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Normal Paradistributive Latticoids

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Abstract. For any filter $\mathcal{P}$ of a paradistributive latticoid, $\mathcal{O}(\mathcal{P})$ is defined and it is proved that $\mathcal{O}(\mathcal{P})$ is a filter if $\mathcal{P}$ is prime. It is also proved that each minimal prime filter belonging to $\mathcal{O}(\mathcal{P})$ is contained in $\mathcal{P}$, and $\mathcal{O}(\mathcal{P})$ is the intersection of all the minimal prime filters contained in $\mathcal{P}$. The concept of a normal paradistributive latticoid is introduced and characterized in terms of the prime filters and minimal prime filters. We proved that every relatively complemented paradistributive latticoid is normal.

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Key Words and Phrases: Paradistributive Latticoid(PDL), Minimal element, Filter.

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1. Introduction

In 1972, Cornish[3] coined the term "normal lattices" for distributive lattices with 0, where each prime ideal contains a unique minimal prime ideal. He established that a distributive lattice A with 0 is normal iff for all $x, y \in A$, $x \wedge y = 0$ implies $x^{\perp}$ and $y^{\perp}$ are comaximal. Pawar[4] characterized normal lattices using the properties of Stone space of prime filters and the Stone space of maximal filters of a bounded distributive lattice. In 1977, Pawar and Thakare[5] introduced pm-lattices as bounded distributive lattices where each prime ideal is uniquely contained within a maximal ideal. In 1980, Simmons[9] demonstrated the equivalence between pm-lattices and normal lattices, revealing that a bounded distributive lattice is a pm-lattice precisely when it is normal. In pm-lattices,

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the map from prime ideals to their unique maximal ideals is continuous, serving as the exclusive retraction from prime to maximal spectrum. Moreover, the pm-property signifies prime spectrum normality and ensures the maximal spectrum satisfies the T_2 separation axiom. In 2012, Borumand Saeid and Mohtashamnia[8] introduced the notion of (right and left) stabilizer in residuated lattices and proved some theorems which gives the relationship between this notion and all types of filters in residuated lattices. After that they constructed quotient of residuated lattices via stabilizer and studied its properties. Rasouli and Dehghani[6] investigated the notion of an mp-residuated lattice, and extracted their topological characterizations. Rasouli and Kondo[7], introduced and investigated the notion of n-normal residuated lattice, as a subclass of residuated lattices in which every prime filter contains at most n minimal prime filters. Bandaru et al.[2], generalized the the concept of distributive lattice and introduced the concept of Paradistributive Latticoid(PDL) and investigated its properties. They also obtained subdirect representation of a paradistributive latticoid. Recently, Bandaru et al.[1], introduced the concept of a parapseudo-complementation in a paradistributive latticoid(PDL) and investigated its elementary properties. Additionally, authors established necessary conditions for a PDL with a minimal element to be parapseudo-complemented and explored the properties required for parapseudo-complementation to be equationally definable. Moreover, they established a one-to-one correspondence between the set of all minimal elements and the set of all parapseudo-complementations

In this paper, we introduce the concept of annihilator filters and study their properties. We define the set $\mathcal{O}(\mathcal{P})$ in a PDL and derive that $\mathcal{O}(\mathcal{P})$ is intersection of all minimal primr filters of V contained in $\mathcal{P}$, in case of $\mathcal{P}$ is a prime filter. Also, observe the relations between prime ideals and prime filters. We also derive that, every maximal ideal is prime ideal and converse holds true if V is relatively complemented PDL. Also, we introduce the concept of normal PDL in the terms of prime filters and characterize it, in element wise and its annihilators.

2. Preliminaries

First we recall the necessary definitions and results from [2].

