EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 2, 2024, 870-904
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Modeling and multi-objective optimal control of the dynamics of counterterrorism in the Sahel region in Africa

Mathieu Romaric Pooda[1,*], Yacouba Simpore[2,3], Oumar Traore[1,2]

[1] *Laboratoire de Sciences et Technologie, Université Thomas SANKARA, 12 BP 417 Ouagadougou 12, Burkina Faso.*
[2] *Laboratoire d'Analyse Mathématiques et d'Informatique, Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso.*
[3] *DeustoTech Fundación Deusto Avda Universidades, 24, 48007, Bilbao, Basque Country, Spain.*

Abstract. Terrorist activity in the Sahel region has been on the increase for almost a decade. Groups advocating extremist ideologies with a political or religious base are carrying out attacks against states with the aim of imposing a totalitarian ideology, sometimes against a backdrop of political crisis and famine. In this state of crisis and sometimes communal conflict, it is questionable whether ideological terrorism can really be eradicated in the Sahel. In this study, we develop a model of the dynamics of ideological terrorism based on the terrorism situation in the Sahel. In particular, this model incorporates popular resistance to terrorism through the class of volunteers for the defense of the homeland, but we also take into account the indoctrination of certain sections of the population vulnerable to fanatical ideology. We estimate $\mathcal{R}_0$, the number of elementary replications of extremist behavior, which enables us to predict the evolution of extremism. We also identify four thresholds $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ of sufficient conditions for the eradication of ideological terrorism and brigandage. A multi-objective optimal control of a counter-terrorism strategy is also presented. Finally, we perform a numerical simulation of the analysis and control results to test our hypotheses.

2020 Mathematics Subject Classifications: 49K15, 93B05, 93C15, 93D23

Key Words and Phrases: Ideological terrorism, fanatical behavior, local and global asymptotic stability, global threshold, optimal control

---

1. Introduction

Africa's Sahel region faces major security challenges, not least terrorism, which poses a considerable threat to regional stability. Several terrorist groups operate in the region,

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i2.5122

Email addresses: math7roma8@gmail.com (M. R. Pooda), simplesaint@gmail.com (Y. Simpore), oumar.traore@uts.bf (O. Traore)

including Al-Qaeda in the Islamic Maghreb (AQIM), the Group for Support of Islam and Muslims (GSIM), Boko Haram and the Islamic State in the Greater Sahara (ISGS). Their motivations vary, ranging from radical ideological claims to local rivalries for control of resources and territory, to funding opportunities through illicit trafficking. These groups employ a variety of tactics, including bomb attacks, ambushes and kidnappings, often targeting security forces, critical infrastructure and civilian populations. Fragile institutions, poverty, high unemployment and ethnic tensions contribute to the region's vulnerability. International responses include efforts to support Sahel governments through training, capacity building and security cooperation, but a holistic, long-term approach is needed to effectively tackle the root causes of instability and insecurity.

Let us note that, evolving in time and space, terrorism is not an easy concept to define. International law is unable to give it a clear definition. When we speak of terrorism, we generally mean terrorist methods. Oscillating and unpredictable, the means used by terrorists have varied greatly over time, as have their objectives. The threat seems to come from nowhere to strike anywhere. This raises the question of the purpose of terrorist organizations, which operate in the shadows, making them difficult to catch. There are four main types of terrorism: individual terrorism, caused by rebels, anarchists or nihilists (who accept moral freedom); organized terrorism, advocated by groups defending different ideologies; state terrorism and cyber terrorism. These various forms of terrorism are motivated by vengeful hatred (hatred based on a person's determination to avenge the abuses for which their enemies are responsible), deterrence (so that the terrorized population can exert pressure on their government), propaganda (to strike an emotional chord), and provocation (to push a government to react excessively). Thus, terrorist methods have evolved, and although no definitive definition is currently established, we can attempt to understand what terrorist methods and motivations entail. Additionally, terrorist groups are generally driven by significant ideologies that form the basis of their recruitment.

Terrorism represents a significant threat in the contemporary world, garnering serious attention from numerous countries. Among those affected are Burkina Faso, Mali, Niger, Nigeria, and the United States of America, among others. Consequently, these nations spare no effort in implementing measures to mitigate the risk of attacks. They tailor local and global security initiatives to their specific circumstances in order to protect themselves effectively. Despite serious attempts to establish frameworks for analyzing the dynamics of human behavior [14], [6], [26], [18], there is still no comprehensive mathematical framework or approach for systematically studying human behavior. Similarly, there is no comprehensive mathematical framework or approach for the systematic study of the transmission of ideas [9]. Nevertheless, a wide range of mathematical modeling problems related to evolution, contagion or propagation phenomena have been explored. These efforts encompass programs using paradigms rooted in evolutionary biology, with notable contributions from sources such as [6], [26], [18]. However, there is no longer any doubt that mathematical modeling can play a very important role in understanding and solving these critical problems. We can cite work on migration and crowd behavior [7],[28],[20], work on crime [12], [13], [24], [17], [15], [16], [27], [22], gang membership [30], [2], [8]. We can also cite works on the dynamics of war [3], [11], [10] and many others.

In this work, we study the dynamics of the transmission of extreme behaviors as a kind of contact process in epidemiology combined with local and global security initiatives, as well as the mode of recruitment of terrorists. In particular, we have designed a model of the dynamics of ideological terrorism in a population developing anti-terrorism and anti-brigandage initiatives. This model, which describes the dynamics of ideological terrorism or fanatical insurgency, is inspired on the one hand by the terrorism situation in the Sahel countries, and is characterized by a sequence of thresholds that provide quantitative and qualitative information on the evolution of the permanent insecurity situation. A first local threshold makes it possible to evaluate the number of elementary reproductions of the behavior extremism $\mathcal{R}_0$, well known in epidemiology, and to predict the evolution of violent extremism. In addition to this first local threshold, we identify four global thresholds $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ that essentially give us sufficient conditions for the eradication or extinction of the radical core subpopulation of terrorism, fanatical ideology and brigandage respectively. We conclude this study with a numerical simulation and an optimal control analysis. The specifics of the model are described in the next section.

2. Model formulation

To aid comprehension of this model, we introduce the following definitions and assumptions: the host population is categorized into two subpopulations. One is a non-radical subpopulation denoted as $\mathcal{G}(t)$, while the other is a radical subpopulation characterized by violent extremism and fanaticism, defined as $\mathcal{D}(t)$.

The non-radical subpopulation comprises individuals not indoctrinated into extreme ideologies and not involved in criminal activities, enjoying their freedom. This subset $\mathcal{G}(t)$ is further divided into four classes: civilians not indoctrinated in extreme ideology and non-combatants denoted as $C(t)$, the defense and security forces referred to as $A(t)$, a highly organized group formed as part of a state's global security initiative, composed of individuals trained in warfare techniques, combat, and counter-terrorism activities, along with elites. Additionally, there are volunteers for homeland defense denoted as $V(t)$, part of a local initiative for vigilance and defense against terrorism and criminal activities, and those who have been removed from the ranks of the defense and security forces designated as $R(t)$.

On the other hand, the radical subpopulation, mainly comprising individuals indoctrinated with extremist ideologies, is divided into six classes, with four being hierarchical. This hierarchy is determined by the level of individual commitment to the ideology, with fanatical individuals assumed to be the most effective in propagating the ideology to vulnerable members of the population, making recruitment a crucial factor. Recruitment in this context is modeled based on previous works [9], [21], [23], [31]. The classes in the radical subpopulation include: the naive or vulnerable class $S(t)$, consisting of individuals frequently in contact with members of the radical core but not yet converted; the semi-fanatical class $E(t)$, comprising individuals newly converted to the ideology or not fully committed; the fanatical class $F(t)$, consisting of individuals fully embracing the extreme ideology; and the terrorist class $T(t)$, encompassing all armed individuals participating in acts of terrorism targeting symbols of the state.

In addition to these hierarchical classes within the radical subpopulation, there is the brigand class $B(t)$, comprising individuals not indoctrinated with radical ideologies but engaged in violent criminal activities such as robbery and looting, often operating in gangs. Finally, there is the prisoner class $P(t)$. The dynamics of ideological terrorism or fanatical insurgency are described in the diagram below.

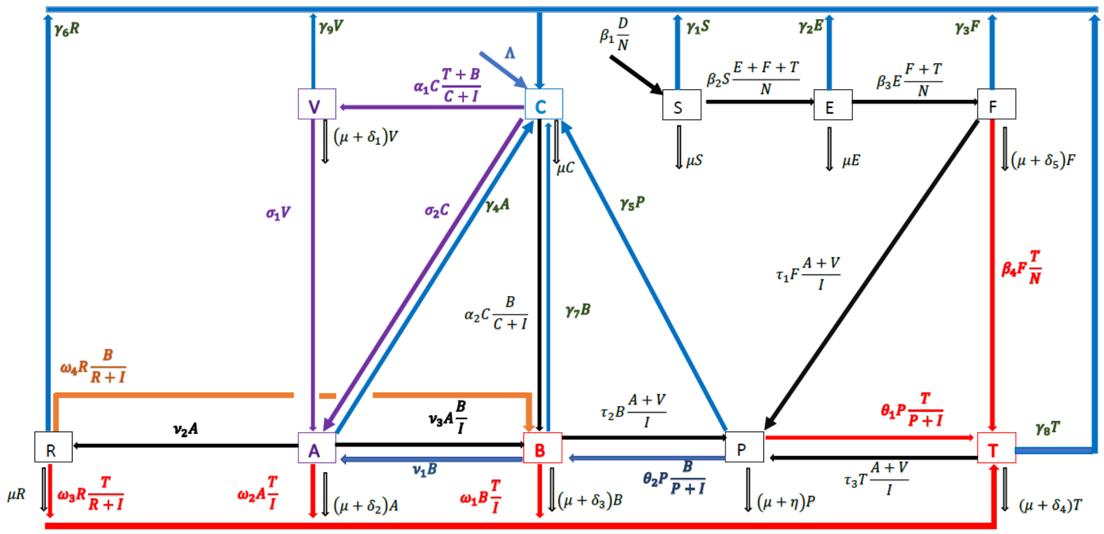

Figure 1: Diagram of the dynamics of ideological terrorism or fanatic insurgency

Where Λ represents the turnover constant of class C, $D = S + E + F + T$ designates the radical core strongly dominated by indoctrination, $I = A + V + B + T$ the combatant class, and $N = C + R + A + V + S + E + F + B + T + P$ designates the total population. The conversion rate from class $C, V, A, B, R,$ and P to class S is $\beta_1 \dfrac{D}{N}$, where $\beta_1 = \pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P$ and π_i, $i \in \{1, 2, 3, 4, 5, 6\}$ represent the intensity of the recruitment force associated with D on C for $i = 1$, V for $i = 2$, A for $i = 3$, B for $i = 4$, R for $i = 5$, P for $i = 6$. This conversion or recruitment occurs during individual contacts between members of D and individuals not in class D. A contact has a broad meaning here as it includes distant contacts (phone calls, emails, etc.). This type of contact is distinguished from contacts in epidemiology. We define $\beta_2 \dfrac{E+F+T}{N}$, $\beta_3 \dfrac{F+T}{N}$,

$\beta_4\dfrac{T}{N}$ as the conversion rates from class S to class E, from class E to class F, and from class F to class T respectively, where β_2, β_3, and β_4 designate the contact frequency or recruitment force intensity associated with D for classes S, E, F, and T respectively.

Similarly, $\omega_1\dfrac{T}{I}$, $\omega_2\dfrac{T}{I}$, $\omega_3\dfrac{T}{R+I}$, $\theta_1\dfrac{T}{P+I}$ represent the recruitment rates into class T of individuals from class B, A, R, and P respectively. Additionally, $\omega_4\dfrac{B}{R+I}$ denotes the conversion rate from class R to class B, with ω_4 representing the intensity of the conversion force in class B acting on individuals in class R. We designate by $\theta_2\dfrac{B}{P+I}$ the conversion rate from class P to class B, $\nu_3\dfrac{B}{I}$ the conversion rate from class A to class B, where ν_3 indicates the intensity of the conversion force in class B acting on individuals in class A. The transfer rates in class P for individuals of classes F, B, and T under the influence of individuals in class A or V are $\tau_1\dfrac{A+V}{F+I}$, $\tau_2\dfrac{A+V}{I}$, $\tau_3\dfrac{A+V}{I}$ respectively. Furthermore, $\alpha_2\dfrac{B}{C+I}$ is the rate of conversion from class C to class B, and $\alpha_1\dfrac{T+B}{C+I}$ is the rate of conversion from class C to class V. These rates are proportional to the number of contacts per unit of time and the probability of success.

Moreover, ν_1, σ_1, σ_2 represent the recruitment rates of class B, V, and C respectively into class A, while ν_2 denotes the attrition rate in class A. Additionally, γ_i, $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, represents the recovery rate or return to normal civilian life for individuals in classes $S, E, F, A, P, R, B, T, V$ respectively. The assumption about the hierarchy of certain classes in the radical subpopulation implies that γ_3 is extremely small, indicating that the average durations of individuals in the fanatical and terrorist classes are very long ($\dfrac{1}{\gamma_3} >> 0$).

We denote by δ_i, $i \in \{1, 2, 3, 4, 5\}$, the probability of dying following a fight of individuals in class A for $i = 1$, of class V for $i = 2$, of class F for $i = 3$, of class B for $i = 4$, and of class T for $i = 5$. These probabilities are assumed to be proportional to the numbers of contacts between combatants and the intensity of the nuisance force that each combatant class exerts on its opponents. Hence, we have $\delta_1 = \zeta_1\dfrac{T+B}{I}$, $\delta_2 = \zeta_2\dfrac{T+B}{I}$, $\delta_3 = \zeta_3\dfrac{A+V}{I}$, $\delta_4 = \zeta_4\dfrac{A+V}{I}$, $\delta_5 = \zeta_5\dfrac{A+V}{F+I}$, where ζ_i, $i \in \{1, 2, 3, 4, 5\}$, represents the intensity of the nuisance force of classes T and B on class A and V respectively for $i = 1$ and $i = 2$, and of A and V on classes F, B, and T respectively for $i = 3$, $i = 4$, and $i = 5$. Additionally, η represents the probability of dying in prison as a result of torture or any other form of maltreatment.

