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Finite Groups with Certain Weakly S-permutable Subgroups

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Abstract. Let G be a finite group. A subgroup H of G is said to be weakly S-permutable in G if G has a subnormal subgroup T such that $G = HT$ and $T \cap H \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-permutable in G. In this paper, we prove the following: For a Sylow p-subgroup P of G $(p > 2)$, suppose that P has a subgroup D such that $1 < |D| < |P|$ holds and all subgroups H of P with $|H| = |D|$ are weakly S-permutable in G. Then, the commutator subgroup G' is p-nilpotent. We certainly believe that this result will improve and extend a current and classical theories in the literature.

2020 Mathematics Subject Classifications: 20D10, 20D10, 20D20.

Key Words and Phrases: Sylow subgroups , S-permutable subgroups, weakly S-permutable subgroups, p-nilpotent groups

1. Introduction

All groups considered in this paper will be finite. A subgroup H of a group G is said to be permutable in G if H permutes with every subgroup of G, that is, $HK \leqslant G$ for all $K \leqslant G$. A subgroup H of G is called S-permutable in G provided H permutes with all Sylow subgroups of G, i.e., $HP = PH$ for any Sylow subgroup P of G. This concept was proposed by Kegel in [8]. In 1996, Wang [10], defined the concept of c-normality as follows: A subgroup H of a group G is said to be c-normal in G if G has a normal subgroup K such that $G = HK$ and $H \cap K \leqslant H_G$, where $H_G = Core_G(H)$ is the largest normal subgroup of G contained in H. As a generalization of c-normality, a subgroup H of G is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and

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$H \cap K \leqslant H_G$, where $H_G = Core_G(H)$ is the largest normal subgroup of G contained in H (see [1]).

A number of scholars have studied influence of special types of subgroups behavior on the group structure. For instance, Gaschütz and Itö ([7], Satz 5.7, p. 436) proved that a group G is solvable if all its minimal subgroups are normal (a minimal subgroup is a subgroup of prime order). In [4], Heliel proved a group G is solvable if each subgroup of prime odd order of G is c-supplemented in G. In 2015, Hijazi [5] proved that if each Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-permutable (or c-normal) in G, then G is solvable. It is remarkable to mention that the research on c -normal subgroups has formed a series, which is similar to the series of S-permutable subgroups, however the two series are independent of each other. In 2019, Hijazi and Charaf [6] continued the above mentioned studies and proved: Let P be a Sylow p-subgroup of a group G, where p is an odd prime, and suppose P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-permutable in G. Then $G^{'}$ is p-nilpotent. In [9], Skiba generalized both of the concepts S-permutability and c-normality as follows : A subgroup H of G is said to be weakly S-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leqslant H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-permutable in G.

Our main purpose here is to use this more general concept, weakly S-permutable, to take the above mentianed investigations further. More precisely, we prove:

Main theorem. Let p be an odd prime and let P be a Sylow p-subgroup of G. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are weakly S-permutable in G. Then $G^{'}$ is p-nilpotent.

2. Preliminaries

In this section, we state some known results from the literature which will be used in proving our results.

Lemma 1. (See [6, Theorem 2]) Let P be a Sylow p-subgroup of a group G, where p is an odd prime. If each subgroup of P of order p is S-permutable in G, then $G^{'}$ is p-nilpotent.

Lemma 2. (See [9, Theorem 1.4]) Let $\mathfrak{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) not having a supersolvable supplement in G are weakly S-permutable in G. Then $G \in \mathfrak{F}$

Lemma 3. (See [2, Theorem 10.6 (a), p. 36]) Let G be a finite group:

(i) $C_G(F(G))F(G)/F(G)$ contains no non-trivial solvable normal subgroup of $G/F(G)$. In particuler, $C_G(F(G)) \leq F(G)$ when G solvable.

(ii) If N is a minimal normal subgroup of G, then $F(G) \leq C_G(N)$, furthermore, if N is abelian, then $N \leq Z(F(G)$.