Definition 1. *An algebra $(V, \vee, \wedge, 1)$ of type (2,2,0) is called a Paradistributive Latticoid, abbreviated as PDL, if it assures the subsequent axioms:*
$(LD\vee)$ $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$,
$(RD\vee)$ $(p \wedge q) \vee r = (p \vee r) \wedge (q \vee r)$,
(L_1) $(p \vee q) \wedge q = q$,
(L_2) $(p \vee q) \wedge p = p$,
(L_3) $p \vee (p \wedge q) = p$,
(I_1) $p \vee 1 = 1$,
for any $p, q, r \in V$.

For any $p, q \in V$, we say that p is less than or equal to q and write $p \leq q$ if $p \wedge q = p$ or equivalently $p \vee q = q$ and it can be easily observed that $\leq$ is a partial order on V. The element 1, in Definition 1, is called the greatest element.

Example 1. *Let V be a non-empty set. Fix some element $y_0 \in V$. Then, for any $x, y \in V$ define $\vee$ and $\wedge$ on V by*

$$x \vee y = \begin{cases} x & y \neq y_0 \\ y_0 & y = y_0 \end{cases}$$

and

$$x \wedge y = \begin{cases} y & y \neq y_0 \\ x & y = y_0 \end{cases}$$

Then $(V, \vee, \wedge, y_0)$ is a disconnected PDL with y_0 as its greatest element.

Lemma 1. *Let $(V, \vee, \wedge, 1)$ be a PDL. Then for any $p, q, r, s \in V$, we have the following:*
(1) $1 \wedge p = p$,
(2) $p \wedge 1 = p$,
(3) $1 \vee p = 1$,
(4) $(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$,
(5) $p \vee (q \wedge r) = p \vee (r \wedge q)$,
(6) *the operation $\vee$ is associative in V i.e., $p \vee (q \vee r) = (p \vee q) \vee r$,*
(7) *the set $V_a = \{p \in V \mid a \leq p\} = \{a \vee p \mid p \in V\}$ is a distributive lattice under induced operations $\vee$ and $\wedge$ with a as its least element,*
(8) $s \vee \{p \wedge (q \wedge r)\} = s \vee \{(p \wedge q) \wedge r\}$,
(9) $p \vee (q \vee r) = p \vee (r \vee q)$,
(10) $p \vee q = 1$ *if and only if* $q \vee p = 1$,
(11) $p \wedge q = q \wedge p$ *whenever* $p \vee q = 1$.

Theorem 1. *An algebra $(V, \vee, \wedge, 1)$ of type $(2, 2, 0)$ is a PDL if and only if it satisfies the following:*
$(LD\vee)$ $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$,
$(RD\vee)$ $(p \wedge q) \vee r = (p \vee r) \wedge (q \vee r)$,
$(RD\wedge)$ $(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$,
(L_1) $(p \vee q) \wedge q = q$,
(L_3) $p \vee (p \wedge q) = p$,
(I_1) $p \vee 1 = 1$,
(I_2) $1 \wedge p = p$,
for all $p, q, r \in V$.

Definition 2. *A Paradistributive Latticoid $(V, \vee, \wedge, 1)$ is said to be associative if it satisfies the following condition*

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r$$

for all $p, q, r \in V$.

Definition 3. *Let V be a PDL. Then, an element $a \in V$ is said to be a minimal element if for any $u \in V$, $u \leq a \Rightarrow u = a$.*

Lemma 2. *Let V be a PDL. Then, for any $a \in V$, the following are equivalent:*
(1) *a is minimal,*
(2) *$p \wedge a = a$ for all $p \in V$,*
(3) *$p \vee a = p$ for all $p \in V$.*

Definition 4. *A non-empty subset $\mathcal{F}$ of a PDL V is said to be a filter if it satisfies the following:*

$$p, q \in \mathcal{F} \Rightarrow p \wedge q \in \mathcal{F},$$
$$p \in \mathcal{F}, a \in V \Rightarrow a \vee p \in \mathcal{F}.$$

Theorem 2. *Let $\mathcal{S}$ be a non-empty subset of V. Then*

$$[\mathcal{S}) \;=\; \{p \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in \mathcal{S}, p \in V, 1 \leq i \leq n \text{ and } n \text{ is a positive integer } \}$$

is the smallest filter of V containing $\mathcal{S}$.

