1-x_i^a\right)^{2b}},$$

and

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} \frac{6\left(1-x_i^a\right)^{b} - 6\left(1-x_i^a\right)^{2b} - 1}{1-\lambda+6\lambda\left(1-x_i^a\right)^{b} - 6\lambda\left(1-x_i^a\right)^{2b}}.$$

Now, by setting $\frac{\partial l}{\partial a} = 0$, $\frac{\partial l}{\partial b} = 0$, and $\frac{\partial l}{\partial \lambda} = 0$, and simultaneously solving the resulting equations, the maximum likelihood estimate of the expression $\hat{\Omega} = \left(\hat{a}, \hat{b}, \hat{\lambda}\right)'$ of $\Omega = (a, b, \lambda)'$ is discovered. The resulting equations could not be solved analytically, thus we used the R packages "bbmle", Bolker and Bolker [3] to solve them numerically. The asymptotic distribution of the MLE as $n \to \infty$, is given by

$$\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{\lambda} \end{pmatrix} \sim N\left[\begin{pmatrix} a \\ b \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix}\right].$$

The estimates, $\hat{a}$, $\hat{b}$, and $\hat{\lambda}$'s asymptotic variance-covariance matrix

$$V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}$$

is produced by inverting the Hessian matrix (see Appendix). For a, b, and λ, the following formulas provide roughly 100(1-α)% two-sided confidence intervals:

$$\hat{a} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}, \quad \hat{b} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}}, \quad and \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}},$$

where the α percentile of the standard normal distribution is represented by the constant Z_{α}.

6. Simulation Study

The efficacy of the MLEs for the three CTKw distributional parameters a, b, and λ is evaluated in this section. The CTKw distribution may be simulated using equation (15). We presumed the starting parameter values $a = 1$, $b = 3$, and $\lambda = -1$ under the support of these parameters, with varied sample sizes of 50, 100, 200, 500, and 1000 to create random samples for the CTKw distribution. For each sample size, the MLEs are obtained, and the procedure is repeated 1000 times in total. After that, two sets of initial parameter values including $a = 4.07$, $b = 6.44$, $\lambda = -1$ and $a = 0.9$, $b = 0.86$, $\lambda = -1$ have considered and the above process have similarly repeated. Table 3 displays the computed mean and MSE for these 1000 data of each of the three arbitrarily set initial values. We have demonstrated from the table 3 that the parameter estimates are quite close to the actual values and that the MSE decreases as the sample size rises as illustrated in Figure 5. This shows that the estimating method is effective.

Table 3: Average estimates of model parameters and corresponding MSEs

Sample Size	Initial Parameter	Estimate a	Estimate b	Estimate λ	MSE a	MSE b	MSE λ
50	$a = 1$	0.936	2.780	-0.682	0.034	0.640	0.365
100		0.942	2.801	-0.740	0.016	0.296	0.172
200	$b = 3$	0.956	2.840	-0.817	0.008	0.143	0.077
500		0.964	2.866	-0.861	0.003	0.064	0.036
1000	$\lambda = -1$	0.978	2.915	-0.916	0.001	0.030	0.015
50	$a = 4.07$	4.052	6.533	-0.855	0.413	4.622	0.262
100		4.021	6.352	-0.903	0.173	1.765	0.106
200	$b = 6.44$	4.043	6.372	-0.928	0.082	0.820	0.049
500		4.049	6.376	-0.951	0.031	0.313	0.018
1000	$\lambda = -1$	4.047	6.375	-0.969	0.016	0.154	0.009
50	$a = 0.9$	0.897	0.858	-0.848	0.043	0.032	0.297
100		0.892	0.852	-0.891	0.017	0.014	0.111
200	$b = 0.86$	0.891	0.854	-0.933	0.008	0.006	0.049
500		0.893	0.853	-0.952	0.003	0.002	0.019
1000	$\lambda = -1$	0.895	0.857	-0.969	0.001	0.001	0.009