Finally, we assume that all individuals in the total population have the same natural mortality rate μ. Therefore, the model is expressed as follows:

$$\begin{aligned}
\frac{dC}{dt} &= \Lambda + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \Big(\pi_1 \frac{D}{N} + \alpha_1 \frac{T+B}{C+I} + \alpha_2 \frac{B}{C+I} + \sigma_2 + \mu\Big) C & (1)\\
\frac{dR}{dt} &= \nu_2 A - \Big(\pi_5 \frac{D}{N} + \omega_3 \frac{T}{R+I} + \omega_4 \frac{B}{R+I} + \gamma_6 + \mu\Big) R & (2)\\
\frac{dA}{dt} &= \sigma_1 V + \sigma_2 C + \nu_1 B - \Big(\pi_3 \frac{D}{N} + \nu_3 \frac{B}{I} + \omega_2 \frac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \frac{T+B}{I}\Big) A & (3)\\
\frac{dV}{dt} &= \alpha_1 C \frac{T+B}{C+I} - \Big(\pi_2 \frac{D}{N} + \gamma_9 + \sigma_1 + \mu + \zeta_2 \frac{T+B}{I}\Big) V & (4)\\
\frac{dS}{dt} &= \Big(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P\Big) \frac{D}{N} - (\beta_2 \frac{E+F+T}{N} + \gamma_1 + \mu) S & (5)\\
\frac{dE}{dt} &= \beta_2 S \frac{E+F+T}{N} - (\beta_3 \frac{F+T}{N} + \gamma_2 + \mu) E & (6)\\
\frac{dF}{dt} &= \beta_3 E \frac{F+T}{N} - \Big(\gamma_3 + \tau_1 \frac{A+V}{F+I} + \beta_4 \frac{T}{N} + \mu + \zeta_3 \frac{A+V}{F+I}\Big) F & (7)\\
\frac{dB}{dt} &= \alpha_2 C \frac{B}{C+I} + \omega_4 R \frac{B}{R+I} + \nu_3 A \frac{B}{I} + \theta_2 P \frac{B}{P+I} - \Big(\pi_4 \frac{D}{N} + \omega_1 \frac{T}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_4 \frac{A+V}{I}\Big) B & (8)\\
\frac{dT}{dt} &= \beta_4 F \frac{T}{N} + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I} - \Big(\tau_3 \frac{A+V}{I} + \gamma_8 + \mu + \zeta_5 \frac{A+V}{I}\Big) T & (9)\\
\frac{dP}{dt} &= \tau_1 F \frac{A+V}{F+I} + \tau_2 B \frac{A+V}{I} + \tau_3 T \frac{A+V}{I} - \Big(\pi_6 \frac{D}{N} + \theta_1 \frac{T}{P+I} + \theta_2 \frac{B}{P+I} + \gamma_5 + \mu + \eta\Big) P & (10)
\end{aligned}$$

With non-negative initial conditions given by :

$$C(0) > 0; S(0) \geq 0; E(0) \geq 0; F(0) \geq 0; V(0) \geq 0; A(0) > 0; R(0) \geq 0; B(0) \geq 0; P(0) \geq 0; T(0) \geq 0, N(0) \leqslant \frac{\Lambda}{\mu} \quad (11)$$

The parameters of the system $(1) - (10)$ are assumed to be all non-negative.

3. Mathematical analysis of the model

3.1. Existence and uniqueness of solution

The $(1)-(10)$ model is described by a system of first order nonlinear differential equations. It is rewritten as follows:

$$X'(t) = f(X(t)) \quad (12)$$

where $X(t)$ is a column vector of the number of individuals by class, and $f : \mathbb{R}^{10} \to \mathbb{R}^{10}$ is a fonction.

$$X(t) = \begin{bmatrix} C(t) \\ R(t) \\ A(t) \\ V(t) \\ S(t) \\ E(t) \\ F(t) \\ B(t) \\ T(t) \\ P(t) \end{bmatrix} \quad (13)$$

and

$$f(x)=\begin{bmatrix} \Lambda+\gamma_1x_5+\gamma_2x_6+\gamma_3x_7+\gamma_4x_3+\gamma_5x_{10}+\gamma_6x_2+\gamma_7x_8+\gamma_8x_9+\gamma_9x_4-\Big(\pi_1\frac{x_{12}}{x_{13}}+\alpha_1\frac{x_9+x_{10}}{x_1+x_{11}}+\alpha_2\frac{x_8}{x_1+x_{11}}+\sigma_2+\mu\Big)x_1 \\ \nu_2x_3-\Big(\pi_5\frac{x_{12}}{x_{13}}+\omega_3\frac{x_9}{x_2+x_{11}}+\omega_4\frac{x_8}{x_2+x_{11}}+\gamma_6+\mu\Big)x_2 \\ \sigma_1x_4+\sigma_2x_1+\nu_1x_8-\Big(\pi_3\frac{x_{12}}{x_{13}}+\nu_3\frac{x_8}{x_{11}}+\omega_2\frac{x_9}{x_{11}}+\gamma_4+\nu_2+\mu+\zeta_1\frac{x_9+x_8}{x_{11}}\Big)x_3 \\ \alpha_1x_1\frac{x_9+x_8}{x_1+x_{11}}-\Big(\pi_2\frac{x_{12}}{x_{13}}+\gamma_9+\sigma_1+\mu+\zeta_2\frac{x_9+x_8}{x_{11}}\Big)x_4 \\ \Big(\pi_1x_1+\pi_2x_4+\pi_3x_3+\pi_4x_8+\pi_5x_2+\pi_6x_{10}\Big)\frac{x_{12}}{x_{13}}-(\beta_2\frac{x_6+x_7+x_9}{x_{13}}+\gamma_1+\mu)x_5 \\ \beta_2x_5\frac{x_6+x_7+x_9}{x_{13}}-(\beta_3\frac{x_7+x_9}{x_{13}}+\gamma_2+\mu)x_6 \\ \beta_3x_6\frac{x_7+x_9}{x_{13}}-\Big(\gamma_3+\tau_1\frac{x_3+x_4}{x_7+x_{11}}+\beta_4\frac{x_9}{x_{13}}+\mu+\zeta_3\frac{x_3+x_4}{x_7+x_{11}}\Big)x_7 \\ \alpha_2x_1\frac{x_8}{x_1+x_{11}}+\omega_4x_2\frac{x_8}{x_2+x_{11}}+\nu_3x_3\frac{x_8}{x_{11}}+\theta_2x_{10}\frac{x_8}{x_{10}+x_{11}}-\Big(\pi_4\frac{x_{12}}{x_{13}}+\omega_1\frac{x_9}{x_{11}}+\tau_2\frac{x_3+x_4}{x_{11}}+\gamma_7+\nu_1+\mu+\zeta_4\frac{x_3+x_4}{x_{11}}\Big)x_8 \\ \beta_4x_7\frac{x_9}{x_{13}}+\omega_1x_8\frac{x_9}{x_{11}}+\omega_2x_3\frac{x_9}{x_{11}}+\omega_3x_2\frac{x_9}{x_2+x_{11}}+\theta_1x_{10}\frac{x_9}{x_{10}+x_{11}}-\Big(\tau_3\frac{x_3+x_4}{x_{11}}+\gamma_8+\mu+\zeta_5\frac{x_3+x_4}{x_{11}}\Big)x_9 \\ \tau_1x_7\frac{x_3+x_4}{x_7+x_{11}}+\tau_2x_8\frac{x_3+x_4}{x_{11}}+\tau_3x_9\frac{x_3+x_4}{x_{11}}-\Big(\pi_6\frac{x_{12}}{x_{13}}+\theta_1\frac{x_9}{x_{10}+x_{11}}+\theta_2\frac{x_8}{x_{10}+x_{11}}+\gamma_5+\mu+\eta\Big)x_{10} \end{bmatrix} \tag{14}$$

with

$$x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})\in\mathbb{R}^{10}$$

and

$$\begin{cases} x_{11} &= x_3+x_4+x_8+x_9 \\ x_{12} &= x_5+x_6+x_7+x_9 \\ x_{13} &= x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10} \end{cases}$$

The function f is clearly locally lipschitzian with respect to x. We then deduce the existence and the uniqueness of the maximal solution to the Cauchy problem associated to the differential equation $(1)-(10)$ related to the initial condition (11).

3.2. Positivity of the solutions

For this model of the dynamics of ideological terrorism to be realistic, it is necessary to show that all state variables remain positive at all times.

Proposition 1. ***(Positivity)*** *The positive orthan* $\mathbb{R}^{10}_{\geq 0}$ *is positively invariant for the system* $(1)-(10)$*, and the initial condition* (11) *ensures the positivity of the solutions of the system* $(1)-(10)$ *for any time* $t>0$.

Proof: We use the barrier theorem [5].
Let us show that the set $\Big\{C\geq 0\Big\}$ is positively invariant. Let $x=\Big(C,R,A,V,S,E,F,B,T,P\Big)$ and consider L an application defined by

$$L(x)=-C \tag{15}$$

The application L thus defined is differentiable and we have:

$$\nabla L(x) = (-1, 0, 0, 0, 0, 0, 0, 0, 0, 0) \neq 0_{\mathbb{R}^{10}}. \tag{16}$$

The vector field for $\{C = 0\}$ is given by

$$X(x) = \begin{bmatrix} \Lambda + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V \\ \nu_2 A - \left(\pi_5 \dfrac{D}{N} + \omega_3 \dfrac{T}{R+I} + \omega_4 \dfrac{B}{R+I} + \gamma_6 + \mu\right) R \\ \sigma_1 V + \nu_1 B - \left(\pi_3 \dfrac{D}{N} + \nu_3 \dfrac{B}{I} + \omega_2 \dfrac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \dfrac{T+B}{I}\right) A \\ -\left(\pi_2 \dfrac{D}{N} + \gamma_9 + \sigma_1 + \mu + \zeta_2 \dfrac{T+B}{I}\right) V \\ \left(\pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P\right) \dfrac{D}{N} - (\beta_2 \dfrac{E+F+T}{N} + \gamma_1 + \mu) S \\ \beta_2 S \dfrac{E+F+T}{N} - (\beta_3 \dfrac{F+T}{N} + \gamma_2 + \mu) E \\ \beta_3 E \dfrac{F+T}{N} - \left(\gamma_3 + \tau_1 \dfrac{A+V}{F+I} + \beta_4 \dfrac{T}{N} + \mu + \zeta_3 \dfrac{A+V}{F+I}\right) F \\ \omega_4 R \dfrac{B}{R+I} + \nu_3 A \dfrac{B}{I} + \theta_2 P \dfrac{B}{P+I} - \left(\pi_4 \dfrac{D}{N} + \omega_1 \dfrac{T}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_4 \dfrac{A+V}{I}\right) B \\ \beta_4 F \dfrac{T}{N} + \omega_1 B \dfrac{T}{I} + \omega_2 A \dfrac{T}{I} + \omega_3 R \dfrac{T}{R+I} + \theta_1 P \dfrac{T}{P+I} - \left(\tau_3 \dfrac{A+V}{I} + \gamma_8 + \mu + \zeta_5 \dfrac{A+V}{I}\right) T \\ \tau_1 F \dfrac{A+V}{I} + \tau_2 B \dfrac{A+V}{I} + \tau_3 T \dfrac{A+V}{I} - \left(\pi_6 \dfrac{D}{N} + \theta_1 \dfrac{T}{P+I} + \theta_2 \dfrac{B}{P+I} + \gamma_5 + \mu + \eta\right) P \end{bmatrix} \tag{17}$$

From (16) and (17), we have

$$\langle X(x), \nabla L(x) \rangle = -\Big(\Lambda + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V\Big) \leq 0 \tag{18}$$

From (16) and (18), we infer that the sets $\{C \geq 0\}$, $\{R \geq 0\}$, $\{A \geq 0\}$, $\{V \geq 0\}$, $\{S \geq 0\}$, $\{E \geq 0\}$, $\{F \geq 0\}$, $\{B \geq 0\}$, $\{T \geq 0\}$, and $\{P \geq 0\}$ are positively invariant, as established by the application of the barrier theorem. Hence, $\mathbb{R}^{10}_{\geq 0}$ is positively invariant. Additionally, by the initial condition (11), we have $x(0) \in \mathbb{R}^{10}_{\geq 0}$. Since $\mathbb{R}^{10}_{\geq 0}$ is positively invariant, this ensures that all solutions of the system $(1)-(10)$ remain positive for all time $t > 0$. □

3.3. Invariant region

Theorem 1. *For initial conditions* (11)*, the solutions of the system* $(1)-(10)$ *are contained in the positively invariant, compact and attractive region*

$$\Psi = \left\{ \Big(C, S, E, F, V, A, R, B, P, T\Big) \in \mathbb{R}^{10}_{\geq 0} : N(t) \leq \frac{\Lambda}{\mu} \right\} \tag{19}$$

Proof:

Summing the equations (1) to (10), we find :

$$\frac{dN}{dt} = \Lambda - \mu N - \delta_1 V - \delta_2 A - \delta_3 B - \delta_4 T - \delta_5 F - \eta P$$

Since A, V, B, T, F, P are positive functions and using the positivity of the functions δ_1, δ_2, δ_3, δ_4, δ_5, given that the constants ζ_1, ζ_2, ζ_3, ζ_4, ζ_5 and η are strictly positive as well, we get:

$$\frac{dN}{dt} \leq \quad \Lambda - \mu N$$

Then

$$\frac{d}{dt}\Big(N - \frac{\Lambda}{\mu}\Big) \leq \quad -\mu\Big(N - \frac{\Lambda}{\mu}\Big)$$

So the Gromwall inequality give

$$N(t) - \frac{\Lambda}{\mu} \leq \Big(N(0) - \frac{\Lambda}{\mu}\Big)e^{-\mu t}$$

Thus

$$N(t) \leq \frac{\Lambda}{\mu} + \Big(N(0) - \frac{\Lambda}{\mu}\Big)e^{-\mu t}$$

Since $N(0) \leq \dfrac{\Lambda}{\mu}$, then $0 \leq N(t) \leq \dfrac{\Lambda}{\mu}$.