Lemma 3. *Let V be a PDL and $\mathcal{F}$ be a filter of V. Then for any $p, q \in V$, we have the following:*
(1) *$[p) = \{x \vee p \mid x \in V\}$,*
(2) *$p \in [q)$ if and only if $p = p \vee q$ for all $p, q \in V$,*
(3) *$p \vee q \in \mathcal{F}$ if and only if $q \vee p \in \mathcal{F}$,*
(4) *$[p \vee q) = [q \vee p)$,*
(5) *$[p \wedge q) = [q \wedge p) = [p) \vee [q)$.*

Theorem 3. *The collection $\mathcal{F}(V)$ of all filters of a PDL V forms a distributive lattice under set inclusion, in which, the glb and lub of any tow filters $\mathcal{F}$ and $\mathcal{G}$ are given by $\mathcal{F} \wedge \mathcal{G} = \mathcal{F} \cap \mathcal{G}$ and $\mathcal{F} \vee \mathcal{G} = \{p \wedge q \mid p \in \mathcal{F}$ and $q \in \mathcal{G}\}$, respectively.*

Definition 5. *A non-empty subset $\mathcal{I}$ of a PDL V is said to be an ideal if it satisfies the following:*

$$p, q \in \mathcal{I} \Rightarrow p \vee q \in \mathcal{I},$$
$$p \in \mathcal{I}, a \in V \Rightarrow p \wedge a \in \mathcal{I}.$$

Theorem 4. *Let $\mathcal{S}$ be a non-empty subset of V. Then*

$$(\mathcal{S}] \;=\; \{(\bigvee_{i=1}^{n} s_i) \wedge p \mid s_i \in \mathcal{S}, p \in V, 1 \leq i \leq n \text{ and } n \text{ is a positive integer } \}$$

is the smallest ideal of V containing $\mathcal{S}$.

Lemma 4. *Let V be a PDL and $\mathcal{I}$ be an ideal of V. Then, for any $p, q \in V$, we have the following:*
(1) *$(p] = \{p \wedge x \mid x \in V\}$,*
(2) *$p \in (q]$ if and only if $p = q \wedge p$,*
(3) *$p \wedge q \in \mathcal{I}$ if and only if $q \wedge p \in \mathcal{I}$,*
(4) *$(p \wedge q] = (q \wedge p]$,*
(5) *$(p \wedge q] = (q \wedge p] = (p] \wedge (q]$.*

Theorem 5. *The collection $\mathcal{I}(V)$ of all ideals of a PDL V forms a distributive lattice under set inclusion, in which, the glb and lub of any two ideals $\mathcal{I}$ and $\mathcal{J}$ are given by $\mathcal{I} \wedge \mathcal{J} = \mathcal{I} \cap \mathcal{J}$ and $\mathcal{I} \vee \mathcal{J} = \{p \vee q \mid p \in \mathcal{I}$ and $q \in \mathcal{J}\}$, respectively.*

A proper filter(ideal) $\mathcal{P}$ of V is said to be a prime filter(ideal) if for any $x, y \in V$, $x \vee y \in \mathcal{P}(x \wedge y \in \mathcal{P}) \Rightarrow x \in \mathcal{P}$ or $y \in \mathcal{P}$. A proper filter(ideal) $\mathcal{M}$ of V is said to be maximal if it is not properly contained in any proper filter(ideal) of V. A prime filter $\mathcal{P}$ of V is said to be minimal, if it is minimal among all the prime filters of V. A prime filter $\mathcal{P}$ is said to be a minimal prime filter belonging to a filter $\mathcal{I}$, if it is minimal among all the prime filters of V containing $\mathcal{I}$. A prime filer $\mathcal{P}$ of V is a minimal prime filter if and only if for each $x \in \mathcal{P}$, there exists $y \notin \mathcal{P}$ such that $x \vee y = 1$.