Lemma 4. (See [9, Theorem 2.20]) Let A be a p-group of automorphisms of the p-group P of odd order. Assume that every subgroup of P with prime order is A-invariant. Then A is cyclic.

Lemma 5. (See [9, Lemma 2.10]) Let G be a group and $H \leqslant K \leqslant G$. Then:

(i) If H is S-permutable in G, then H is weakly S-permutable in G.

(ii) Suppose that H is normal in G. Then K/H is weakly S-permutable in G/H if and only if K is weakly S-permutable in G.

(iii) If H is weakly S-permutable in G, then H is weakly S-permutable in K.

(iv) Suppose that H is normal in G. Then the subgroup HE/H is weakly S-permutable in G/H for every weakly S-permutable subgroup E in G satisfying $(|H|, |E|) = 1$.

(v) Suppose that H is a p-subgroup for some prime p and H is not S-permutable in G. Assume that H is weakly S-permutable in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.

Lemma 6. (See [11, Lemma 2.3, p. 214]) If G is solvable and $\Phi(G) = 1$, then Fit(G) is the direct product of (Abelian) minimal normal subgroups of G.

Lemma 7. (See [9, Theorem 2.11]) Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly S-permutable in G. Then some maximal subgroup of N is normal in G.

Lemma 8. (See [3, Theorem 3.2, p. 228]) If $O_{p'}(G) = 1$, then $C_G(O_p(G)) \subseteq O_p(G)$.

3. Results

We first prove the following theorem:

Theorem 1. *Let P be a Sylow p-subgroup of a group G, where p is an odd prime and suppose that each subgroup of P of order p is weakly S-permutable in G. Then $G^{'}$ is p-nilpotent.*

Proof. We prove the theorem by induction on $|G|$. If each subgroup of P of order p is S-permutable in G, then $G^{'}$ is p-nilpotent by Lemma 1. Thus we may assume that there exists a subgroup H of P of order p such that H is not S-permutable in G. By hypothesis, H is weakly S-permutable in G. So, there exists subnormal subgroup K of G such that $G = HK$ and $K \cap H \leq H_{sG}$. However, H_{sG} is the subgroup of H generated by all those subgroups of H which are S-permutable in G, and $|H| = p$, then $H_{sG} = 1$ and so

$K \cap H = 1$. Clearly, $K \lhd G$. By induction on $|G|$, $K^{'}$ is p-nilpotent. Hence, if $O_{p'}(G) \neq 1$, the group $G/O_{p'}(G)$ satisfies the hypothesis of theorem and so $G^{'}/(G^{'} \cap O_{p'}(G))$ is p-nilpotent, by induction on $|G|$, which implies that $G^{'}$ is p-nilpotent. Thus we may assume that $O_{p'}(G) = 1$. Now we have that $K^{'} char K \lhd G$ which implies that $K^{'} \lhd G$ and moreover $K^{'}$ is p-group as $O_{p'}(G) = 1$. Then K has a normal Sylow p-subgroup, say P_1, and so $P_1 \lhd G$ (note that P_1 is characteritstic in K and $K \lhd G$). Also, K possesses a $p^{'}$-Hall subgroup, say K_1. The subgroup K_1 is Abelian since $K/K^{'}$ is Abelian with $K^{'} = P_1$ and $K/P_1 \cong K_1$. It is clear that G is solvable. If $P \lhd G$, then $G/P \cong K_1$ and therefore, by Lemma 2, (taking $E = P$, $F^*(E) = F^*(P) = F(P)$ since P is solvable, $F(P) = P$ from the definition, $D = H$ with order p, $1 < p < p^n$, and all subgroups H of P are weakly S-permutable) G is supersolvable, in particular, $G^{'}$ is p-nilpotent. So we may assume that P is not normal in G. As $O_{p'}(G) = 1$, P_1 is characteristic in G and $P \ntrianglelefteq G$, we have $F(G) = P_1$ and since G is solvable, we have, by Lemma 3 (1), that $C_G(F(G)) \leq F(G) = P_1$. Clearly, K_1 is a $p^{'}$-group of automorophisms of $F(G) = P_1$. Hence, if each subgroup of P_1 is S-permutable in K, then K_1 is cyclic, by Lemma 6, and so p is the largest prime dividing $|G|$ (otherwise we have a contradiction). This means that $P \lhd G$, a contradiction. Thus P_1 contains a subgroup L of order p such that L is not S-permutable in K and consequently L is not S-permutable in G. By hypothesis, L is weakly S-permutable in G. Hence, there exists a subgroup K^* of G such that $G = LK^*$, $L \cap K^* = 1$ and $K^* \lhd G$. As above $P_2 \lhd G$, where P_2 is a Sylow p-subgroup of K^*. But $P_1 \neq P_2$ because $L \leq P_1$ and $L \lneqq P_1$, then $P = P_1P_2 \lhd G$, a contradiction completing the proof of the theorem.