Therefore, all feasible solutions of the model $(1)-(10)$ converge in the region Ψ. □

4. Analysis of the equilibrium without terrorist, nor brigand nor fanatic x^*, and basic reproduction rumber $\mathcal{R}_0$

4.1. Equilibrium without terrorist, nor brigand, nor fanatic x^*

The uninfected compartments are C, R, A, V and the infected compartments are S, E, F, B, T, P. Given that we are at equilibrium without terrorists, fanatics, or robbers then we can discard the P compartment and the infected compartments being S, E, F, B, T, then an equilibrium solution with S = E = F = B = T=0 has the form:

$$x^* \quad = \quad \Big(C^*, R^*, A^*, 0, 0, 0, 0, 0, 0\Big) \tag{20}$$

with

$$\begin{aligned}
C^* &= \frac{\Lambda(\gamma_6+\mu)(\gamma_4+\nu_2+\mu)}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]}\\
R^* &= \frac{\Lambda\nu_2\sigma_2}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]}\\
A^* &= \frac{\Lambda\sigma_2(\gamma_6+\mu)}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]}
\end{aligned}$$

4.2. Matrix of next generation $\mathcal{K}$, and basic reproduction number $\mathcal{R}_0$

The Jacobian matrix of the system $(1)-(10)$ is decomposed into $J_x(x^*) = D\mathcal{F}(x^*) + D\mathcal{V}(x^*)$ with

$$\mathcal{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \Big(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R\Big)\dfrac{D}{N} \\ 0 \\ 0 \\ \alpha_2 C\dfrac{B}{C+I} + \omega_4 R\dfrac{B}{R+I} + \nu_3 A\dfrac{B}{I} \\ \beta_4 F\dfrac{T}{N} + \omega_1 B\dfrac{T}{I} + \omega_2 A\dfrac{T}{I} + \omega_3 R\dfrac{T}{R+I} \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} \Lambda + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \Big(\pi_1\dfrac{D}{N} + \alpha_1\dfrac{T+B}{C+I} + \alpha_2\dfrac{B}{C+I} + \sigma_2 + \mu\Big)C \\ \nu_2 A - \Big(\pi_5\dfrac{D}{N} + \omega_3\dfrac{T}{R+I} + \omega_4\dfrac{B}{R+I} + \gamma_6 + \mu\Big)R \\ \sigma_1 V + \sigma_2 C + \nu_1 B - \Big(\pi_3\dfrac{D}{N} + \nu_3\dfrac{B}{I} + \omega_2\dfrac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1\dfrac{T+B}{I}\Big)A \\ \alpha_1 C\dfrac{T+B}{C+I} - \Big(\pi_2\dfrac{D}{N} + \gamma_9 + \sigma_1 + \mu + \zeta_2\dfrac{T+B}{I}\Big)V \\ -(\beta_2\dfrac{E+F+T}{N} + \gamma_1 + \mu)S \\ \beta_2 S\dfrac{E+F+T}{N} - (\beta_3\dfrac{F+T}{N} + \gamma_2 + \mu)E \\ \beta_3 E\dfrac{F+T}{N} - \Big(\gamma_3 + \tau_1\dfrac{A+V}{F+I} + \beta_4\dfrac{T}{N} + \mu + \zeta_3\dfrac{A+V}{F+I}\Big)F \\ -\Big(\pi_4\dfrac{D}{N} + \omega_1\dfrac{T}{I} + \tau_2\frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_4\dfrac{A+V}{I}\Big)B \\ -\Big(\tau_3\dfrac{A+V}{I} + \gamma_8 + \mu + \zeta_5\dfrac{A+V}{I}\Big)T \end{bmatrix}$$

$$D\mathcal{F}(x^*) = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{F} \end{bmatrix} \quad ; \quad D\mathcal{V}(x^*) = \begin{bmatrix} J_1 & J_2 \\ 0 & \mathbb{V} \end{bmatrix} \quad \text{with} \quad \mathbb{F} = \left[\frac{\partial F_i(x^*)}{\partial x_j}\right]_{5\le i,j\le 9}$$

$$J_1 = \left[\frac{\partial \mathcal{V}_i(x^*)}{\partial x_j}\right]_{1\le i,j\le 4} \text{ and } J_2 = \left[\frac{\partial \mathcal{V}_i(x^*)}{\partial x_j}\right]_{1\le i\le 4;\, 5\le j\le 9} \; ; \; \mathbb{V} = \left[\frac{\partial \mathcal{V}_i(x^*)}{\partial x_j}\right]_{5\le i,j\le 9}$$

Let:

$$f = \pi_1 \frac{C^*}{C^*+R^*+A^*} + \pi_5 \frac{R^*}{C^*+R^*+A^*} + \pi_3 \frac{A^*}{C^*+R^*+A^*}$$

$$g = \alpha_2 \frac{C^*}{C^*+A^*} + \omega_4 \frac{R^*}{R^*+A^*} + \nu_3$$

$$h = \omega_2 + \omega_3 \frac{R^*}{R^*+A^*}$$

We get:

$$\mathbb{F} = \begin{bmatrix} f & f & f & 0 & f \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}$$

$$J_1 = \begin{bmatrix} -(\sigma_2+\mu) & \gamma_6 & \gamma_4 & \gamma_9 \\ 0 & -(\gamma_6+\mu) & \nu_2 & 0 \\ \sigma_2 & 0 & -(\gamma_4+\mu+\nu_2) & \sigma_1 \\ 0 & 0 & 0 & -(\gamma_9+\mu+\sigma_1) \end{bmatrix}$$

$$J_2 = \begin{bmatrix} \gamma_1 & \gamma_2 - \pi_1 C^* & \gamma_3 - \pi_3 C^* & \varpi_1 & \varpi_2 \\ -\pi_5 R^* & -\pi_5 R^* & -\pi_5 R^* & -\omega_4 \dfrac{R^*}{R^*+A^*} & \varpi_3 \\ -\pi_3 A^* & -\pi_3 A^* & -\pi_3 A^* & \nu_1 - \nu_3 & -\pi_3 A^* - \omega_2 \\ 0 & 0 & 0 & \alpha_1 \dfrac{C^*}{C^*+A^*} & \alpha_1 \dfrac{C^*}{C^*+A^*} \end{bmatrix}$$

with

$$\begin{aligned} \varpi_1 &= \gamma_7 - \alpha_1 \frac{C^*}{C^*+A^*} - \alpha_2 \frac{C^*}{C^*+A^*} \\ \varpi_2 &= \gamma_8 - \pi_1 \frac{C^*}{C^*+R^*+A^*} - \alpha_1 \frac{C^*}{C^*+A^*} \\ \varpi_3 &= -\pi_5 \frac{C^*}{C^*+R^*+A^*} - \omega_3 \frac{R^*}{R^*+A^*} - \omega_4 \frac{R^*}{R^*+A^*} \end{aligned}$$

Note that J_1 is a non-singular Metzler matrix (see [4]).

$$\mathbb{V}=\begin{bmatrix}-a & 0 & 0 & 0 & 0\\ 0 & -b & 0 & 0 & 0\\ 0 & 0 & -c & 0 & 0\\ 0 & 0 & 0 & -d & 0\\ 0 & 0 & 0 & 0 & -e\end{bmatrix}$$

with

$$\begin{aligned}
a &= \gamma_1+\mu\\
b &= \gamma_2+\mu\\
c &= \gamma_3+\mu+\tau_1+\zeta_3\\
d &= \gamma_7+\mu+\tau_2+\nu_1+\zeta_4\\
e &= \gamma_8+\mu+\tau_3+\zeta_5
\end{aligned}$$

We also note that $\mathbb{V}$ is a Metzler-Hurwitz matrix and

$$\mathbb{V}^{-1}=\begin{bmatrix}-\frac{1}{a} & 0 & 0 & 0 & 0\\ 0 & -\frac{1}{b} & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{c} & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{d} & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{e}\end{bmatrix}$$

$$\mathbb{V}^{-1}=\begin{bmatrix}-\frac{1}{a} & 0 & 0 & 0 & 0\\ 0 & -\frac{1}{b} & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{c} & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{d} & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{e}\end{bmatrix}\Rightarrow \mathcal{K}=-\mathbb{F}\mathbb{V}^{-1}=\begin{bmatrix}\frac{f}{a} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{g}{d} & 0\\ 0 & 0 & 0 & 0 & \frac{h}{e}\end{bmatrix}$$

where

$$\begin{aligned}
\frac{f}{a} &= \frac{(\gamma_6+\mu)\big[\pi_1(\gamma_4+\nu_2+\mu)+\pi_3\sigma_2\big]+\pi_5\sigma_2\nu_2}{\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2)+\sigma_2\nu_2\big](\gamma_1+\mu)}\\
\frac{g}{d} &= \left(\frac{1}{\gamma_7+\tau_2+\nu_1+\mu+\zeta_4}\right)\left(\alpha_2\frac{\gamma_4+\nu_2+\mu}{\gamma_4+\nu_2+\sigma_2+\mu}+\omega_4\frac{\nu_2}{\gamma_6+\nu_2+\mu}+\nu_3\right)\\
\frac{h}{e} &= \frac{\omega_2(\gamma_6+\nu_2+\mu)+\omega_3\nu_2}{(\gamma_6+\nu_2+\mu)(\gamma_8+\tau_3+\mu+\zeta_5)}
\end{aligned}$$

and

$$\mathcal{R}_0=\rho(\mathcal{K})=\max\left\{\frac{f}{a};\frac{g}{d};\frac{h}{e}\right\} \tag{21}$$

Theorem 2. *The equilibrium without terrorist, nor brigand nor fanatic x^*, is locally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*

Proof: See [32, 33].

Thus $\mathcal{R}_0 < 1$, then an average radical indoctrinates or recruits less than one, which means that radical fanaticism as well as terrorism and robbery will disappear from this population over time. Conversely, if $\mathcal{R}_0 > 1$, then radical fanaticism and insecurity can spread through the population.

Theorem 3. *The equilibrium without terrorist, brigand or fanatic x^*, is globally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*

Proof:
From Theorem 2, when $\mathcal{R}_0 < 1$, the compartments S, E, F, B, T tend to 0 as $t \to \infty$. By setting S, E, F, B, and T to zero, it follows that $(C, R, A, V, S, E, F, B, T, P) \to x^*$ as $t \to \infty$, where x^* is the unique point in the positively invariant, compact, and attractive solution region Ψ, such that $S = E = F = B = T = 0$. □

5. Global thresholds

The $(1) - (10)$ model is characterized qualitatively by a sequence of thresholds that are based on global dynamics. The first global threshold identifies the conditions for the extinction of the radical population. The second threshold identifies the conditions of the eradication of ideological terrorism. The third gives us the conditions of the extinction of fanatical ideology, and the fourth gives us the conditions of the eradication of insecurity without terrorism.

Remark 1. *In all this part, without loss of generality we note*

$$\gamma_i = \gamma_i + \mu, \forall i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

5.1. A sufficient condition of the extinction of radical indoctrinated subpopulations

Now we give a necessary and sufficient condition for extinction or stabilization of the radical core subpopulation.

Theorem 4. *Let $\lambda_1 = \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \omega_1 + \omega_2 + \omega_3 + \theta_1$ and $\gamma = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_8\}$. So for all $\mathcal{R}_1 = \dfrac{\lambda_1}{\gamma} < 1$, we have $\lim\limits_{t\to\infty} D(t) = 0$.*

Proof: We have $D = S + E + F + T$. Then:

$$\begin{aligned}
\frac{dD}{dt} &= \frac{dS}{dt} + \frac{dE}{dt} + \frac{dF}{dt} + \frac{dT}{dt} \\
&= \left(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P\right)\frac{D}{N} + \left(\omega_1 B\frac{T}{I} + \omega_2 A\frac{T}{I} + \omega_3 R\frac{T}{R+I} + \theta_1 P\frac{T}{P+I}\right)
\end{aligned}$$

$$
\begin{aligned}
& -\big(\gamma_1 S+\gamma_2 E+\gamma_3 F+\gamma_8 T\big)-\tau_1\frac{A+V}{F+I}F-\zeta_3\frac{A+V}{F+I}F-\tau_3\frac{A+V}{I}T-\zeta_5\frac{A+V}{I}T\\
\leq\ & \big(\pi_1+\pi_2+\pi_3+\pi_4+\pi_5+\pi_6\big)D+\big(\omega_1+\omega_2+\omega_3+\theta_1\big)D\\
& -\big(\gamma_1 S+\gamma_2 E+\gamma_3 F+\gamma_8 T\big)-\tau_1\frac{A+V}{F+I}F-\zeta_3 F-\tau_3\frac{A+V}{I}T-\zeta_5 T\\
\leq\ & \big(\pi_1+\pi_2+\pi_3+\pi_4+\pi_5+\pi_6+\omega_1+\omega_2+\omega_3+\theta_1\big)D-\gamma D\\
\leq\ & \big(\lambda_1-\gamma\big)D
\end{aligned}
$$

The inequality $\lambda_1 < \gamma$ implies that the rate of decay γ exceeds the association strength λ_1. Consequently, D, representing the radical core subpopulation, decreases exponentially towards zero. This decline is swift and inevitable, highlighting the vulnerability of the radical core subpopulation under these conditions. □

This result underscores a crucial insight: when the combined influence of association strength and recruitment capacity within the radical core subpopulation is outweighed by the rate of recovery or reintegration into civilian life, the radical core subpopulation is destined for extinction. In essence, if the mechanisms driving radicalization and recruitment cannot outpace the natural tendency of individuals to return to normal societal roles, the radical core subpopulation will inevitably perish. This insight underscores the significance of addressing factors that promote radicalization and recruitment, as well as the importance of rehabilitation and reintegration efforts in countering extremist movements.

5.2. A sufficient condition of the eradication of ideological terrorism

Considering the capacities of indoctrination to the fanatic ideology and the capacity of recruitment of the terrorist, we establish in this part a condition of eradication of ideological terrorism.

Theorem 5. *Let $\lambda_2 = \beta_4 + \omega_1 + \omega_2 + \omega_3 + \theta_1$, $\lambda_3 = (\tau_3 + \zeta_5)\kappa + \gamma_8$ with κ the infimum of $\dfrac{A+V}{I}$. So for all $\mathcal{R}_2 = \dfrac{\lambda_2}{\lambda_3} < 1$, we have $\lim\limits_{t\to\infty} T(t) = 0$.*

Proof: From the equation (9) we have:

$$
\begin{aligned}
\frac{dT}{dt} &= \beta_4 F\frac{T}{N}+\omega_1 B\frac{T}{I}+\omega_2 A\frac{T}{I}+\omega_3 R\frac{T}{R+I}+\theta_1 P\frac{T}{P+I}-\left(\tau_3\frac{A+V}{I}+\gamma_8+\zeta_5\frac{A+V}{I}\right)T\\
&= \big(\beta_4\frac{F}{N}+\omega_1\frac{B}{I}+\omega_2\frac{A}{I}+\omega_3\frac{R}{R+I}+\theta_1\frac{P}{P+I}\big)T-\left(\tau_3\frac{A+V}{I}+\gamma_8+\zeta_5\frac{A+V}{I}\right)T\\
&\leq \big(\beta_4\frac{F}{N}+\omega_1\frac{B}{I}+\omega_2\frac{A}{I}+\omega_3\frac{R}{R+I}+\theta_1\frac{P}{P+I}\big)T-\left(\tau_3\kappa+\gamma_8+\zeta_5\kappa\right)T\\
&\leq \big(\beta_4+\omega_1+\omega_2+\omega_3+\theta_1\big)T-\left((\tau_3+\zeta_5)\kappa+\gamma_8\right)T=(\lambda_2-\lambda_3)T
\end{aligned}
$$

The last inequality implies that T decreases exponentially towards zero as soon as $\lambda_2 < \lambda_3$. Thus, when $\mathcal{R}_2 = \dfrac{\lambda_2}{\lambda_3} < 1$, it provides a sufficient condition for the stabilization or eradication of ideological terrorism. □

This observation is crucial as it highlights the pivotal role of recruitment and conversion rates in the dynamics of ideological terrorism propagation. When the recruitment rate is lower than the conversion rate, the number of individuals joining the terrorist cause diminishes over time, eventually leading to the extinction of this form of terrorism. This underscores the importance of implementing policies and strategies aimed at reducing recruitment opportunities and effectively countering extremist ideological propaganda. Consequently, controlling these parameters offers a potentially effective pathway to mitigate and prevent the threats posed by ideological terrorism.

The fact that the invariant superplane $T = 0$ is globally attractive for $\mathcal{R}_2 < 1$, we can then reduce the dimension of the model $(1) - (10)$.