Definition 6. *By a homomorphism of a PDL $(V, \vee, \wedge, 1)$ into a PDL $(V', \vee', \wedge', 1')$, we mean, a mapping $f : V \to V'$ satisfying the following:*
(1) $f(a \vee b) = f(a) \vee' f(b)$,
(2) $f(a \wedge b) = f(a) \wedge' f(b)$,
(3) $f(1) = f(1')$.

3. Normal PDL

In this section, we introduce the notion of a normal paradistributive latticoid and characterize interms of prime filters and minimal prime filters. Throughout this section, a PDL V means a paradistributive latticoid $(V, \vee, \wedge, 1)$ with minimal elements. First, we give the following:

Definition 7. *For any non-empty subset $\mathcal{S}$ of a PDL V, write $(\mathcal{S})^\bullet = \{a \in V | s \vee a = 1 \text{ for all } s \in \mathcal{S}\}$. Then $(\mathcal{S})^\bullet$ is a filter of V, and is called the annihilator of $\mathcal{S}$ in V. If $\mathcal{S} = \{s\}$, we write $(s)^\bullet$ for $(\{s\})^\bullet$.*

The following lemma can be verified routinely.

Lemma 5. *For any $a, b \in V$,*

(1) $a \leq b \implies (a)^\bullet \subseteq (b)^\bullet$,

(2) $(a \vee b)^\bullet = (b \vee a)^\bullet$,

(3) $(a \wedge b)^\bullet = (b \wedge a)^\bullet$,

(4) $(a \wedge b)^\bullet = (a)^\bullet \cap (b)^\bullet$,

(5) $(a)^\bullet \vee (b)^\bullet \subseteq (a \vee b)^\bullet$,

(6) $a \in (x)^\bullet \implies (x)^{\bullet\bullet} \subseteq (a)^\bullet$,

(7) $a \in [b) \implies (b)^\bullet \subseteq (a)^\bullet$,

(8) $[a) \subseteq [b) \implies (b)^\bullet \subseteq (a)^\bullet$.

Definition 8. *Let $\mathcal{S}$ be a subset of a PDL V. Then we define*

$$\mathcal{O}(\mathcal{S}) = \{x \in V | a \vee x = 1 \text{ for some } a \in V \backslash \mathcal{S}\}.$$

Observe that for any non-empty subset $\mathcal{S}$ of V,

$$\mathcal{O}(\mathcal{S}) = \bigcup_{a \in V \backslash \mathcal{S}} (a)^{\bullet}.$$

The following is an important characterization of $\mathcal{O}(\mathcal{S})$.

Lemma 6. *If $\mathcal{S}$ is any non-empty subset of a PDL V and $x \in V$, then $x \in \mathcal{O}(\mathcal{S})$ if and only if $(x)^{\bullet} \nsubseteq \mathcal{S}$.*

Proof. Let $\mathcal{S}$ be any non-empty subset of a PDL V and $x \in V$. Now $x \in \mathcal{O}(S)$ implies that $y \vee x = 1$ for some $y \notin \mathcal{S}$. Hence $y \in (x)^{\bullet}$ for some $y \notin \mathcal{S}$. Therefore $(x)^{\bullet} \nsubseteq \mathcal{S}$. Conversely, $(x)^{\bullet} \nsubseteq \mathcal{S}$ implies that there is $y \in (x)^{\bullet}$ such that $y \notin \mathcal{S}$. Hence $y \vee x = 1$ and $y \notin \mathcal{S}$. Therefore, $x \in \mathcal{O}(\mathcal{S})$.

The following lemma can be proved easily.