As a corollary of Theorem 1:

Corollary 1. *If each subgroup of prime order of G is weakly S-permutable in G, then G is solvable, $L \lhd G^{'}$ and $G^{'}/L$ is nilpotent, where L is a Sylow 2-subgroup of $G^{'}$.*

Proof. By Theorem 1, $G^{'}$ is p-nilpotent for each odd prime p dividing $|G|$. So $G^{'}/L$ is nilpotent, L is a Sylow 2-subgroup of $G^{'}$ and hence G is solvable.

Now, we are equipped to prove the Main Theorem:

Proof. Assume that the result is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$, then, by Lemma 5 (4), $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. Hence $(G/O_{p'}(G))^{'} = G^{'}O_{p'(G)}/O_{p'}(G) \cong G^{'}/G^{'} \cap O_{p'}(G)$ is p-nilpotent by the minimal choice of G and so $G^{'}$ is p-nilpotent, a contradiction.

(2) $|D| > p$.

Assume that $|D| = p$. Then, by Theorem 1, $G^{'}$ is p-nilpotent, a contradiction.

(3) There exists a subgroup H of P with $|H| = |D|$ such that H is not S-permutable in G.

Assume that all subgroups H of P with $|H| = |D|$ are S-permutable in G. Then $G^{'}$ is p-nilpotent, by Lemma 1, a contradiction.

Now, We distinguish two cases:

Case 1. $|P : D| > p$. Then,

(4) G is solvable and $F(G)$ is a maximal subgroup of P.

By (3), There exists a subgroup H of P with $|H| = |D|$ such that H is not S-permutable in G. Then, by the hypothesis, H is weakly S-permutable in G, that is, there exists a subnormal subgroup T of G such that $G = HT$ and $T \cap H \leq H_{sG}$. Since H is not S-permutable in G, we have that $H_{sG} \neq H$ and so $T \neq G$. Then G has a normal subgroup M such that $T \leq M$ and $|G/M| = p$. Let A be a Sylow p-subgroup of M. Since $|P : D| > p$, we have that A has a subgroup D with $1 < D < A$. Then, by the hypothesis, all subgroups L of A with $|L| = |D|$ are weakly S-permutable in G and so all subgroups L of A with $|L| = |D|$ are weakly S-permutable in M by Lemma 5 (iii). Then $M^{'}$ is p-nilpotent by choice of G. Hence, $M^{'} \leq A$ as $O_{p'}(G) = 1$ by (1), and so A is characteristic in M and since M is normal in G, we have $A \lhd G$. Since M/A is Abelian and $|G/M| = p$, we have that G is solvable. Clearly, as A is a normal nilpotent subgroup of G, $A \leq F(G)$. Then, by (1), $F(G)$ is a p-group. Since G is solvable, we have that G has a $p^{'}$-Hall subgroup K and so $K \leq M$ (as $G = HT$ and $T \leq M$). Then K is Abelian. Hence if $P = F(G)$, $G/P \cong K$ and so $P \geq G^{'}$, that is, $G^{'}$ is p-nilpotent, contradiction. Then $A = F(G)$.