In fact the model $(1) - (10)$ is reduced to the equivalent nine dimensional system using the limit equation approach as follows:

$$\begin{aligned}
\frac{dC}{dt} &= \Lambda + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_9 V - \Big(\pi_1 \frac{D}{N} + \alpha_1 \frac{B}{C+I} + \alpha_2 \frac{B}{C+I} + \sigma_2\Big) C && (22)\\
\frac{dR}{dt} &= \nu_2 A - \Big(\pi_5 \frac{D}{N} + \omega_4 \frac{B}{R+I} + \gamma_6\Big) R && (23)\\
\frac{dA}{dt} &= \sigma_1 V + \sigma_2 C + \nu_1 B - \Big(\pi_3 \frac{D}{N} + \nu_3 \frac{B}{I} + \gamma_4 + \nu_2 + \zeta_1 \frac{B}{I}\Big) A && (24)\\
\frac{dV}{dt} &= \alpha_1 C \frac{B}{C+I} - \Big(\pi_2 \frac{D}{N} + \gamma_9 + \sigma_1 + \zeta_2 \frac{B}{I}\Big) V && (25)\\
\frac{dS}{dt} &= \Big(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P\Big) \frac{D}{N} - (\beta_2 \frac{E+F}{N} + \gamma_1) S && (26)\\
\frac{dE}{dt} &= \beta_2 S \frac{E+F}{N} - (\beta_3 \frac{F}{N} + \gamma_2) E && (27)\\
\frac{dF}{dt} &= \beta_3 E \frac{F}{N} - \Big(\gamma_3 + \tau_1 \frac{A+V}{F+I} + \zeta_3 \frac{A+V}{F+I}\Big) F && (28)\\
\frac{dB}{dt} &= \alpha_2 C \frac{B}{C+I} + \omega_4 R \frac{B}{R+I} + \nu_3 A \frac{B}{I} + \theta_2 P \frac{B}{P+I} - \Big(\pi_4 \frac{D}{N} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \zeta_4 \frac{A+V}{I}\Big) B && (29)\\
\frac{dP}{dt} &= \tau_1 F \frac{A+V}{F+I} + \tau_2 B \frac{A+V}{I} - \Big(\pi_6 \frac{D}{N} + \theta_2 \frac{B}{P+I} + \gamma_5 + \eta\Big) P && (30)
\end{aligned}$$

where $D = S + E + F; I = B + A + V$

5.3. A sufficient condition of the eradication of fanatical ideology without terrorism

In this part we give a condition of eradication of fanatic ideology without terrorism. We first note the complexity of conducting an armed struggle against a fanatical ideology since it is difficult to identify a fanatical individual in a given society. We also note that sometimes the need to identify fanatical individuals can lead to stigmatization and exactions in certain cases. Through the result below, we observe that inter-religious dialogue as well as values of good governance and good communication could have a very important weight in the factors that can lead to the extinction of a fanatical ideology by giving more weight to γ_3 compared to β_3 by also taking into account the effect of a citizen's vigilance and police measures of proximity through τ_1 and ζ_3.

Theorem 6. *Let* $\lambda_4 = \gamma_3 + (\tau_1 + \zeta_3)\kappa'$ *with* κ' *infimum of* $\dfrac{A+V}{F+I}$.
So for all $\mathcal{R}_3 = \dfrac{\beta_3}{\lambda_4} < 1$, *we have* $\lim\limits_{t\to\infty} F(t) = 0$.

Proof: From the equation (28) we have:

$$\begin{aligned}
\frac{dF}{dt} &= \beta_3 E\frac{F}{N} - \left(\gamma_3 + \tau_1\frac{A+V}{F+I} + \zeta_3\frac{A+V}{F+I}\right)F \\
&\leq \beta_3 E\frac{F}{N} - \left(\gamma_3 + \tau_1\kappa' + \zeta_3\kappa'\right)F \\
&\leq \beta_3 F - \left(\gamma_3 + (\tau_1 + \zeta_3)\kappa'\right)F \\
&\leq (\beta_3 - \lambda_4)F
\end{aligned}$$

Hence F decreases exponentially to zero as $\beta_3 < \lambda_4$. □

In the same way $\mathcal{R}_3 = \dfrac{\beta_3}{\lambda_4} < 1$, gives a sufficient condition of the stabilization or the eradication of the fanatical ideology.

The fact also that the invariant superplane $F = 0$ is also a global attractor for $\mathcal{R}_3 < 1$, one can then reduce the dimension of the model $(22) - (30)$.

In fact the model $(22) - (30)$ is reduced to the equivalent eight dimensional system using the limit equation approach as follows:

$$\frac{dC}{dt} = \Lambda + \gamma_1 S + \gamma_2 E + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_9 V - \left(\pi_1\frac{D}{N} + \alpha_1\frac{B}{C+I} + \alpha_2\frac{B}{C+I} + \sigma_2\right)C \tag{31}$$

$$\frac{dR}{dt} = \nu_2 A - \left(\pi_5\frac{D}{N} + \omega_4\frac{B}{R+I} + \gamma_6\right)R \tag{32}$$

$$\frac{dA}{dt} = \sigma_1 V + \sigma_2 C + \nu_1 B - \left(\pi_3\frac{D}{N} + \nu_3\frac{B}{I} + \gamma_4 + \nu_2 + \zeta_1\frac{B}{I}\right)A \tag{33}$$

$$\frac{dV}{dt} = \alpha_1 C\frac{B}{C+I} - \left(\pi_2\frac{D}{N} + \gamma_9 + \sigma_1 + \zeta_2\frac{B}{I}\right)V \tag{34}$$

$$\frac{dS}{dt} = \left(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P\right)\frac{D}{N} - (\beta_2\frac{E}{N} + \gamma_1)S \tag{35}$$

$$\frac{dE}{dt} = (\beta_2\frac{S}{N} - \gamma_2)E \tag{36}$$

$$\frac{dB}{dt} = \alpha_2 C\frac{B}{C+I} + \omega_4 R\frac{B}{R+I} + \nu_3 A\frac{B}{I} + \theta_2 P\frac{B}{P+I} - \left(\pi_4\frac{D}{N} + \tau_2\frac{A+V}{I} + \gamma_7 + \nu_1 + \zeta_4\frac{A+V}{I}\right)B \tag{37}$$

$$\frac{dP}{dt} = \tau_2 B\frac{A+V}{I} - \left(\pi_6\frac{D}{N} + \theta_2\frac{B}{P+I} + \gamma_5 + \eta\right)P \tag{38}$$

where $D = S + E; I = B + A + V$

5.4. A sufficient condition of the eradication of brigandage without terrorism nor ideological fanaticism.

First of all, let us note that the fight against brigandage is one of the essential steps in the construction of a state. One of the indicators of a state's strength lies in its ability to enforce its laws throughout its territory. In this section we give a condition for eradicating brigandage without terrorism nor ideological fanaticism.

Theorem 7. *Let $\lambda_5 = \alpha_2 + \omega_4 + \nu_3 + \theta_2$ and $\lambda_6 = \pi_4\kappa'' + \tau_2\kappa + \gamma_7 + \nu_1 + \zeta_4\kappa$ with κ, κ'' respective infimum of $\dfrac{A+V}{I}$ and of $\dfrac{D}{N}$. So for all $\mathcal{R}_4 = \dfrac{\lambda_5}{\lambda_6} < 1$, we have $\lim\limits_{t\to\infty} B(t) = 0$.*

Proof: From the equation (37) we have:

$$\begin{aligned}
\frac{dB}{dt} &= \alpha_2 C\frac{B}{C+I} + \omega_4 R\frac{B}{R+I} + \nu_3 A\frac{B}{I} + \theta_2 P\frac{B}{P+I} - \left(\pi_4\frac{D}{N} + \tau_2\frac{A+V}{I} + \gamma_7 + \nu_1 + \zeta_4\frac{A+V}{I}\right)B \\
&\leq \big(\alpha_2 + \omega_4 + \nu_3 + \theta_2\big)B - \Big(\pi_4\kappa'' + \tau_2\kappa + \gamma_7 + \nu_1 + \zeta_4\kappa\Big)B \\
&\leq \big(\lambda_5 - \lambda_6\big)B
\end{aligned}$$

From this last inequality, B decreases exponentially if $\lambda_5 < \lambda_6$. □

This result underscores the critical notion that the effectiveness of countermeasures against brigandage hinges on the balance between conversion forces within society and the state's capacity to combat insecurity through citizen vigilance and effective governance. Specifically, if the cumulative force of conversion into brigandage, stemming from various sources such as non-indoctrinated civilians, discharged personnel from defense and security forces, individuals released from the justice system, and even the defense and security forces themselves, falls short of the state's capability to address all forms of insecurity, including through citizen engagement and robust governance structures, then we are likely to witness either stabilization or complete eradication of brigandage.

This observation underscores the interconnectedness between societal dynamics and state capacity in combating criminal activities like brigandage. It suggests that a holistic approach, encompassing not only law enforcement and judicial efforts but also social, economic, and governance reforms, is essential for effectively tackling such security challenges. By addressing root causes, fostering community resilience, and ensuring robust governance frameworks, states can create an environment where brigandage finds little fertile ground for recruitment and operation. Therefore, achieving stability and eventual eradication of brigandage requires a concerted effort that goes beyond traditional security measures and encompasses broader societal and governance reforms.

6. Numerical simulation

In this section, we present numerical results to provide further insights into the dynamics of the propagation of fanatic ideology. We conducted numerical simulations using MATLAB with the finite difference method, employing an explicit numerical scheme with a discretization step of 0.02.

Figure 2 illustrates scenarios where the number of basic reproductions of extreme behavior $\mathcal{R}_0$ is less than 1, indicating a situation where the spread of fanatic ideology cannot sustain itself over time. Here, we observe the stabilization or extinction of various classes affected by the propagation of fanatic ideology, consistent with theoretical expectations. This indicates that under conditions where the rate of propagation is lower than the rate of recovery, the spread of fanatic ideology tends to diminish over time, ultimately leading to its eradication.

Conversely, Figure 3 depicts scenarios where $\mathcal{R}_0$ is greater than 1, indicating situations where the spread of fanatic ideology can persist and propagate within the population. In such cases, we observe the persistence of the affected classes, as the rate of propagation outweighs the rate of recovery. This highlights the potential for sustained propagation of fanatic ideology under conditions where recruitment and indoctrination efforts outpace efforts to counter and rehabilitate individuals.

These numerical simulations provide valuable insights into the dynamics of the propagation of fanatic ideology and underscore the importance of understanding and controlling key parameters such as recruitment, indoctrination, and recovery rates in mitigating its spread. Additionally, they emphasize the critical role of effective counterterrorism strategies and governance structures in addressing and preventing the proliferation of radical movements. To initialize our simulations, we set the following initial conditions: $C(0) = 150000$, $R(0) = 8$, $A(0) = 150$, $V(0) = 150$, $S(0) = 25000$, $E(0) = 1500$, $F(0) = 400$, $B(0) = 100$, $T(0) = 150$, $P(0) = 20$.

These initial conditions reflect the initial composition of the population in the different classes affected by the spread of fanatical ideology. In addition, the parameter values used in our simulations have been carefully estimated and are detailed in Table 1.

Table 1: Parameter values estimated

Parameters	value for extinction	value for persistence
Λ	22500	22500
γ_1	0.46	0.46
γ_2	0.28	0.28
γ_3	0.000111	0.00111
γ_4	0.12	0.137
γ_5	0.0000016	0.0016
γ_6	0.026	0.09
γ_7	0.002	0.06
γ_8	0.000011	0.00011
γ_9	0.011	0.011
π_1	0.000007	0.0000006
π_2	0.0000534	0.000534
π_3	0.0000002	0.0000002
π_4	0.000074	0.0074
π_5	0.0000004	0.0000004
π_6	0.44	0.44
θ_1	0.000032	0.000032
θ_2	0.000032	0.000032
η	0.15	0.15
ζ_1	0.27	0.27
ζ_2	0.27	0.27
ζ_3	0.17	0.17
ζ_4	0.17	0.47
ζ_5	0.17	0.37
μ	0.08	0.08
ν_1	0.002	0.02
ν_2	0.0002	0.0002
ν_3	0.2	0.2
τ_1	0.2	0.2
τ_2	0.45	0.45
τ_3	0.45	0.45
β_2	0.95	0.95
β_3	0.72	0.72
β_4	0.0078	0.0078
σ_1	0.011	0.011
σ_2	0.0001	0.0001
α_1	0.0002	0.0002
α_2	0.000534	4
ω_1	0.25	1.2
ω_2	0.8	0.2
ω_3	0.38	1.2
ω_4	0.38	2

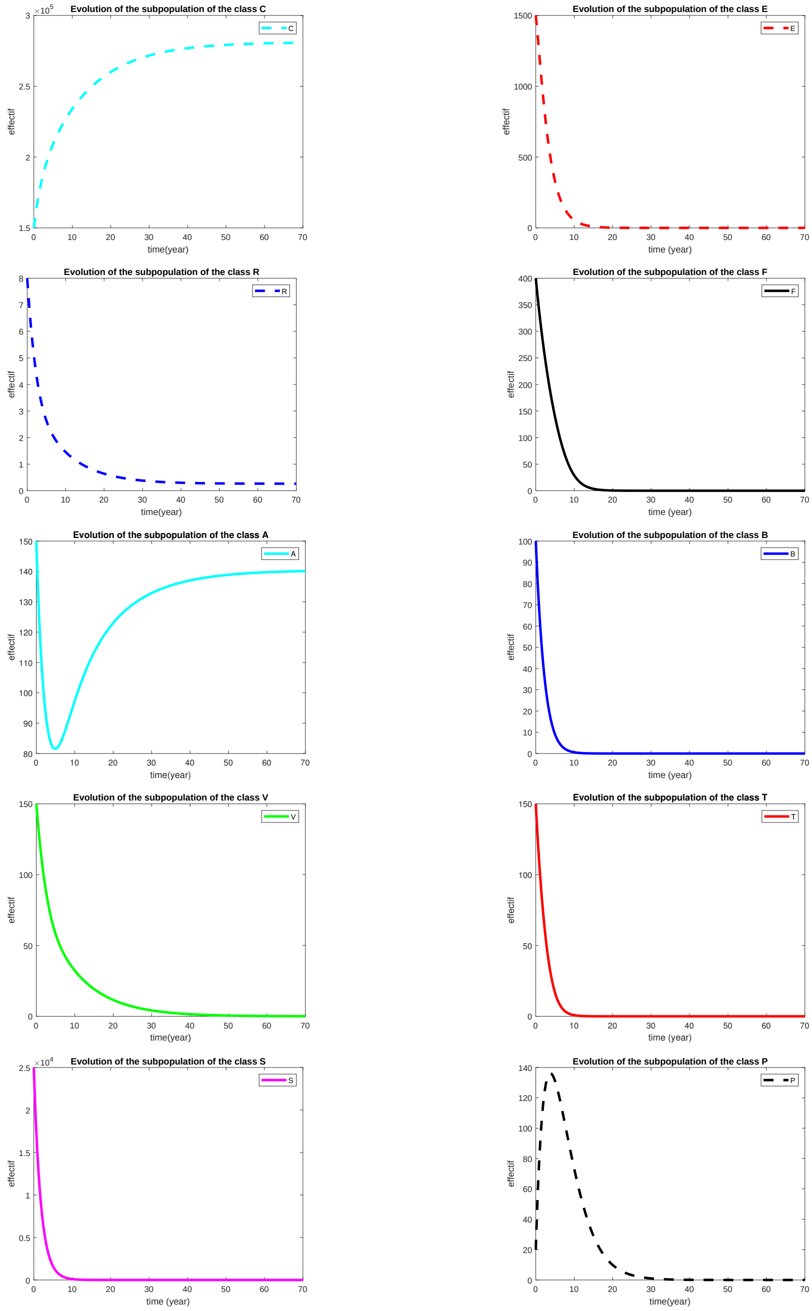

Figure 2: Evolution of the different classes of the model $(1)-(10)$ with the extinction values. We get $\mathcal{R}_0 = 0.3666$, which is less than unity.