7. Numerical Results

The effectiveness of the suggested CTKw distribution is assessed in this section. Here, two real-world data sets with censored and lifetime data have been taken into consideration to apply to the suggested distribution. Then, to apply to the same data sets, three analogous distributions, including the CRTKw, TKw, and Kw distributions, have also been taken into consideration. Eventually, the superior flexibility of the CTKw distribution has been proved through the measure of goodness of fit including the Akaike

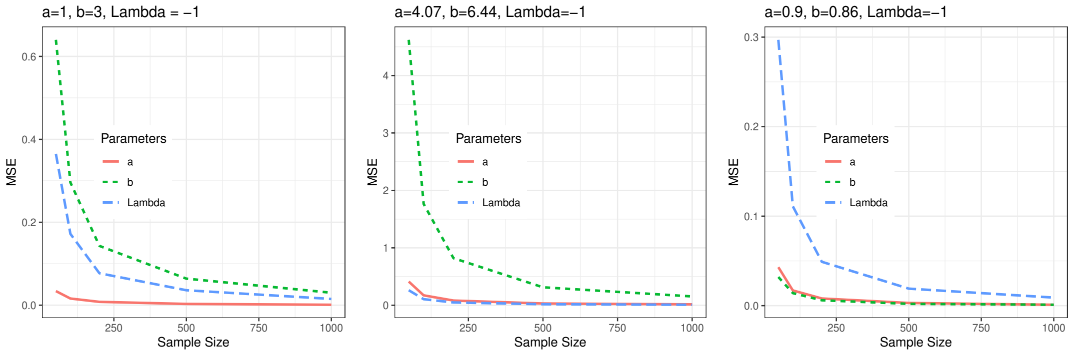

Figure 5: Plots of the decreasing MSE according to the sample size

Information Criterion (AIC), the Second Order Estimate of Akaike Information Criterion (AICc), the Bayesian Information Criterion (BIC), Log–likelihood (LL), 2Log–likelihood (2LL), Kolmogorov–Smirnov (KS) Statistic, and Cramer–von Mises (C-vM) Statistic, and Anderson–Darling (A) Statistic as well as the graphical illustration like Empirical Cumulative Distribution Function (ECDF) and Empirical Probability Distribution Function (EPDF) (see Brewer et al. [4], and Evans et al. [11] for details). The values of these test statistics (KS, A, C-vM) measure the discrepancy between the empirical distribution derived from the data and the theoretical distribution assumed by the model. Lower values of these statistics indicate better fits, implying that the empirical distribution is closer to the theoretical distribution. The summary statistics of the two data sets are presented in the following Table 4.

Table 4: The summarizing statistics as well as the data sets skewness and kurtosis

Data set	Min.	Q_1	Median	Q_3	Max.	Mean	Skewness	Kurtosis
Censored	0.127	0.425	0.640	0.723	0.825	0.577	-0.625	-0.701
Lifetime	0.020	0.143	0.507	0.892	0.990	0.494	0.062	-1.773

In addition, in Figure 6 and 7, four plots are displayed for the censored and lifetime data respectively. Among them the Probability-Probability (P-P) and Quantile-Quantile (Q-Q) plots asses that the proposed CTKw model strongly agrees with both the data. Also, the Total Time on Test (TTT) plot is used as a tool for identifying the graphical behavior of the Hazard Rate Function (HRF) for the above two data sets. It has been observed from the TTT plots of the two data sets that the concave and convex TTT plots predict increasing hazard curves for both data sets.