(5) $\Phi(G) \neq 1$.

Assume that $\Phi(G) = 1$. Since G is solvable, from (4), it follows, by Lemma 6, that $A = F(G)$ is a direct product of Abelian minimal normal subgroups of G. But $A = F(G)$ has a maximal subgroup B such that B is normal in G. Then, by [3, A. (913), p. 33], for some minimal normal subgroup L of G contained in $A = F(G)$, we have $|L| = p$. Then $G/C_G(L)$ is Abelian and $G^{'} \leq C_G(L)$. Since $|D| > p$ from (2), then G/L satifies the hypothesis of the Lemma 5, and so $(G/L)^{'} \cong G^{'}/(G^{'} \cap L)$ is p-nilpotent by the choice of G and since $G^{'} \leq C_G(L)$, we have that $G^{'}$ is p-nilpotent, contradiction.

(6) $|\Phi(G)| \geq |D|$.

Assume that $|\Phi(G)| < |D|$. By (4), $\Phi(G) < F(G) = A < P$. Then $G/\Phi(G)$ satifies the hypothesis of the Lemma 5, so $(G/\Phi(G))^{'} = G^{'}\Phi(G)/\Phi(G) \cong G^{'}/(G^{'} \cap \Phi(G))$ is p-nilpotent by the choice of G and it follows easily that $G^{'}$ is p-nilpotent, contradiction.

(7) Let L be a minimal normal subgroup of G such that $L \leq \Phi(G)$. Then $|L| \leq |D|$.

Assume that $|L| > |D|$. Then every subgroup of L with order equals to $|D|$ is weakly S-permutable in G and so, by Lemma 7, some maximal subgroup of L is normal in G. Then $|L| = p > |D|$ which contradicts (2). Thus $|L| \leq |D|$.

(8) There exists a subgroup L_1 of $A = F(G)$ with $|L_1| = |D|$ such that is not S-permutable.

Assume that all subgroups of L_1 of A with $|L_1| = |D|$ are S-permutable. By (5), $\Phi(G) \neq 1$ and so $\Phi(G)$ contains a minimal normal subgroup L such that $|L| \leq |D|$ by (7). Consider the factor group G/L. If $|L| = |D|$, then every subgroup of A/L of order p is S-permutable in G/L. Since $L \leq \Phi(G)$, we have that $F(G/L) = F(G)/L = A/L$. Since G is solvable by (4), we have that $C_{(G/\Phi(G))}(F(G/L)) = C_{(G/\Phi(G))}(F(G)/L) \leq F(G/L) = F(G)/L = A/L$. Then $\bar{K} = KL/L \cong K$ is a $p^{'}$-group of automorphisms of A/L and every subgroup of A/L of prime order is $\bar{K}$ invariant. Then $\bar{K} \cong K$ is cyclic by Lemma 6. Also, as G is solvable, G contains a Hall subgroup PQ, where Q is a Sylow q-subgroup of G and $q \neq p$. Hence, if $p < q$, PQ is p-nilpotent and so $Q \leq C_G(A) = C_G(F(G)) \leq F(G) = A$, a contradiction. Thus p is the largest prime dividing $|G|$. Since K is cyclic, we have by Burnside's Theorem [5, Satz 2.8, p. 420], that P is normal in G, a contradiction. Thus that assume $|L| < |D|$. It is easy to see that G/L satisfies the hypothesis of the theorem and so $(G/L)^{'} \cong G^{'}/G^{'} \cap L$ is p-nilpotent by the choice of G and, since $G^{'} \cap L \leq \Phi(G)$, we have that $G^{'}$ is p-nilpotent, a contradiction.