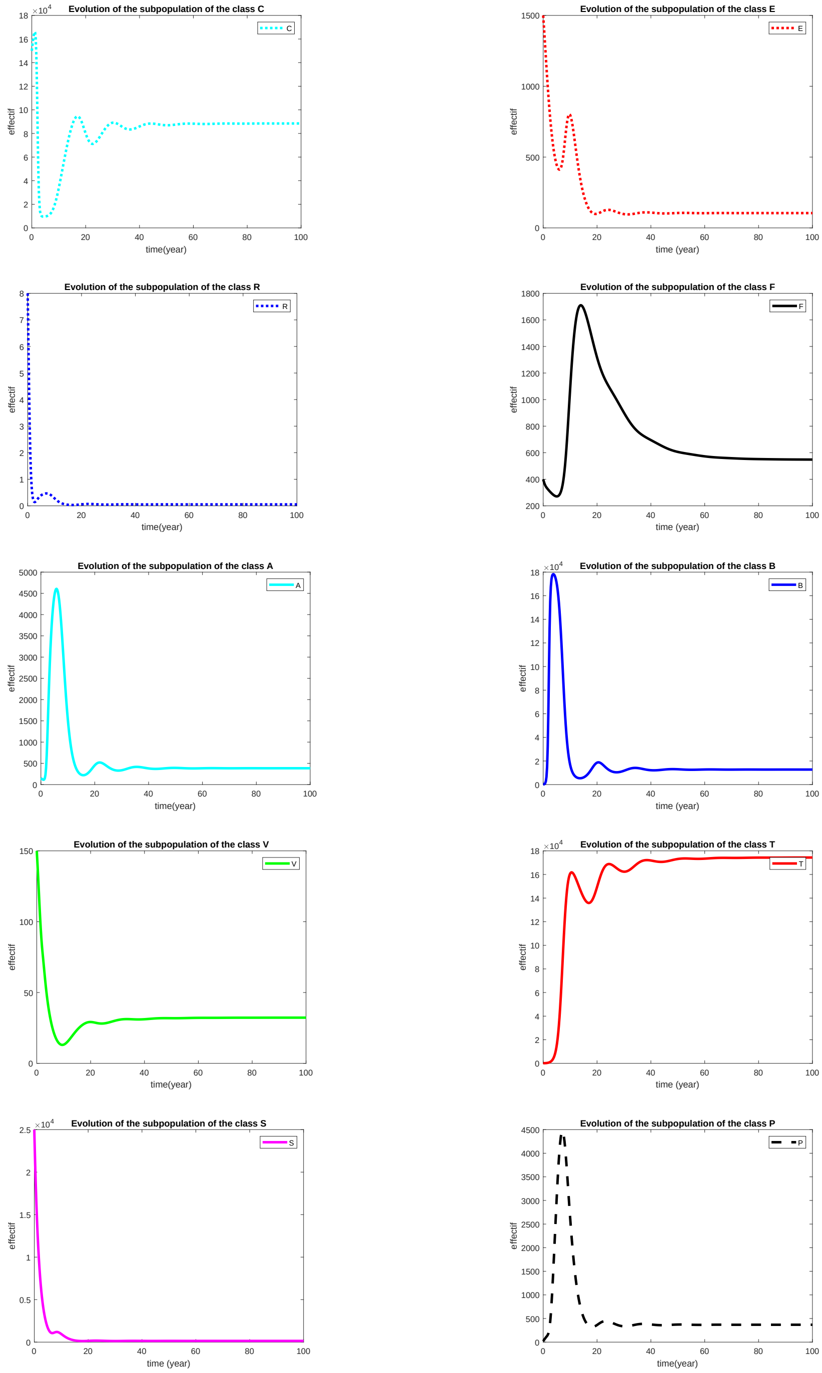

Figure 3: Evolution of the different classes of the model $(1)-(10)$ with the persistence values. We get $\mathcal{R}_0 = 11.5538$, which is greater than unity.

7. Mathematical analysis of a strategy to fight against terrorism, fanaticism and brigandage

7.1. Strategy to fight against terrorism, fanaticism and brigandage

Based on the findings from our analysis, we apply optimal control theory to the model (1) − (10) in order to combat fanatical insurgency and brigandage. This involves introducing five time-dependent control variables, denoted as $u_1(t)$, $u_2(t)$, $u_3(t)$, $u_4(t)$, and $u_5(t)$, each representing specific strategies aimed at addressing radicalization, violent extremism, and insecurity. Let's delve into the details of these strategies:

(i) $u_1(t)$ represents a preventive strategy designed to hinder the indoctrination into radical ideology. This strategy may involve advocacy efforts to bolster social cohesion, socio-economic integration of vulnerable communities, educational initiatives, and facilitating employment opportunities. Notably, $u_1(t) = 1$ indicates the strategy's effectiveness against radicalization, while $u_1(t) = 0$ signifies its failure. The focus here is on significantly reducing the vulnerability of certain populations to radicalization.

(ii) Similarly, $u_2(t)$ serves as another preventive strategy, which, in addition to the measures outlined in (i), emphasizes the state's presence in vulnerable communities and endeavors to instill hope among disenchanted youth. This involves offering more attractive alternatives than those provided by radical groups, thereby fostering a sense of optimism. A value of $u_2(t) = 1$ denotes the strategy's effectiveness, while $u_2(t) = 0$ indicates failure.

(iii) $u_3(t)$ represents deradicalization strategies, which may entail intensified efforts towards inter-religious dialogue, community reconciliation, and engagement with radical groups by the state. A value of $u_3(t) = 1$ signifies the effectiveness of the deradicalization strategy, while $u_3(t) = 0$ indicates failure.

(iv) The control variable $u_4(t)$ is a strategy aimed at combating organized crime, brigandage, and corruption. It encompasses police actions, such as community policing, investigations, and protection measures. This strategy also involves improving the capacity and equipment of defense and security forces and facilitating the rehabilitation and reintegration of offenders into society. A value of $u_4(t) = 1$ denotes the effectiveness of the strategy against brigandage, while $u_4(t) = 0$ signifies failure.

(v) Finally, $u_5(t)$ represents the counterterrorism strategy, which, in addition to preventive measures and efforts against organized crime, focuses on disrupting terrorist financing and strengthening the capabilities of defense and security forces. This entails enhancing firepower, intelligence systems, and executing coordinated actions to mitigate terrorist threats. A value of $u_5(t) = 1$ indicates the strategy's effectiveness against terrorism, while $u_5(t) = 0$ denotes failure.

7.2. Mathematical analysis of strategy optimality

Let

$$c_i(t) = 1 - u_i(t), \qquad \forall i \in \{1, 2, 3, 4, 5\}. \tag{39}$$

Consequently, the optimal control model with the five aforementioned time-dependent variables is given by the following differential equations

$$\left\{\begin{array}{lcl}
\frac{dC}{dt} & = & \Lambda - \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \Big(c_1\pi_1 \frac{D}{N} + \alpha_1 \frac{T+B}{C+I} + c_4\alpha_2 \frac{B}{C+I} + \sigma_2 + \mu\Big) C \\
\frac{dR}{dt} & = & \nu_2 A - \Big(c_1\pi_5 \frac{D}{N} + c_5\omega_3 \frac{T}{R+I} + c_4\omega_4 \frac{B}{R+I} + \gamma_6 + \mu\Big) R \\
\frac{dA}{dt} & = & \sigma_1 V + \sigma_2 C + \nu_1 B - \Big(c_1\pi_3 \frac{D}{N} + c_4\nu_3 \frac{B}{I} + c_5\omega_2 \frac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \frac{T+B}{I}\Big) A \\
\frac{dV}{dt} & = & \alpha_1 C \frac{T+B}{C+I} - \Big(c_1\pi_2 \frac{D}{N} + \gamma_9 + \sigma_1 + \mu + \zeta_2 \frac{T+B}{I}\Big) V \\
\frac{dS}{dt} & = & c_1\Big(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P\Big)\frac{D}{N} - \Big(c_2\beta_2 \frac{E+F+T}{N} + \gamma_1 + \mu\Big) S \\
\frac{dE}{dt} & = & c_2\beta_2 S \frac{E+F+T}{N} - \Big(c_3\beta_3 \frac{F+T}{N} + \gamma_2 + \mu\Big) E \\
\frac{dF}{dt} & = & c_3\beta_3 E \frac{F+T}{N} - \Big(\gamma_3 + \tau_1 \frac{A+V}{F+I} + c_5\beta_4 \frac{T}{N} + \mu + \zeta_3 \frac{A+V}{F+I}\Big) F \\
\frac{dB}{dt} & = & c_4\Big(\alpha_2 \frac{C}{C+I} + \omega_4 \frac{R}{R+I} + \nu_3 \frac{A}{I} + \theta_2 \frac{P}{P+I}\Big) B - \Big(c_1\pi_4 \frac{D}{N} + c_5\omega_1 \frac{T}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_4 \frac{A+V}{I}\Big) B \\
\frac{dT}{dt} & = & c_5\Big(\beta_4 \frac{F}{N} + \omega_1 \frac{B}{I} + \omega_2 \frac{A}{I} + \omega_3 \frac{R}{R+I} + \theta_1 \frac{P}{P+I}\Big) T - \Big(\tau_3 \frac{A+V}{I} + \gamma_8 + \mu + \zeta_5 \frac{A+V}{I}\Big) T \\
\frac{dP}{dt} & = & \tau_1 F \frac{A+V}{F+I} + \tau_2 B \frac{A+V}{I} + \tau_3 T \frac{A+V}{I} - \Big(c_1\pi_6 \frac{D}{N} + c_5\theta_1 \frac{T}{P+I} + c_4\theta_2 \frac{B}{P+I} + \gamma_5 + \mu + \eta\Big) P
\end{array}\right. \tag{40}$$

With non-negative initial conditions given by (11). This system can be written in matrix form as follows:

$$X'(t) = g(t, X, c) \tag{41}$$

Where X is defined in (13), $c = (c_1(t), c_2(t), c_3(t), c_4(t), c_5(t)) \in \mathbb{R}^5$ verifies (39), and $g : \mathbb{R} \times \mathbb{R}^{10} \times \mathbb{R}^5 \to \mathbb{R}^{10}$ is a nonlinear function which is written as in (14) but introducing the command $c(t)$ so as to verify (40).

The purpose of introducing the five control variables is to seek the optimal solution required to minimize the number of individuals in the radical subpopulation or core of violent extremism and fanatical behavior as well as brigands. Therefore, the objective function for this control problem is given by

$$\mathcal{J}(u_1, u_2, u_3, u_4, u_5) = \min_{0 \leqslant u_1, u_2, u_3, u_4, u_5 \leqslant 1} \int_0^{T_f} \Big(j(t) + \frac{1}{2} k(t)\Big) dt \tag{42}$$

where

$$\begin{aligned} j(t) &= w_1 S(t) + w_2 E(t) + w_3 F(t) + w_4 B(t) + w_5 T(t) + w_6 P(t) \\ k(t) &= \Big[w_7 u_1^2(t) + w_8 u_2^2(t) + w_9 u_3^2(t) + w_{10} u_4^2(t) + w_{11} u_5^2(t)\Big] \end{aligned}$$

where the constants $w_i, i = 1, 2, ..., 11$ are positive weights needed to balance the corresponding terms of the objective function. We choose quadratic costs on the orders, where $\frac{1}{2} w_7 u_1^2(t)$, $\frac{1}{2} w_8 u_2^2(t)$, $\frac{1}{2} w_9 u_3^2(t)$, $\frac{1}{2} w_{10} u_4^2(t)$, $\frac{1}{2} w_{11} u_5^2(t)$ are the total cost of implementing the preventive measure and the police-military response to manage active cases of armed insurgency over the time interval $[0, T_f]$. Specifically, we look for the optimal quintuple control $u^* = \left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*\right)$ is sought such that

$$\mathcal{J}\left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*\right) = \min\left\{\mathcal{J}\left(u_1, u_2, u_3, u_4, u_5\right) : u_1, u_2, u_3, u_4, u_5 \in \mathcal{U}\right\}, \tag{43}$$

where, $\mathcal{U}$ is the non-empty control set defined by

$$\mathcal{U} = \left\{ \left(u_1, u_2, u_3, u_4, u_5\right) \;\middle|\; \begin{array}{l} \quad u_i(t) \text{ is a piecewise continuous function on } [0, T_f] \\ \text{and} \quad 0 \leqslant u_i \leqslant 1, \quad \forall \in t \in [0, T_f], \quad i = 1, 2, 3, 4, 5 \end{array} \right\} \tag{44}$$

Thus, to determine the necessary conditions that the optimal control quintuple must satisfy, we use Pontryagin's maximum principle [29], which transforms the control problem (43) subject to model (40) into a pointwise minimization problem of a Hamiltonian $\mathcal{H}$. This Hamiltonian is given by

$$\begin{aligned}
\mathcal{H} \;=\;& w_1 S + w_2 E + w_3 F + w_4 B + w_5 T + w_6 P + \frac{1}{2}\left[w_7 u_1^2(t) + w_8 u_2^2(t) + w_9 u_3^2(t) + w_{10} u_4^2(t) + w_{11} u_5^2(t)\right] \\
+\;& \lambda_1\left[\Lambda + \gamma_1 S + \gamma_2 E + \gamma_3 F + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \left(c_1\pi_1\frac{D}{N} + \alpha_1\frac{T+B}{C+I} + c_4\alpha_2\frac{B}{C+I} + \sigma_2 + \mu\right)C\right] \\
+\;& \lambda_2\left[\nu_2 A - \left(c_1\pi_5\frac{D}{N} + c_5\omega_3\frac{T}{R+I} + c_4\omega_4\frac{B}{R+I} + \gamma_6 + \mu\right)R\right] \\
+\;& \lambda_3\left[\sigma_1 V + \sigma_2 C + \nu_1 B - \left(c_1\pi_3\frac{D}{N} + c_4\nu_3\frac{B}{I} + c_5\omega_2\frac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1\frac{T+B}{I}\right)A\right] \\
+\;& \lambda_4\left[\alpha_1 C\frac{T+B}{C+I} - \left(c_1\pi_2\frac{D}{N} + \gamma_9 + \sigma_1 + \mu + \zeta_2\frac{T+B}{I}\right)V\right] \\
+\;& \lambda_5\left[c_1\left(\pi_1 C + \pi_2 V + \pi_3 A + \pi_4 B + \pi_5 R + \pi_6 P\right)\frac{D}{N} - \left(c_2\beta_2\frac{E+F+T}{N} + \gamma_1 + \mu\right)S\right] \\
+\;& \lambda_6\left[c_2\beta_2 S\frac{E+F+T}{N} - \left(c_3\beta_3\frac{F+T}{N} + \gamma_2 + \mu\right)E\right] \\
+\;& \lambda_7\left[c_3\beta_3 E\frac{F+T}{N} - \left(\gamma_3 + \tau_1\frac{A+V}{F+I} + c_5\beta_4\frac{T}{N} + \mu + \zeta_3\frac{A+V}{F+I}\right)F\right] \\
+\;& \lambda_8\left[c_4\left(\alpha_2\frac{C}{C+I} + \omega_4\frac{R}{R+I} + \nu_3\frac{A}{I} + \theta_2\frac{P}{P+I}\right)B - \left(c_1\pi_4\frac{D}{N} + c_5\omega_1\frac{T}{I} + \tau_2\frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_4\frac{A+V}{I}\right)B\right] \\
+\;& \lambda_9\left[c_5\left(\beta_4\frac{F}{N} + \omega_1\frac{B}{I} + \omega_2\frac{A}{I} + \omega_3\frac{R}{R+I} + \theta_1\frac{P}{P+I}\right)T - \left(\tau_3\frac{A+V}{I} + \gamma_8 + \mu + \zeta_5\frac{A+V}{I}\right)T\right] \\
+\;& \lambda_{10}\left[\tau_1 F\frac{A+V}{F+I} + \tau_2 B\frac{A+V}{I} + \tau_3 T\frac{A+V}{I} - \left(c_1\pi_6\frac{D}{N} + c_5\theta_1\frac{T}{P+I} + c_4\theta_2\frac{B}{P+I} + \gamma_5 + \mu + \eta\right)P\right]
\end{aligned} \tag{45}$$

where $\lambda_i, i = 1, 2, ..., 10$, represent the adjoint variables associated with the state variables of the model (40). The standard existence result for minimizing control problem as appeared in [19] is adapted as follows.