7.1. Application of Cubic Transmuted Kumaraswamy Distribution for Type II Censored Data

This data was used by Tu and Gui [22], and it is slightly negatively skewed as shown in Table 4. By employing 21 observations, a generalized progressive hybrid censored sample was created for the data below:

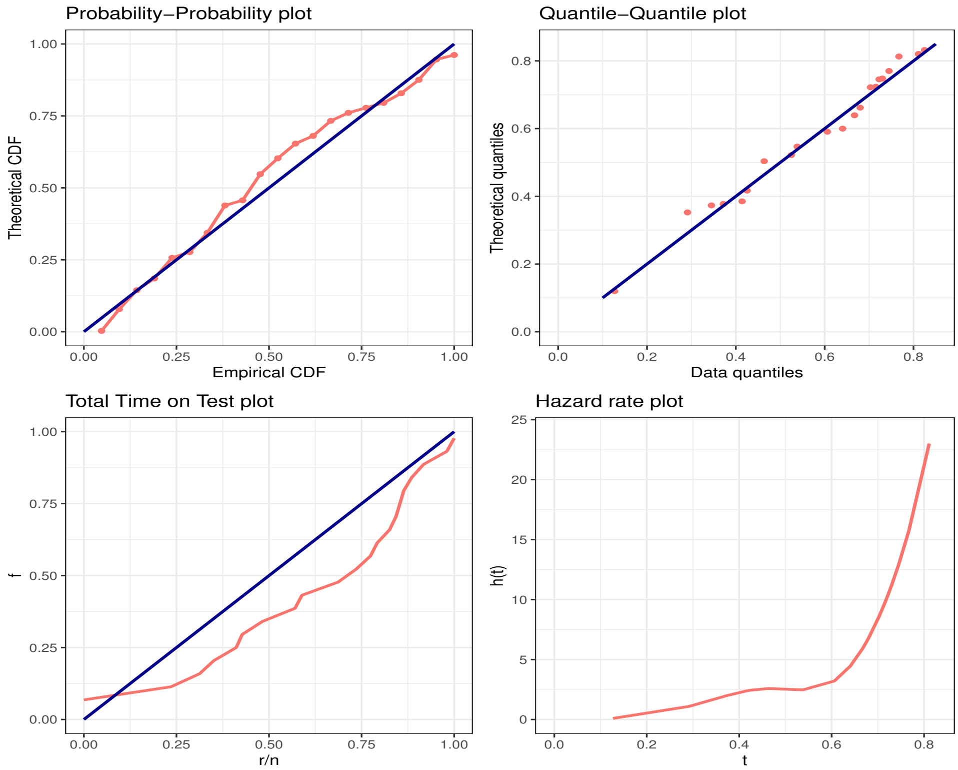

Figure 6: P-P (upper left), Q-Q (upper right), TTT (lower left), and hazard (lower right) plots (respectively) of the proposed CTKw distribution for censored data

0.126977, 0.291172, 0.345075, 0.371904, 0.414087, 0.425334, 0.463726, 0.524947, 0.538082, 0.605979, 0.640395, 0.667157, 0.679829, 0.703119, 0.715158, 0.722613, 0.729986, 0.744881, 0.767135, 0.811159, 0.824860.

For the suggested as well as alternative comparative models, Table 5 displays the calculated values of the model parameters, together with their accompanying standard errors and log-likelihood. Table 6 also shows the values of the model selection criteria taken into consideration for the proposed and comparable distributions. It is evident from examining all of the results of censored data for these distributions that the suggested second order transmuted Kumaraswamy distribution outperforms the CRTKw, TKw, and Kw distributions. The lines for the proposed and competing models are now depicted in Figure 8 together with the empirical distribution function (ECDF, EPDF) for the data pertaining to censored. The suggested CTKw distribution exhibits a greater fit to the censored data, indicating that it is significantly more closely related to the empirical data than the other competing models, as seen in Figure 8.