(9) Finishing the proof of Case 1.
By (8), there exists a subgroup L_1 of $A = F(G)$ with $|L_1| = |D|$ such that is not S-permutable. By the hypothesis, L_1 is weakly S-permutable in G. Then there exists a subnormal subgroup T_1 of G such that $G = L_1T_1$ and $T_1 \cap L_1 \leq (L_1)_{sG} \neq L_1$ and so $T_1 \neq G$. Hence, there exists normal subgroup M_1 of G such that $T_1 \leq M_1$ and $|G/M_1| = p$. By Lemma 5, all subgroups L_2 of P_2, where P_2 is a Sylow p-subgroup of M_1 with $|L_2| = |D|$ are weakly S-permutable in M_1. Then $M_1^{'}$ is p-nilpotent by the choice of G. As $O_{p^{'}} = 1$ from (1), we have $M_1^{'} \leq P_2$. Then P_2 is characteristic in M_1, and since $M_1 \trianglelefteq G$, it follows that $P_2 \trianglelefteq G$. Since $G = L_1T_1 = L_1M_1, P_2 \trianglelefteq G$, and $L_1 \trianglelefteq A \trianglelefteq M \trianglelefteq G$, we have that $P = L_1P_2$ is a subnormal Hall subgroup of G and so $P \trianglelefteq G$, a contradiction.

Case 2. $|P : D| = p$. Then,

(10) There exists a maximal subgroup L of P with $|L| = |D|$ such that L is not S-permutable in G.
Assume that all maximal subgroups L of P with $|L| = |D|$ are S-permutable in G. Then by Lemma 1, $G^{'}$ is p-nilpotent, a contradiction.

(11) There exists a proper normal subgroup M of G such that $|G/M| = p$, $G = LT$, $L \cap T = (L)_{sG}$, where T is a subnormal subgroup of G and $T \leq M < G$.
By (10), L is not S-permutable in G. Then by the hypothesis, L is weakly S-permutable in G. Hence there exists a subnormal subgroup T in G such that $G = LT$ and $T \cap L \leq (L)_{sG} \neq L$, since L is not weakly S-permutable in G. So $T \neq G$ and there exists a normal subgroup M of G such that $|G/M| = p$ and $T \leq M < G$.

(12) $(L)_{sG} \neq 1$.
Assume that $(L)_{sG} = 1$. Then, by (11), $G = LT$ and $T \cap L = 1$, where T is a subnormal subgroup of G and there exists a normal subgroup M of G such that $|G/M| = p$ and $T \leq M < G$. Let P_1 be a Sylow p-subgroup of M. If P_1 is S-permutable in