Theorem 8. ***(Existence and well-posedness of control problem)***
There exits an optimal quintuple control $\left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*\right) \in \mathcal{U}$ *satysfing* (42) *subject to the control system* (40) *with non-negative initial conditions given by* (11).

Proof: The existence of an optimal control is obtained thanks to the Fleming and Rishel [19]. Thanks to a result of Lukes [25] which ensures the existence of solutions for the state system (40) with constant coefficients, the set of controls and corresponding solutions is non empty. In addition the set of controls $\mathcal{U}$ is a closed convex by definition and the vector field of the system (40) is bounded. Also the integrand of the objective function is clearly convex and $g(t, X, c)$ in (41) is convex with respect to c. On the other hand there exist $a_1, a_2 > 0$ and $\beta > 1$ such that

$$w_1 S + w_2 E + w_3 F + w_4 B + w_5 T + w_6 P + \frac{1}{2}\left[w_7 u_1^2(t) + w_8 u_2^2(t) + w_9 u_3^2(t) + w_{10} u_4^2(t) + w_{11} u_5^2(t)\right]$$
$$\geq a_1 \left(|u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2 + |u_5|^2\right)^{\frac{\beta}{2}} - a_2$$

since the state variables are bounded.
Then, we deduce the existence of an optimal control $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*)$ that minimizes the objective function $J(u_1, u_2, u_3, u_4, u_5)$. $\square$

Theorem 9. *Given that* $\left(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*\right)$ *minimizes the objective functional* (42) *subject to the corresponding state system* (40)*, then the adjoint variables* $\lambda_i, i = 1, 2, ..., 10$*, satisfy the following system:*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= (\lambda_1 - \lambda_5)c_1\pi_1 \frac{D(N-C)}{N^2} + (\lambda_1 - \lambda_4)\alpha_1 \frac{(T+B)I}{(C+I)^2} + (\lambda_1 - \lambda_8)c_4\alpha_2 \frac{BI}{(C+I)^2} + (\lambda_1 - \lambda_3)\sigma_2 + \lambda_1\mu \\
&\quad + (\lambda_5 - \lambda_2)c_1\pi_5 \frac{DR}{N^2} + (\lambda_5 - \lambda_3)c_1\pi_3 \frac{DA}{N^2} + (\lambda_5 - \lambda_4)c_1\pi_2 \frac{DV}{N^2} + (\lambda_8 - \lambda_5)c_1\pi_4 \frac{DB}{N^2} + (\lambda_5 - \lambda_{10})c_1\pi_6 \frac{DP}{N^2} \\
&\quad + (\lambda_6 - \lambda_5)c_2\beta_2 \frac{S(E+F+T)}{N^2} + (\lambda_7 - \lambda_6)c_3\beta_3 \frac{E(F+T)}{N^2} + (\lambda_9 - \lambda_7)c_5\beta_4 \frac{TF}{N^2} \\
\frac{d\lambda_2}{dt} &= (\lambda_2 - \lambda_1)\gamma_6 + (\lambda_2 - \lambda_1)c_1\pi_1 \frac{DC}{N^2} + (\lambda_2 - \lambda_1)c_1\pi_5 \frac{D(N-R)}{N^2} + (\lambda_2 - \lambda_6)c_5\omega_3 \frac{TI}{(R+I)^2} + \lambda_2\mu \\
&\quad + (\lambda_2 - \lambda_8)c_4\omega_4 \frac{BI}{(R+I)^2} + (\lambda_5 - \lambda_3)c_1\pi_3 \frac{DA}{N^2} + (\lambda_5 - \lambda_4)c_1\pi_2 \frac{DV}{N^2} + (\lambda_5 - \lambda_8)c_1\pi_4 \frac{DB}{N^2} \\
&\quad + (\lambda_5 - \lambda_{10})c_1\pi_6 \frac{DP}{N^2} + (\lambda_6 - \lambda_5)c_2\beta_2 \frac{S(E+F+T)}{N^2} + (\lambda_7 - \lambda_6)c_3\beta_3 \frac{E(F+T)}{N^2} + (\lambda_9 - \lambda_7)c_5\beta_4 \frac{TF}{N^2}
\end{aligned} \tag{46}$$

$$
\begin{aligned}
\frac{d\lambda_3}{dt} &= (\lambda_3-\lambda_1)\gamma_4 + (\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2} + (\lambda_4-\lambda_1)\alpha_1\frac{(T+B)C}{(C+I)^2} + (\lambda_8-\lambda_1)c_4\alpha_2\frac{BC}{(C+I)^2} + (\lambda_3-\lambda_2)\nu_2 \\
&+ (\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2} + (\lambda_9-\lambda_2)c_5\omega_3\frac{TR}{(R+I)^2} + (\lambda_8-\lambda_2)c_4\omega_4\frac{BR}{(R+I)^2} + (\lambda_3-\lambda_5)c_1\pi_3\frac{D(N-A)}{N^2} \\
&+ (\lambda_3-\lambda_8)c_4\nu_3\frac{B(V+T+B)}{I^2} + (\lambda_3-\lambda_9)c_5\omega_2\frac{T(V+T+B)}{I^2} + \lambda_3\zeta_1\frac{(T+B)(V+T+B)}{I^2} + \lambda_3\mu \\
&(\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2} - \lambda_4\zeta_2\frac{(T+B)V}{I^2} + (\lambda_5-\lambda_8)c_1\pi_1\frac{DB}{N^2} + (\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2} + (\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2} \\
&+ (\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2} + (\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2} + (\lambda_7-\lambda_{10})\tau_1\frac{F(F+T+B)}{(F+I)^2} + (\lambda_8-\lambda_{10})c_4\theta_2\frac{PB}{(P+I)^2} \\
&+ \lambda_7\zeta_3\frac{F(F+T+B)}{(F+I)^2} + (\lambda_8-\lambda_{10})\tau_2\frac{B(T+B)}{I^2} + (\lambda_9-\lambda_8)c_5\omega_1\frac{TB}{I^2} + \lambda_8\zeta_4\frac{B(T+B)}{I^2} \\
&+ (\lambda_9-\lambda_{10})\theta_1\frac{TP}{(P+I)^2} + (\lambda_9-\lambda_{10})\tau_3\frac{T(T+B)}{I^2} + \lambda_9\zeta_5\frac{T(T+B)}{I^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_4}{dt} &= (\lambda_3-\lambda_1)\gamma_9 + (\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2} + (\lambda_4-\lambda_1)\alpha_1\frac{(T+B)C}{(C+I)^2} + (\lambda_8-\lambda_1)c_4\alpha_2\frac{BC}{(C+I)^2} + (\lambda_4-\lambda_3)\sigma_1 \\
&+ (\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2} + (\lambda_9-\lambda_2)c_5\omega_3\frac{TR}{(R+I)^2} + (\lambda_8-\lambda_2)c_4\omega_4\frac{BR}{(R+I)^2} + (\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2} \\
&+ (\lambda_8-\lambda_3)c_4\nu_3\frac{BA}{I^2} + (\lambda_9-\lambda_3)c_5\omega_2\frac{TA}{I^2} - \lambda_3\zeta_1\frac{(T+B)A}{I^2} + \lambda_4\mu + (\lambda_4-\lambda_5)c_1\pi_2\frac{D(N-V)}{N^2} \\
&+ \lambda_4\zeta_2\frac{(T+B)(A+T+B)}{I^2} + (\lambda_5-\lambda_8)c_1\pi_4\frac{DB}{N^2} + (\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2} + (\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2} \\
&+ (\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2} + (\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2} + (\lambda_7-\lambda_{10})\tau_1\frac{F(F+T+B)}{(F+I)^2} + (\lambda_8-\lambda_{10})c_4\theta_2\frac{PB}{(P+I)^2} \\
&+ \lambda_7\zeta_3\frac{F(F+T+B)}{(F+I)^2} + (\lambda_8-\lambda_{10})\tau_2\frac{B(T+B)}{I^2} + (\lambda_9-\lambda_8)c_5\omega_1\frac{TB}{I^2} + \lambda_8\zeta_4\frac{B(T+B)}{I^2} \\
&+ (\lambda_9-\lambda_{10})\theta_1\frac{TP}{(P+I)^2} + (\lambda_9-\lambda_{10})\tau_3\frac{T(T+B)}{I^2} + \lambda_9\zeta_5\frac{T(T+B)}{I^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_5}{dt} &= -w_1 + (\lambda_5-\lambda_1)\gamma_1 + (\lambda_1-\lambda_5)c_1\pi_1\frac{C(N-D)}{N^2} + (\lambda_2-\lambda_5)c_1\pi_5\frac{R(N-D)}{N^2} + (\lambda_3-\lambda_5)c_1\pi_3\frac{A(N-D)}{N^2} \\
&+ (\lambda_4-\lambda_5)c_1\pi_2\frac{V(N-D)}{N^2} + (\lambda_8-\lambda_5)c_1\pi_4\frac{B(N-D)}{N^2} + (\lambda_{10}-\lambda_5)c_1\pi_6\frac{P(N-D)}{N^2} + \lambda_5\mu \\
&+ (\lambda_5-\lambda_6)c_2\beta_2\frac{(E+F+T)(N-S)}{N^2} + (\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2} + (\lambda_9-\lambda_7)c_5\pi_4\frac{TF}{N^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_6}{dt} &= -w_2 + (\lambda_6-\lambda_1)\gamma_2 + (\lambda_1-\lambda_5)c_1\pi_1\frac{C(N-D)}{N^2} + (\lambda_2-\lambda_5)c_1\pi_5\frac{R(N-D)}{N^2} + (\lambda_3-\lambda_5)c_1\pi_3\frac{A(N-D)}{N^2} \\
&+(\lambda_4-\lambda_5)c_1\pi_2\frac{V(N-D)}{N^2} + (\lambda_8-\lambda_5)c_1\pi_4\frac{B(N-D)}{N^2} + (\lambda_{10}-\lambda_5)c_1\pi_6\frac{P(N-D)}{N^2} + \lambda_6\mu \\
&+(\lambda_5-\lambda_6)c_2\beta_2\frac{S\Big(N-(E+F+T)\Big)}{N^2} + (\lambda_6-\lambda_7)c_3\beta_3\frac{(F+T)(N-E)}{N^2} + (\lambda_9-\lambda_7)c_5\pi_4\frac{TF}{N^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_7}{dt} &= -w_3 + (\lambda_7-\lambda_1)\gamma_2 + (\lambda_1-\lambda_5)c_1\pi_1\frac{C(N-D)}{N^2} + (\lambda_2-\lambda_5)c_1\pi_5\frac{R(N-D)}{N^2} + (\lambda_3-\lambda_5)c_1\pi_3\frac{A(N-D)}{N^2} \\
&+(\lambda_4-\lambda_5)c_1\pi_2\frac{V(N-D)}{N^2} + (\lambda_8-\lambda_5)c_1\pi_4\frac{B(N-D)}{N^2} + (\lambda_{10}-\lambda_5)c_1\pi_6\frac{P(N-D)}{N^2} + \lambda_7\mu \\
&+(\lambda_5-\lambda_6)c_2\beta_2\frac{S\Big(N-(E+F+T)\Big)}{N^2} + (\lambda_6-\lambda_7)c_3\beta_3\frac{(F+T)(N-E)}{N^2} + (\lambda_7-\lambda_9)c_5\beta_4\frac{T(N-F)}{N^2} \\
&+(\lambda_7-\lambda_{10})\tau_1\frac{(A+V)I}{(F+I)^2} + \lambda_7\zeta_3\frac{(A+V)I}{(F+I)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_8}{dt} &= -w_4 + (\lambda_8-\lambda_1)\gamma_7 + (\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2} + (\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2} + (\lambda_9-\lambda_2)c_5\omega_3\frac{TR}{(R+I)^2} + (\lambda_8-\lambda_3)\nu_1 \\
&+(\lambda_5-\lambda_{10})c_1\pi_6\frac{DP}{N^2} + (\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2} + (\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2} + (\lambda_1-\lambda_4)c_4\alpha_1\frac{C(C+A+V)}{(C+I)^2} \\
&+(\lambda_1-\lambda_8)c_4\alpha_2\frac{C(C+A+V+T)}{(C+I)^2} + (\lambda_2-\lambda_8)c_4\omega_4\frac{R(R+A+V+T)}{(R+I)^2} + \lambda_8\mu + (\lambda_3-\lambda_9)c_5\omega_2\frac{TA}{I^2} \\
&+(\lambda_3-\lambda_8)c_4\nu_3\frac{A(A+V+T)}{I^2} + \lambda_3\zeta_1\frac{A(A+V)}{I^2} + \lambda_4\zeta_2\frac{V(A+V)}{I^2} + (\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2} \\
&+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2} + (\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2} + (\lambda_{10}-\lambda_7)\tau_1\frac{F(A+V)}{(F+I)^2} - \lambda_7\zeta_3\frac{F(A+V)}{(F+I)^2} \\
&+(\lambda_8-\lambda_{10})\tau_2\frac{(A+V)(A+V+T)}{I^2} + \lambda_8\zeta_4\frac{(A+V)(A+V+T)}{I^2} + (\lambda_8-\lambda_{10})c_4\theta_2\frac{P(P+A+V+T)}{(P+I)^2} \\
&+(\lambda_9-\lambda_8)c_5\omega_1\frac{TB}{I^2} + (\lambda_9-\lambda_{10})\theta_1\frac{TP}{(P+I)^2} + (\lambda_{10}-\lambda_9)\tau_3\frac{T(A+V)}{I^2} - \lambda_9\zeta_5\frac{T(A+V)}{I^2} \\
&+(\lambda_8-\lambda_5)c_1\pi_4\frac{D(N-B)}{N^2} + (\lambda_8-\lambda_9)c_5\omega_1\frac{T(A+V+T)}{I^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_9}{dt} \quad = \quad & -w_5+(\lambda_9-\lambda_1)\gamma_8+(\lambda_1-\lambda_5)c_1\pi_1\frac{C(N-D)}{N^2}+(\lambda_1-\lambda_4)c_1\alpha_1\frac{C(C+A+V)}{(C+I)^2}+(\lambda_8-\lambda_1)c_4\alpha_2\frac{BC}{(C+I)^2} \\
& +(\lambda_2-\lambda_5)c_1\pi_5\frac{R(N-D)}{N^2}+(\lambda_2-\lambda_9)c_5\omega_3\frac{R(R+A+V+B)}{(R+I)^2}+(\lambda_8-\lambda_2)c_4\omega_4\frac{BR}{(R+I)^2} \\
& +(\lambda_3-\lambda_5)c_1\pi_3\frac{A(N-D)}{N^2}+(\lambda_8-\lambda_3)c_4\nu_3\frac{BA}{I^2}+(\lambda_3-\lambda_9)c_5\omega_2\frac{A(A+V+B)}{I^2}+\lambda_3\zeta_1\frac{A(A+V)}{I^2} \\
& +(\lambda_4-\lambda_5)c_1\pi_2\frac{V(N-D)}{N^2}+\lambda_4\zeta_2\frac{V(A+V)}{I^2}+(\lambda_5-\lambda_6)c_2\beta_2\frac{S\Big(N-(E+F+T)\Big)}{N^2} \\
& +(\lambda_6-\lambda_7)c_3\beta_3\frac{E\Big(N-(F+T)\Big)}{N^2}+(\lambda_{10}-\lambda_7)\tau_1\frac{F(A+V)}{(F+I)^2}+(\lambda_7-\lambda_{10})c_5\beta_4\frac{F(N-T)}{N^2}-\lambda_7\zeta_3\frac{F(A+V)}{(F+I)^2} \\
& +(\lambda_8-\lambda_{10})c_4\theta_2\frac{PB}{(P+I)^2}+(\lambda_8-\lambda_5)c_1\pi_4\frac{B(N-D)}{N^2}+(\lambda_8-\lambda_9)c_5\omega_1\frac{B(A+V+B)}{I^2}-\lambda_8\zeta_4\frac{B(A+V)}{I^2} \\
& +(\lambda_{10}-\lambda_8)\tau_2\frac{B(A+V)}{I^2}+(\lambda_{10}-\lambda_9)c_5\theta_1\frac{P(P+A+V+B)}{(P+I)^2}+(\lambda_9-\lambda_{10})\tau_3\frac{(A+V)(A+V+B)}{I^2} \\
& +\lambda_9+(\lambda_{10}-\lambda_5)c_1\pi_6\frac{P(N-B)}{N^2}+\lambda_9\zeta_5\frac{(A+V)(A+V+B)}{I^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_{10}}{dt} \quad = \quad & -w_6+(\lambda_{10}-\lambda_1)\gamma_5+(\lambda_5-\lambda_1)c_1\pi_1\frac{DC}{N^2}+(\lambda_5-\lambda_2)c_1\pi_5\frac{DR}{N^2}+(\lambda_5-\lambda_3)c_1\pi_3\frac{DA}{N^2}+(\lambda_5-\lambda_4)c_1\pi_2\frac{DV}{N^2} \\
& +(\lambda_5-\lambda_8)c_1\pi_4\frac{DB}{N^2}+(\lambda_{10}-\lambda_5)c_1\pi_6\frac{D(N-P)}{N^2}+(\lambda_6-\lambda_5)c_2\beta_2\frac{S(E+F+T)}{N^2}+(\lambda_7-\lambda_6)c_3\beta_3\frac{E(F+T)}{N^2} \\
& +(\lambda_9-\lambda_7)c_5\beta_4\frac{TF}{N^2}+(\lambda_{10}-\lambda_8)c_4\theta_2\frac{BI}{(P+I)^2}+(\lambda_{10}-\lambda_9)c_5\theta_1\frac{TI}{(P+I)^2}+\lambda_{10}\mu+\lambda_{10}\eta
\end{aligned}
$$