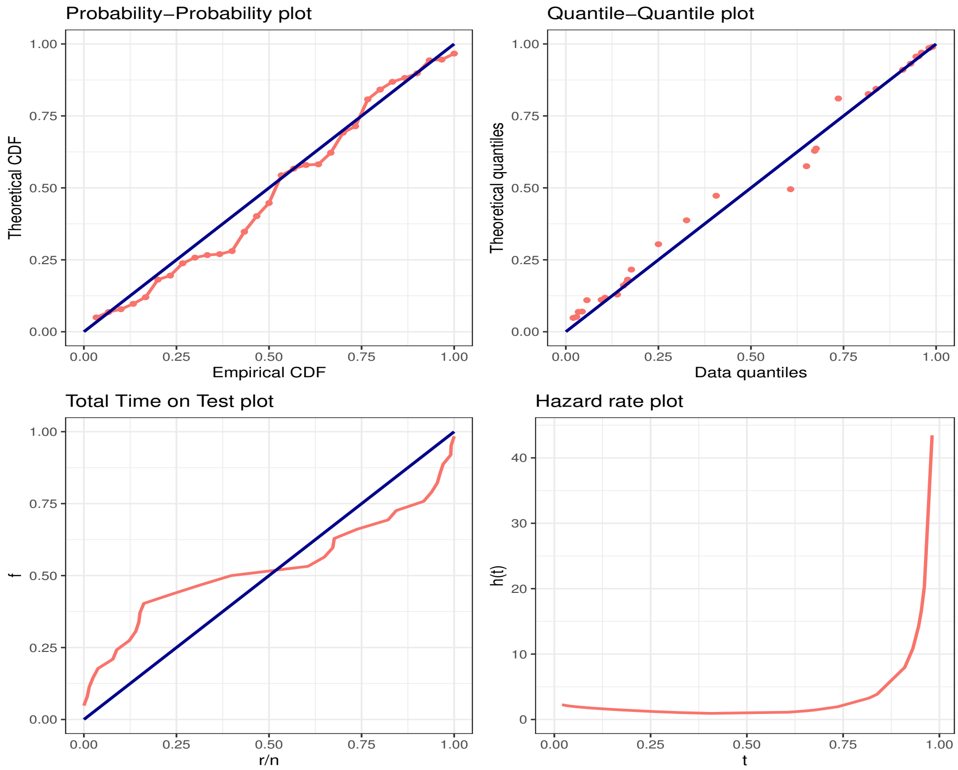

Figure 7: P-P (upper left), Q-Q (upper right), TTT (lower left), and hazard (lower right) plots (respectively) of the proposed CTKw distribution for lifetime data

7.2. Application of Cubic Transmuted Kumaraswamy Distribution for Lifetime Data

By utilizing an actual data set, we have provided an implementation of the suggested CTKw distribution in this subsection. 30 electrical gadgets' lifetime (measured in days) are covered by the dataset, which was used by Rahman et al. [18] as given below:

0.020, 0.029, 0.034, 0.044, 0.057, 0.096, 0.106, 0.139, 0.156, 0.164, 0.167, 0.177, 0.250, 0.326, 0.406, 0.607, 0.650, 0.672, 0.676, 0.736, 0.817, 0.838, 0.910, 0.931, 0.946, 0.953, 0.961, 0.981, 0.982, 0.990.

The summary statistics of the censored data are given in Table 4 show that the data has a somewhat positively skewed distribution. All of the results of this lifetime data computed for the proposed and competing models are then displayed in Table 7 and 8 respectively which shows better applicability of the CTKw distribution.

Eventually, some lines for the lifetime data, for both the ECDF and the EPDF, are shown in Figure 9. These lines include empirical lines as well as all the distributions taken into consideration here. It can be seen from the plotted Figure 9 that the suggested

Table 5: Log-likelihood estimates for various distributions together with the estimated parameter and related SEs

Distribution	Parameter	Estimate	SE	LL
CTKw	a	4.073	0.222	8.213
	b	6.435	6.246	
	λ	-1.000	0.586	
CRTKw	a	3.330	3.383	7.896
	b	5.021	6.831	
	λ_1	1.000	2.461	
	λ_2	-0.286	3.769	
TKw	a	8.878	2.780	6.989
	b	0.774	3.450	
	λ	0.140	-0.402	
Kw	a	3.145	1.434	6.791
	b	3.412	0.023	

Table 6: Estimation of the selection criteria for models

Distribution	2LL	AIC	AICc	BIC	KS	A	C-vM
CTKw	16.425	-10.425	-9.014	-7.292	0.082	2.989	0.262
CRTKw	15.791	-7.791	-5.291	-3.613	0.096	3.303	0.272
TKw	13.977	-7.977	-6.566	-4.844	0.094	3.746	0.330
Kw	13.582	-9.582	-8.915	-7.493	0.103	3.818	0.339

model's line is significantly closer to the data points than the lines for the other models, demonstrating that it has the best fit overall for lifetime data.