G, then $P_1 \trianglelefteq G$. If $\Phi(P_1) \neq 1$, then $G/\Phi(P_1)$ satisfies the hypothesis of the theorem and hence $(G/\Phi(P_1))^{'} \cong G^{'}/G^{'} \cap \Phi(P_1)$ is p-nilpotent by the choice of G, and since $G^{'} \cap \Phi(P_1) \leq \Phi(G)$, we have that $G^{'}$ is p-nilpotent, a contradiction. Thus $\Phi(P_1) = 1$. Now it is clear that $P_1 \cap T$ is a normal Sylow p-subgroup of T of order p. Then $T/C_T(P_1 \cap T)$ is Abelian and $T^{'} \leq C_T(P_1 \cap T)$ which implies that $T^{'} = P_1 \cap T$. By Shur-Zassenhaus Theorem, $T = (P_1 \cap T)K$, where K is an abelian $p^{'}$-Hall subgroup of T. Since $G = L(P_1 \cap T)K = PK$, that is, G is product of two nilpotent groups, then G is solvable by Kegel-Wielandt Theorem. Thus $P_1 = O_p(G) = F(G)$. We can assume that $P \ntrianglelefteq G$ (otherwise, $G^{'} \leq P$, a contradiction). If $\Phi(G) \neq 1$, then $\Phi(G) < P_1 = F(G)$. By choice of G, $(G/\Phi(G))^{'} = G^{'}/G^{'} \cap \Phi(G)$ is p-nilpotent and so $G^{'}$ is p-nilpotent, a contradiction. Thus $\Phi(G) = 1$. Then $P_1 = F(G)$ a direct product of minimal normal subgroups of G. Hence, if L_1 and L_2 are two distinct minimal normal subgroups of G, then $(G/L_1)^{'}$ and $(G/L_2)^{'}$ are p-nilpotent by choice of G and so $G^{'}$ is p-nilpotent, a contradiction. Thus $P_1 = O_p(G) = F(G)$ is the unique minimal normal subgroups of G by [2, A, 14.3], $P_1 \leq N_G(T)$ which implies that $T \trianglelefteq M$. Now as G is solvable, G contains a Hall subgroup PQ, where Q is a Sylow q-subgroup of G and $p \neq q$. Hence, if $p < q$, PQ is p-nilpotent and so $Q \leq C_G(F(G)) \leq F(G)$, a contradiction. Thus p is the largest prime dividing $|G|$. Now $T = (P_1 \cap T)K$ and K is not normal in $T = (P_1 \cap T)K$, otherwise $K \trianglelefteq T \trianglelefteq M$, that is, $K \leq O_{p^{'}}(G) = 1$, a contradiction. Hence T is a Frobenius group and so K is cyclic which implies that $P \trianglelefteq G$, a contradiction. Thus P_1 is not S-permutable in G. By the hypothesis, P_1 is weakly S-permutable in G. Then there exists a subnormal subgroup K_1 of G such that $G = P_1 K_1$ and $P_1 \cap K_1 \leq (P_1)_{sG} < P_1$. Hence, if $(P_1)_{sG} = 1$, then $G = P_1 K_1$ and $P_1 \cap K_1 = 1$. Clearly, $M = P_1(M \cap K_1)$ and $M \cap K_1$ is subnormal in G. Since $M \cap K_1$ is a $p^{'}$-group, we have $M \cap K_1 \leq O_{p^{'}}(G) = 1$ by (1), that is, $M \cap K_1 = 1$ which means that $P_1 \trianglelefteq G$, a contradiction. Thus we may assume that $(P_1)_{sG} \neq 1$. Then $(P_1)_{sG} \leq O_p(G) \neq 1$. Assume that $\Phi(O_p(G)) \neq 1$. Then $G^{'}/G^{'} \cap \Phi(O_p(G))$ is p-nilpotent by the choice of G and so $G^{'}$ is p-nilpotent, a contradiction. Thus $\Phi(O_p(G)) = 1$ and so $O_p(G) = F(G)$ is elementary abelian. Assume that $\Phi(O_p(G)) \nleq M$. If $O_p(G) < |D|$, then $G^{'}/G^{'} \cap \Phi(O_p(G)) = G^{'}$ is p-nilpotent by the choice of G, a contradiction. Thus $O_p(G) = |D|$. So $O_p(G) \cap M = 1$ which means that $|P| = p^2$ and $O_p(G) = |D| = p$ and this contradiction (2). Thus $O_p(G) \leq M$ and $O_p(G)$ is a Sylow p-subgroup of M and $O_p(G) = P_1$, a contradiction as P_1 is not S-permutable in G. Thus $(L)_{sG} \neq 1$.

(13) Finishing the proof of Case 2.

By (12), $(L)_{sG} \neq 1$. Then $(L)_{sG} \leq O_p(G) \neq 1$. Assume that $\Phi(O_p(G)) \neq 1$. Then $G^{'}/G^{'} \cap \Phi(O_p(G))$ is p-nilpotent by the choice of G, and so $G^{'}$ is p-nilpotent, a contradiction. Thus $\Phi(O_p(G)) = 1$, and so $O_p(G)$ is elementary abelian. If $O_p(G) < |D|$, then $G^{'}/G^{'} \cap O_p(G)$ is p-nilpotent by the choice of G and so G is p-solvable. Since $O_{p^{'}}(G) = 1$ by (1), we have, by Lemma 8, that $C_G(O_p(G)) = O_p(G)$. If $O_p(G) \cap \Phi(G) \neq 1$, then $G^{'}$ is p-nilpotent, a contradiction. Thus $O_p(G)$ is the unique minimal normal subgroups of G. Since $M \trianglelefteq G$, we have $O_p(G) \cap M = 1$ or $O_p(G) \leq M$. If $O_p(G) \cap M = 1$, then $G^{'}$ is p-nilpotent, a contradiction. Assume that $O_p(G) \leq M$ and let P_1 be a Sylow p-subgroup of M. By hypothesis, P_1 is weakly S-permutable in G. Then there exists a subnormal subgroup K_1