with transversality conditions

$$\lambda_i(T_f)=0, \quad i=1,2,...,10. \tag{47}$$

Further, the optimal control quintuple $\left(u_1^*,u_2^*,u_3^*,u_4^*,u_5^*\right)$ is given as follows

$$\begin{cases} u_1^* = \max\left\{0, \min\left\{1, \dfrac{\Big((\lambda_5-\lambda_1)\pi_1 C + (\lambda_5-\lambda_2)\pi_5 R + (\lambda_5-\lambda_3)\pi_3 A + (\lambda_5-\lambda_4)\pi_2 V + (\lambda_5-\lambda_8)\pi_4 B + (\lambda_5-\lambda_{10})\pi_6 P\Big) D}{w_7 N}\right\}\right\} \\[2ex] u_2^* = \max\left\{0, \min\left\{1, \dfrac{(\lambda_6-\lambda_5)\beta_2 S(E+F+T)}{w_8 N}\right\}\right\} \\[2ex] u_3^* = \max\left\{0, \min\left\{1, \dfrac{(\lambda_7-\lambda_6)\beta_3 E(F+T)}{w_9 N}\right\}\right\} \\[2ex] u_4^* = \max\left\{0, \min\left\{1, \dfrac{(\lambda_8-\lambda_1)\alpha_2 \frac{BC}{C+I} + (\lambda_8-\lambda_2)\omega_4 \frac{BR}{R+I} + (\lambda_8-\lambda_3)\nu_3 \frac{BA}{I} + (\lambda_8-\lambda_{10})\theta_2 \frac{BP}{P+I}}{w_{10}}\right\}\right\} \\[2ex] u_5^* = \max\left\{0, \min\left\{1, \dfrac{(\lambda_9-\lambda_2)\omega_3 \frac{TR}{R+I} + (\lambda_9-\lambda_3)\omega_2 \frac{TA}{I} + (\lambda_9-\lambda_7)\beta_4 \frac{TF}{N} + (\lambda_9-\lambda_8)\omega_1 \frac{TB}{I} + (\lambda_9-\lambda_{10})\theta_1 \frac{TP}{P+I}}{w_{11}}\right\}\right\} \end{cases} \tag{48}$$

Proof: As mentioned earlier, the characterization of the optimal solution is obtained by applying the Pontryagin's maximum principle to the Hamiltonian of the system $\mathcal{H}$. The system of ordinary differential equations (46) governing the adjoint variables is derived by differentiating the Hamiltonian. Further, the control characterizations in (48) are derived by solving, on the interior of the control set $\mathcal{U}$, the partial differentials of the Hamiltonian $\mathcal{H}$ with respect to each of the controls u_1, u_2, u_3, u_4 and u_5. Hence, by standard arguments involving control bounds, it follows that:

$$u_1^* = \begin{cases} 0 & \text{if} \quad r_1^* \leq 0 \\ r_1^* & \text{if} \quad 0 < r_1^* < 1 \\ 1 & \text{if} \quad r_1^* \geq 1 \end{cases}$$
$$u_2^* = \begin{cases} 0 & \text{if} \quad r_2^* \leq 0 \\ r_2^* & \text{if} \quad 0 < r_2^* < 1 \\ 1 & \text{if} \quad r_2^* \geq 1 \end{cases}$$
$$u_3^* = \begin{cases} 0 & \text{if} \quad r_3^* \leq 0 \\ r_3^* & \text{if} \quad 0 < r_3^* < 1 \\ 1 & \text{if} \quad r_3^* \geq 1 \end{cases}$$
$$u_4^* = \begin{cases} 0 & \text{if} \quad r_4^* \leq 0 \\ r_4^* & \text{if} \quad 0 < r_1^* < 1 \\ 1 & \text{if} \quad r_4^* \geq 1 \end{cases}$$
$$u_5^* = \begin{cases} 0 & \text{if} \quad r_5^* \leq 0 \\ r_5^* & \text{if} \quad 0 < r_5^* < 1 \\ 1 & \text{if} \quad r_5^* \geq 1 \end{cases}$$

where,

$$\begin{cases} r_1^* &= \dfrac{\Big((\lambda_5-\lambda_1)\pi_1 C + (\lambda_5-\lambda_2)\pi_5 R + (\lambda_5-\lambda_3)\pi_3 A + (\lambda_5-\lambda_4)\pi_2 V + (\lambda_5-\lambda_8)\pi_4 B + (\lambda_5-\lambda_{10})\pi_6 P\Big)D}{w_7 N} \\ r_2^* &= \dfrac{(\lambda_6-\lambda_5)\beta_2 S(E+F+T)}{w_8 N} \\ r_3^* &= \dfrac{(\lambda_7-\lambda_6)\beta_3 E(F+T)}{w_9 N} \\ r_4^* &= \dfrac{(\lambda_8-\lambda_1)\alpha_2\dfrac{BC}{C+I} + (\lambda_8-\lambda_2)\omega_4\dfrac{BR}{R+I} + (\lambda_8-\lambda_3)\nu_3\dfrac{BA}{I} + (\lambda_8-\lambda_{10})\theta_2\dfrac{BP}{P+I}}{w_{10}} \\ r_5^* &= \dfrac{(\lambda_9-\lambda_2)\omega_3\dfrac{TR}{R+I} + (\lambda_9-\lambda_3)\omega_2\dfrac{TA}{I} + (\lambda_9-\lambda_7)\beta_4\dfrac{TF}{N} + (\lambda_9-\lambda_8)\omega_1\dfrac{TB}{I} + (\lambda_9-\lambda_{10})\theta_1\dfrac{TP}{P+I}}{w_{11}} \end{cases}$$

This ends the proof. □

7.3. Numerical simulations of the control system

In this section we perform a numerical simulation to illustrate the impacts of counter-radicalization and counter-terrorism strategies on the evolutionary dynamics of the core sub-population of violent extremism. The results we present were obtained by an implementation on MATLAB using an explicit Euler scheme with a discretization step of 0.02. For the readers convenience, they should remembers that each of individuals without control is marked by red line. The individuals with control are marked by blue line. We use the persistence values for the parameters obtained in Table 1.

In Figure 4 we illustrate the dynamics of the individuals of the classes S, E, F, B and T in the case where the preventive measures and the deradicalization strategies u_1, u_2 and u_3 have a very low effectiveness while that of the fight against terrorism and brigandage have a high effectiveness. In Figure 5 we illustrate the dynamics of individuals in classes S, E, F, B, and T in the case where preventive measures and deradicalization strategies u_1, u_2, and u_3 have very high effectiveness while those of counterterrorism and brigandage u_4 and u_5 have very low effectiveness. In Figure 6 we illustrate the dynamics of individuals in the S, E, F, B, and T classes in the case where preventive measures and deradicalization strategies u_1, u_2 and u_3 have a very high effectiveness as well as those of counter-terrorism and brigandage u_4 and u_5.

Figures 4 and 5 show that preventive measures and deradicalization strategies u_1, u_2, and u_3 have a very large effect in combating radicalization and indoctrination to a fanatical ideology, but even with maximum effectiveness of u_1, u_2, and u_3, terrorism and brigandage will persist if counterterrorism measures u_4 and u_5 are weak . While a high effectiveness of counter-terrorism measures, combining the military-police proximity measures u_4 and u_5, ensures stabilization of all subclasses of the radical core of the subpopulation. By contrast, from Figure 6, we see a faster regression in the S, E, F, B and T compartments compared to Figure 5, when preventive measures and deradicalization strategies (u_1, u_2, u_3), as well as counterterrorism and brigandage combining proximity military-police measures (u_4, u_5) are very high.

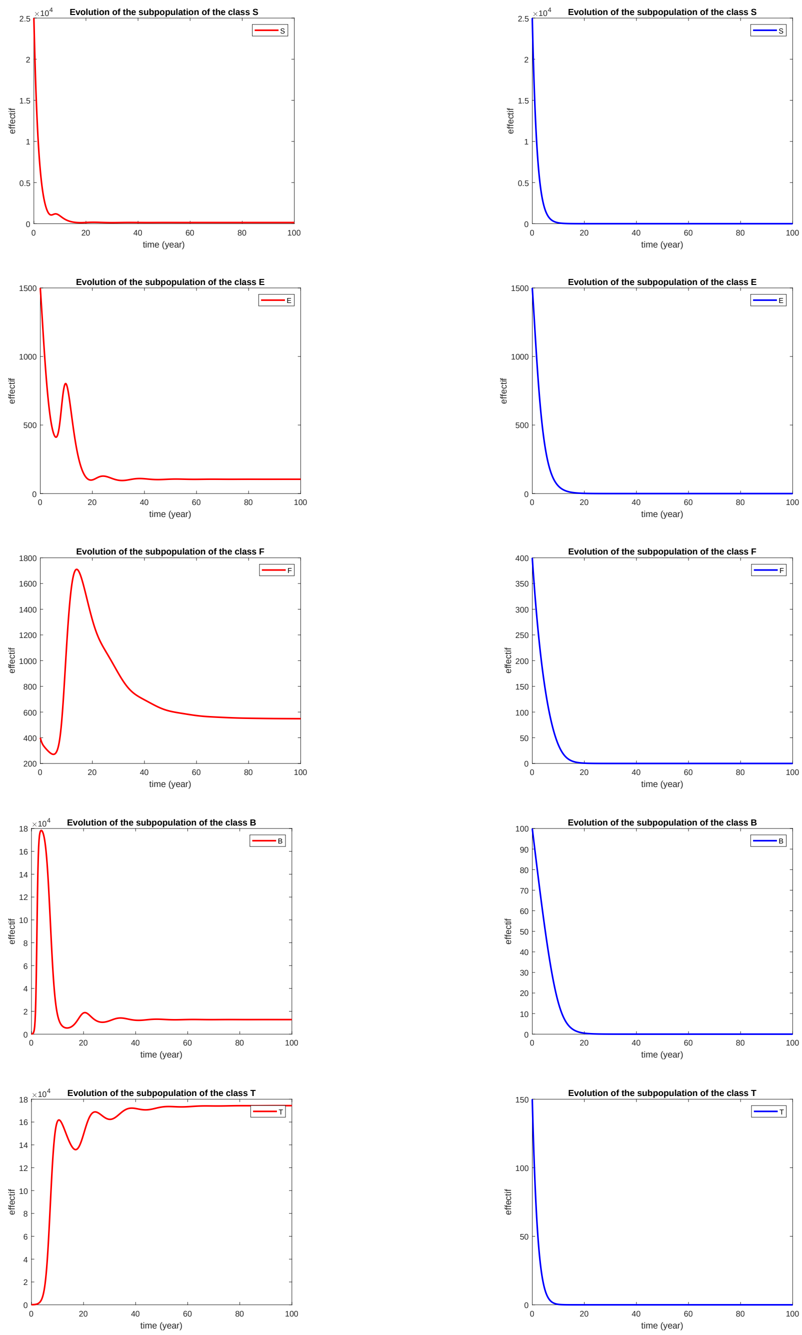

Figure 4: Dynamics of individuals in the S, E, F, B, and T classes with and without control when $u_1 = u_2 = u_3 = 0$ and $u_4 = u_5 = 0.87$.