8. Concluding Remarks

In this study, a cubic transmutation map has been used to introduce the generalization of a second order transmuted Kumaraswamy (CTKw) distribution with three parameters. The characteristics of the suggested distribution are covered, and the analytical forms of the reliability, hazard, distribution, and density functions are illustrated. Additionally, asymptotic log-likelihood inferences are examined in relation to maximum likelihood estimation, and a simulation investigation of the suggested distribution using a simulated random sample is conducted to demonstrate the suitability of the model parameters for this distribution. The CTKw distribution provides a better match than all other comparable distributions in terms of the statistical significance of the model's suitability.

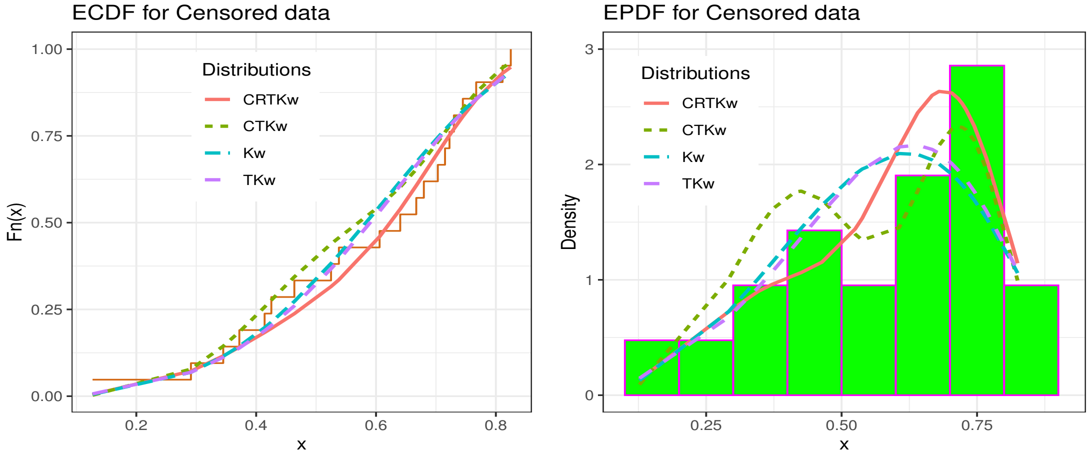

Figure 8: ECDF and EPDF for Censored Data

Table 7: Log-likelihood estimates for various distributions together with the estimated parameter and related SEs

Distribution	Parameter	Estimate	SE	LL
CTKw	a	0.898	0.210	6.532
	b	0.863	0.215	
	λ	-1.000	0.593	
CRTKw	a	0.649	0.210	3.915
	b	0.734	0.111	
	λ_1	1.000	0.617	
	λ_2	0.383	0.998	
TKw	a	0.609	0.178	3.535
	b	0.585	0.175	
	λ	0.125	0.495	
Kw	a	0.588	0.161	3.503
	b	0.612	0.134	

Appendix: The Hessian Matrix for the CTKw Distribution

The Hessian matrix is given as

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},$$

where the variance–covariance matrix V is obtained by

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}^{-1}.$$

Table 8: Estimation of the selection criteria for models

Distribution	2LL	AIC	AICc	BIC	KS	A	C-vM
CTKw	13.065	-7.064	-6.142	-2.861	0.120	3.938	0.264
CRTKw	7.830	0.169	1.769	5.774	0.178	5.410	0.378
TKw	7.069	-1.069	-0.146	3.134	0.157	4.977	0.344
Kw	7.005	-3.005	-2.561	-0.202	0.160	5.008	0.347