of G such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_{sG} \leq P_1$. If $(P_1)_{sG} = P_1$, then $P \trianglelefteq G$ and this means that $O_p(G) = P_1$, a contradiction. Thus $(P_1)_{sG} < P_1$. If $(P_1)_{sG} = 1$, then $P_1 \cap K_1 = 1$ which implies that $O_p(G) \cap K_1 = 1$ and so $O_p(G)K_1 = O_p(G) \times K_1$, a contradiction. Thus we may assume that $(P_1)_{sG} \neq 1$. Then $(P_1)_{sG} \leq O_p(G)$. We agrue that $\Phi(G) = 1$. If not, $O_p(G) \leq \Phi(G)$ which means that $G^{'}/G^{'} \cap \Phi(G)$ is p-nilpotent and so $G^{'}$ is p-nilpotent, a contradiction. Thus $\Phi(G) = 1$. Then there exists a maximal subgroup S of G such that $G = O_p(G)S$, $O_p(G) \cap S = 1$. We agrue that $O_p(G) \nleq K_1$. If not, $O_p(G) \leq K_1$. Then there exists a maximal subgroup V of P such that $O_p(G) \nleq V$ (because if every a maximal subgroup V of P containing $O_p(G)$, then $O_p(G) \leq \Phi(P)$ and so $P = O_p(G)(P \cap S) = \Phi(P)(P \cap S) = \Phi(P)$ and this is impossible). This V is not S-permutable in G and so V is weakly S-permutable in G. Then there exists a subnormal subgroup T of G such that $G = VT$ and $V \cap T \leq (V)_{sG}$. Then $(V)_{sG} \leq O_p(G)$ and this implies that $V \cap T \leq V \cap O_p(G) \leq V \cap T$. Thus $V \cap O_p(G) = V \cap T$ and $V \cap T \leq (V)_{sG} \leq V \cap O_p(G)$. Now $V \cap O_p(G) = (V)_{sG}$ is normal in P and $(V)_{sG}$ is S-permutable in G implies that $(V)_{sG} \trianglelefteq G$. Hence $V \cap T = (V)_{sG} \leq O_p(G) \leq V$, a contradiction (note that $(V)_{sG} \neq 1$ because if $(V)_{sG} = 1$, then $O_p(G) = p$ and $G/C_G(O_p(G))$ is abelian which means $G^{'} \leq C_G(O_p(G))$ and since $G^{'}/G^{'} \cap O_p(G)$ is p-nilpotent, it follows that $G^{'}$ is p-nilpotent, a contradiction). Thus $O_p(G) \nleq K_1$ and $G/(K_1)_G$ is a p-group and since $G^{'}/G^{'} \cap O_p(G)$ is p-nilpotent, a contradiction. Now we can assume that $|O_p(G)| = |D|$. Then $O_p(G)$ is a maximal in P. Also $O_p(G)$ is elementary abelian and $O_p(G) \nleq M$ because $P_1 \in Syl_p(M)$ is not S-permutable in G. Because $\Phi(G)$ is a p-group and $O_p(G) \nleq M$, we have $\Phi(G) < O_p(G)$ and $\Phi(G) = 1$, that is, $O_p(G)$ is the unique minimal normal subgroups of G. Hence $O_p(G) \cap M = 1$ and $|O_p(G)| = |D| = p$, a contradiction with (2).

As immediate consequences of the main theorem we have:

Corollary 2. *([5], Theorem D) Suppose that each Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are c -normal in G. Then G is solvable.*

Corollary 3. *([6], Corollary 1) Let P be a Sylow p-subgroup of G ($p > 2$). Suppose that P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are permutable in G. Then $G^{'}$ is p-nilpotent.*

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