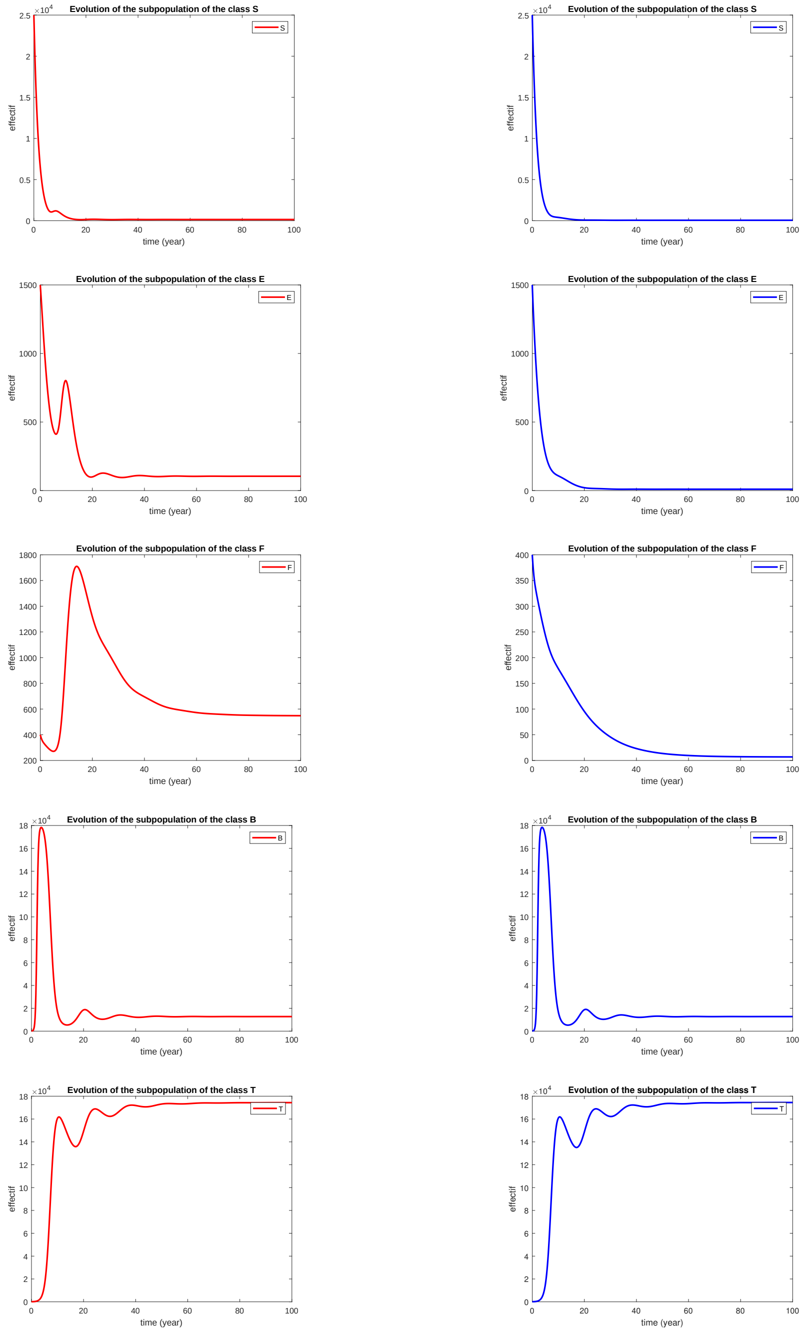

Figure 5: Dynamics of individuals in the S, E, F, B, and T classes with and without control when $u_1 = u_2 = u_3 = 0.87$ and $u_4 = u_5 = 0.13$.

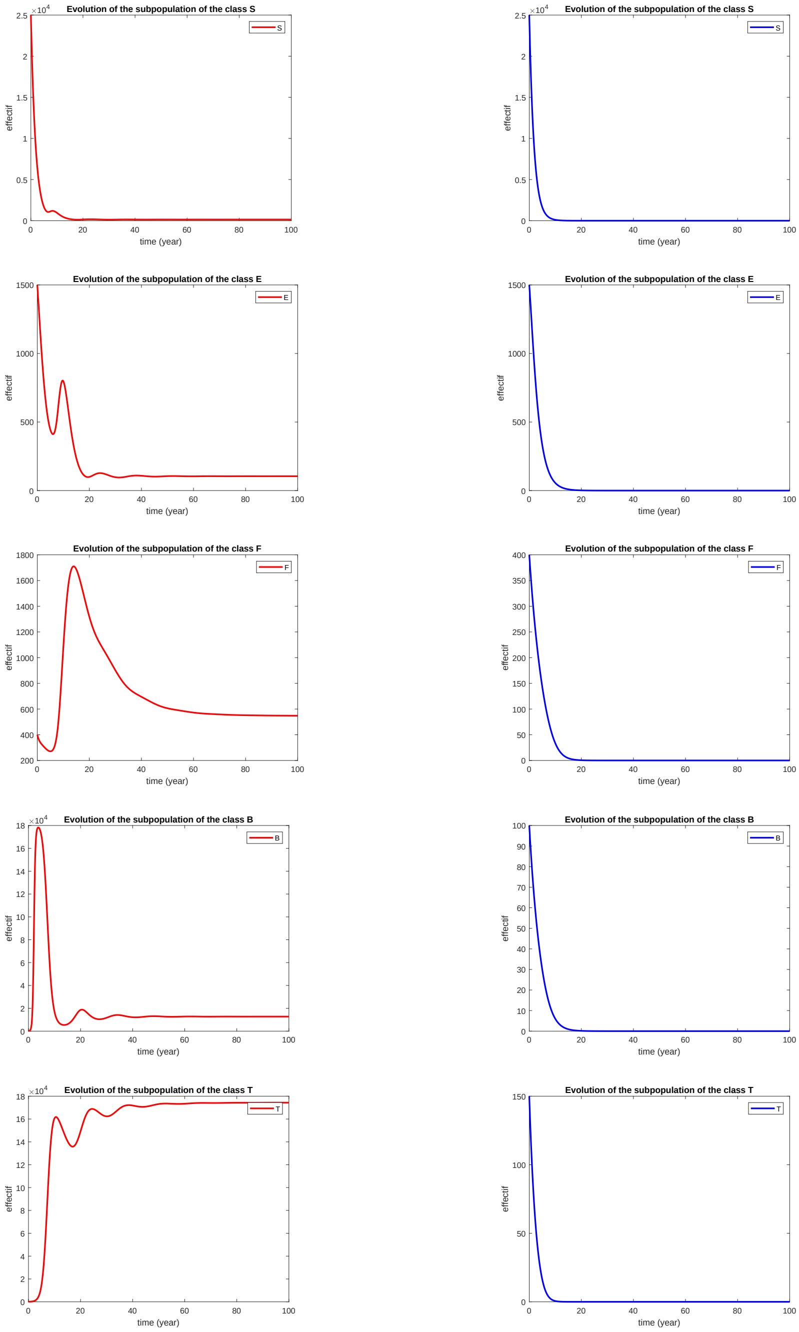

Figure 6: Dynamics of individuals in the S, E, F, B, and T classes with and without control when $u_1 = u_2 = u_3 = 0.87$ and $u_4 = u_5 = 0.87$.

8. Conclusion

In this study, we initially introduced the (1) − (10) model to characterize the dynamics of ideological terrorism, drawing inspiration from the terrorism situation in the Sahel region. This approach enabled us to highlight the influence of local and global security initiatives on the dynamics of ideological terrorism or fanatical insurgency. Within the framework of this model, we defined the threshold $\mathcal{R}_0$, representing the number of fundamental reproductions of extreme behavior, and established the conditions for local and global asymptotic stabilization of the equilibrium devoid of terrorism, brigandage or fanaticism.

In addition, we identified four primordial thresholds $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$, from which we derived conditions leading to the extinction of the radical core subpopulation with respect to violent extremism, ideological terrorism, fanatical ideology, and brigandage without terrorism or fanaticism, respectively. Based on the results of the mathematical analysis of the model, we proposed a control strategy to combat ideological terrorism in the Sahel. We then studied the optimality of this control strategy using mathematical tools such as Pontryagin's maximum principle. Finally, we presented a numerical simulation to complement our theoretical results. It is pertinent to recognize that validation of the model's predictions relies on available data, a challenge frequently encountered. The model predictions are straightforward and based on a theory of the propagation of extreme ideologies [9], as a function of initial conditions. Overall, these results improve our understanding of class evolution by elucidating how specific parameter values can lead either to the stabilization or elimination of extremism, or to its persistence.

To enhance this research, it is worth exploring new mathematical perspectives. This includes examining the influence of social media on the spread of ideological extremism, analyzing the multilevel dynamics of extremism, and conducting social network analysis. By delving deeper into these areas, we can gain a better understanding of the complexity of ideological extremism and develop more effective control strategies. As part of our ongoing project, we are studying the spatio-temporal diffusion of ideological terrorism in the Sahel. We will explore new methods of numerical resolution, drawing on the work [1], [34]. Our aim is to analyze how extremist ideologies spread both spatially and temporally, taking into account factors such as social networks, geographical features, and political climates.

Acknowledgements

The authors would like to thank the The World Bank and the Higher Education Support Project in Burkina Faso for funding this study.

References

[1] Emad Az-Zo'bi. A reliable analytic study for higher-dimensional telegraph equation. *J. Math. Comput. Sci*, 18(4):423–429, 2018.

[2] Alethea BT Barbaro, Lincoln Chayes, and Maria R D'Orsogna. Territorial developments based on graffiti: A statistical mechanics approach. *Physica A: Statistical Mechanics and its Applications*, 392(1):252–270, 2013.

[3] Bijan Berenji, Tom Chou, and Maria R D'Orsogna. Recidivism and rehabilitation of criminal offenders: A carrot and stick evolutionary game. *PloS one*, 9(1):e85531, 2014.

[4] A Berman and RJ Plemmons. Nonnegative matrices in the mathematical sciences, ser. *Classics in applied mathmatics. Philadelphia: SIAM*, 1994.

[5] Jean-Michel Bony. Principe du maximum, inégalité de harnack et unicité du probleme de cauchy pour les opérateurs elliptiques dégénérés. In *Annales de l'institut Fourier*, pages 277–304, 1969.

[6] Robert Boyd and Peter J Richerson. *Culture and the evolutionary process.* University of Chicago press, 1988.

[7] Mario Bunge. Four models of human migration: An exercise in mathematical sociology. *GEN. SYSTEMS*, 16, 1971.

[8] Jose Cadena, Gizem Korkmaz, Chris J Kuhlman, Achla Marathe, Naren Ramakrishnan, and Anil Vullikanti. Forecasting social unrest using activity cascades. *PloS one*, 10(6):e0128879, 2015.

[9] Carlos Castillo-Chavez and Baojun Song. Models for the transmission dynamics of fanatic behaviors. In *Bioterrorism: mathematical modeling applications in homeland security*, pages 155–172. SIAM, 2003.

[10] L Cavalli-Sforza, M Feldman, S Dornbusch, and K-H Chen. Cultural evolution: Anthropology and cultural transmission. *Nature*, 304(5922):124–124, 1983.

[11] Luigi Luca Cavalli-Sforza and Marcus W Feldman. *Cultural transmission and evolution: A quantitative approach.* Princeton University Press, 1981.

[12] Yao-Li Chuang, Tom Chou, and Maria R D'Orsogna. A network model of immigration: Enclave formation vs. cultural integration. *arXiv preprint arXiv:1901.09396*, 2019.

[13] Lawrence E Cohen. Modeling crime trends: A criminal opportunity perspective. *Journal of Research in Crime and Delinquency*, 18(1):138–164, 1981.

[14] DC Dennett. Darwin's dangerous idea simon & schuster. *New York*, 1995.

[15] Maria R D'Orsogna, Ryan Kendall, Michael McBride, and Martin B Short. Criminal defectors lead to the emergence of cooperation in an experimental, adversarial game. *PloS one*, 8(4):e61458, 2013.

[16] Maria R D'Orsogna and Matjaž Perc. Statistical physics of crime: A review. *Physics of life reviews*, 12:1–21, 2015.

[17] Joshua M Epstein. Modeling civil violence: An agent-based computational approach. *Proceedings of the National Academy of Sciences*, 99(suppl_3):7243–7250, 2002.

[18] Marcus W Feldman. *Mathematical evolutionary theory.* Princeton University Press, 1989.

[19] Wendell H Fleming and Raymond W Rishel. *Deterministic and stochastic optimal control*, volume 1. Springer Science & Business Media, 2012.

[20] Babak Fotouhi and Michael G Rabbat. Migration in a small world: A network approach to modeling immigration processes. In *2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 136–143. IEEE, 2012.

[21] Karl P Hadeler and Carlos Castillo-Chávez. A core group model for disease transmission. *Mathematical biosciences*, 128(1-2):41–55, 1995.

[22] Rachel A Hegemann, Laura M Smith, Alethea BT Barbaro, Andrea L Bertozzi, Shannon E Reid, and George E Tita. Geographical influences of an emerging network of gang rivalries. *Physica A: Statistical Mechanics and its Applications*, 390(21-22):3894–3914, 2011.

[23] Herbert W Hethcote and James A Yorke. *Gonorrhea transmission dynamics and control*, volume 56. Springer, 2014.

[24] Mark I Lichbach. Nobody cites nobody else: mathematical models of domestic political conflict. *Defence and Peace Economics*, 3(4):341–357, 1992.

[25] Dahlard L Lukes. Differential equations: classical to controlled. 1982.

[26] Charles J Lumsden and Edward O Wilson. *Genes, mind, and culture-the coevolutionary process*. World Scientific, 2005.

[27] Charles Z Marshak, M Puck Rombach, Andrea L Bertozzi, and Maria R D'Orsogna. Growth and containment of a hierarchical criminal network. *Physical Review E*, 93(2):022308, 2016.

[28] Jie Pan and Anna Nagurney. Using markov chains to model human migration in a network equilibrium framework. *Mathematical and computer modelling*, 19(11):31–39, 1994.

[29] Lev Semenovich Pontryagin. *Mathematical theory of optimal processes*. CRC press, 1987.

[30] Laura M Smith, Andrea L Bertozzi, P Jeffrey Brantingham, George E Tita, and Matthew Valasik. Adaptation of an ecological territorial model to street gang spatial patterns in los angeles. *Discrete and Continuous Dynamical Systems*, 32(9):3223–3244, 2012.

[31] Baojun Song. *Dynamical epidemic models and their applications*. Cornell University, 2002.

[32] Richard S Varga. Matrix iterative analysis. prentice hall, englewood cliffs. *New Jersey*, 1962.

[33] RS Varga. Factorization and normalized iterative methods, boundary problems in differential equations (r. langer, ed.), 1960.

[34] Hamzeh Zureigat, Mohammad A Tashtoush, Ali F Al Jassar, Emad A Az-Zo'bi, Mohammad W Alomari, et al. A solution of the complex fuzzy heat equation in terms of complex dirichlet conditions using a modified crank–nicolson method. *Advances in Mathematical Physics*, 2023, 2023.