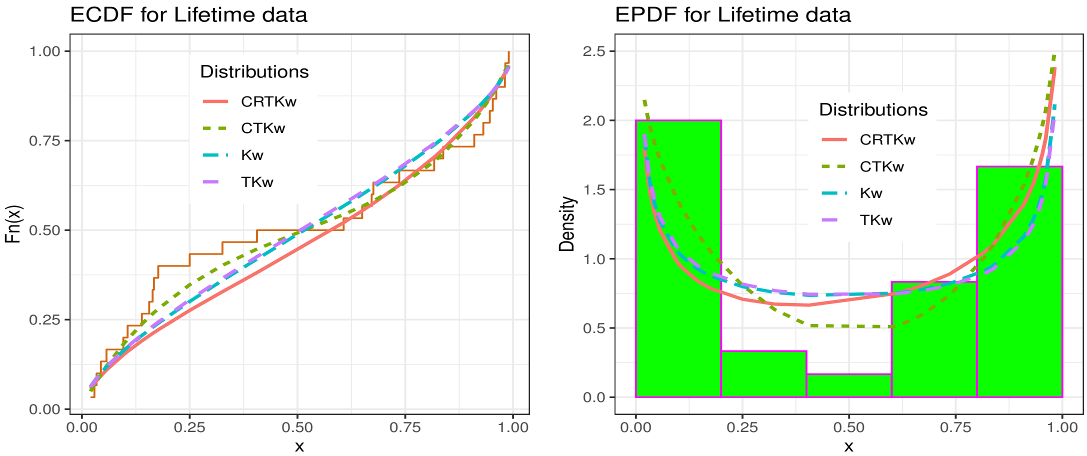

Figure 9: ECDF and EPDF for Lifetime Data

Then the elements of H are given as

$$
\begin{aligned}
H_{11} = -\frac{\partial^2 l}{\partial a^2} \quad &= \quad \frac{n}{a^2} - (b-1)\sum_{i=1}^{n}\left(-\frac{x_i^a \log^2(x_i)}{1-x_i^a} - \frac{x_i^{2a}\log^2(x_i)}{(1-x_i^a)^2}\right) \\
&- \sum_{i=1}^{n}\left[-\frac{\left(12b\lambda x_i^a \log(x_i)(1-x_i^a)^{2b-1} - 6b\lambda x_i^a \log(x_i)(1-x_i^a)^{b-1}\right)^2}{\left(6\lambda(1-x_i^a)^b - 6\lambda(1-x_i^a)^{2b} - \lambda + 1\right)^2}\right. \\
&+ \frac{6(b-1)b\lambda x_i^{2a}\log^2(x_i)(1-x_i^a)^{b-2} - 6b\lambda x_i^a \log^2(x_i)(1-x_i^a)^{b-1}}{6\lambda(1-x_i^a)^b - 6\lambda(1-x_i^a)^{2b} - \lambda + 1} \\
&\left. - \frac{12b\lambda x_i^a \log^2(x_i)(1-x_i^a)^{2b-1} - 12b(2b-1)\lambda x_i^{2a}\log^2(x_i)(1-x_i^a)^{2b-2}}{6\lambda(1-x_i^a)^b - 6\lambda(1-x_i^a)^{2b} - \lambda + 1}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{12} = -\frac{\partial^2 l}{\partial a \partial b} \quad &= \quad -\sum_{i=1}^{n}\left[\frac{\left(6b\lambda x_i^a \log(x_i)(1-x_i^a)^{b-1} - 12b\lambda x_i^a \log(x_i)(1-x_i^a)^{2b-1}\right)}{\left(-6\lambda(1-x_i^a)^b + 6\lambda(1-x_i^a)^{2b} + \lambda + 1\right)^2}\right. \\
&\times \left(6\lambda(1-x_i^a)^b \log(1-x_i^a) + 12\lambda(1-x_i^a)^{2b}\log(1-x_i^a)\right) \\
&+ \frac{-6\lambda x_i^a \log(x_i)(1-x_i^a)^{b-1} - 6b\lambda x_i^a \log(x_i)(1-x_i^a)^{b-1}\log(1-x_i^a)}{6\lambda(1-x_i^a)^b - 6\lambda(1-x_i^a)^{2b} - \lambda + 1} \\
&\left. + \frac{12\lambda x_i^a \log(x_i)(1-x_i^a)^{2b-1} + 24b\lambda x_i^a \log(x_i)(1-x_i^a)^{2b-1}\log(1-x_i^a)}{6\lambda(1-x_i^a)^b - 6\lambda(1-x_i^a)^{2b} - \lambda + 1}\right]
\end{aligned}
$$

$$+\sum_{i=1}^{n}\frac{x_i^a\log(x_i)}{1-x_i^a},$$

$$\begin{aligned}H_{13}=-\frac{\partial^2 l}{\partial a\partial\lambda} &= \sum_{i=1}^{n}\left[-\frac{12bx_i^a\log(x_i)(1-x_i^a)^{2b-1}-6bx_i^a\log(x_i)(1-x_i^a)^{b-1}}{6\lambda(1-x_i^a)^b-6\lambda(1-x_i^a)^{2b}-\lambda+1}\right.\\ &\quad+\frac{\left(6(1-x_i^a)^b-6(1-x_i^a)^{2b}-1\right)}{(6\lambda(1-x_i^a)^b-6\lambda(1-x_i^a)^{2b}-\lambda+1)^2)}\\ &\quad\left.\times\left(12b\lambda x_i^a\log(x_i)(1-x_i^a)^{2b-1}-6b\lambda x_i^a\log(x_i)(1-x_i^a)^{b-1}\right)\right],\end{aligned}$$

$$\begin{aligned}H_{22}=-\frac{\partial^2 l}{\partial b^2} &= \sum_{i=1}^{n}\left[\frac{\left(6\lambda(1-x_i^a)^b\log(1-x_i^a)-12\lambda(1-x_i^a)^{2b}\log(1-x_i^a)\right)^2}{(6\lambda(1-x_i^a)^b-6\lambda(1-x_i^a)^{2b}-\lambda+1)^2}\right.\\ &\quad\left.-\frac{6\lambda(1-x_i^a)^b\log^2(1-x_i^a)-24\lambda(1-x_i^a)^{2b}\log^2(1-x_i^a)}{6\lambda(1-x_i^a)^b-6\lambda(1-x_i^a)^{2b}-\lambda+1}\right]-\frac{n}{b^2},\end{aligned}$$

$$\begin{aligned}H_{23}=-\frac{\partial^2 l}{\partial b\partial\lambda} &= \sum_{i=1}^{n}\left[-\frac{6(1-x_i^a)^b\log(1-x_i^a)-12(1-x_i^a)^{2b}\log(1-x_i^a)}{6\lambda(1-x_i^a)^b-6\lambda(1-x_i^a)^{2b}-\lambda+1}\right.\\ &\quad+\frac{\left(6(1-x_i^a)^b-6(1-x_i^a)^{2b}-1\right)}{(6\lambda(1-x_i^a)^b-6\lambda(1-x_i^a)^{2b}-\lambda+1)^2}\\ &\quad\left.\times\left(6\lambda(1-x_i^a)^b\log(1-x_i^a)-12\lambda(1-x_i^a)^{2b}\log(1-x_i^a)\right)\right],\end{aligned}$$

and

$$H_{33}=-\frac{\partial^2 l}{\partial\lambda^2} = \sum_{i=1}^{n}\frac{\left(6(1-x_i^a)^b-6(1-x_i^a)^{2b}-1\right)^2}{(6\lambda(1-x_i^a)^b-6\lambda(1-x_i^a)^{2b}-\lambda+1)^2}.$$

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