Lemma 7. *For any prime filter $\mathcal{P}$ of a PDL V, $\mathcal{O}(\mathcal{P})$ is a filter of V and $\mathcal{O}(\mathcal{P}) \subseteq \mathcal{P}$.*

Lemma 8. *If $\mathcal{P}$ is a prime filter of a PDL V, then each minimal prime filter belonging to $\mathcal{O}(\mathcal{P})$ is contained in $\mathcal{P}$.*

Proof. Let $\mathcal{P}$ be a prime filter of a PDL V and $\mathcal{Q}$ be any minimal prime filter belonging to $\mathcal{O}(\mathcal{P})$. We have to prove that $\mathcal{Q} \subseteq \mathcal{P}$. Suppose that $\mathcal{Q} \nsubseteq \mathcal{P}$. Then there exists an element $x \in \mathcal{Q}$ such that $x \notin \mathcal{P}$. Since $\mathcal{Q}$ is a minimal prime filter, there is an element $y \notin \mathcal{Q}$ such that $x \vee y = 1$ and hence $y \in \mathcal{O}(\mathcal{P})$. Since $\mathcal{O}(\mathcal{P}) \subseteq \mathcal{Q}$, we get $y \in \mathcal{Q}$. This is a contradiction. Therefore, we get $\mathcal{Q} \subseteq \mathcal{P}$. Thus each minimal prime filter belonging to $\mathcal{O}(\mathcal{P})$ is contained in $\mathcal{P}$.

Lemma 9. *Let $\mathcal{P}$ be a prime filter of a PDL V. Then*

$$x \in V \backslash \mathcal{P} \implies (x)^{\bullet} \subseteq \mathcal{P}.$$

Corollary 1. *Let $\mathcal{P}$ be a prime filter of a PDL V. Then*

$$x \in V \backslash \mathcal{O}(\mathcal{P}) \text{ if and only if } (x)^{\bullet} \subseteq \mathcal{P}.$$

Lemma 10. *If V is a PDL then every proper filter of V is contained in a maximal filter.*

Proof. Let $\mathcal{F}$ be a proper filter of a PDL V and $\mathcal{S} = \{\mathcal{G} | \mathcal{G} \text{ is a filter of } V \text{ containing } \mathcal{F}\}$. clearly $\mathcal{F} \in \mathcal{S}$. Therefore $\mathcal{S} \neq \emptyset$. Let $\{\mathcal{G}_\alpha | \alpha \in \Delta\}$ be a chain in $\mathcal{S}$. Now we prove that $\bigcup_{\alpha \in \Delta} \mathcal{G}_\alpha$ is an upper bound of $\{\mathcal{G}_\alpha | \alpha \in \Delta\}$. Let $x, y \in \bigcup_{\alpha \in \Delta} \mathcal{G}_\alpha$. Then $x \in \mathcal{G}_i$ and $y \in \mathcal{G}_j$ for some $i, j \in \Delta$. Since $\{\mathcal{G}_\alpha | \alpha \in \Delta\}$ is a chain, take $\mathcal{G}_i \subseteq \mathcal{G}_j$. Therefore $x, y \in \mathcal{G}_j$. Since $\mathcal{G}_j$ is a filter of V, $x \vee y \in \mathcal{G}_j \subseteq \bigcup_{\alpha \in \Delta} \mathcal{G}_\alpha$. Again, let $x \in \bigcup_{\alpha \in \Delta} \mathcal{G}_\alpha$ and $r \in V$. Then $x \in \mathcal{G}_\alpha$ for some

$\alpha \in \Delta$. Therefore, $x \wedge r \in \mathcal{G}_\alpha \subseteq \bigcup_{\alpha\in\Delta} \mathcal{G}_\alpha$. Since $\mathcal{F} \subseteq \mathcal{G}_\alpha$ for each α, we get $\mathcal{F} \subseteq \bigcup_{\alpha\in\Delta} \mathcal{G}_\alpha$. Therefore $\cup_{\alpha\in\Delta} \mathcal{G}_\alpha$ is an upper bound of $\{\mathcal{G}_\alpha | \alpha \in \Delta\}$ in $\mathcal{S}$. Thus by Zorn's Lemma $\mathcal{S}$ has a maximal element.

In the lattice theory, due to lattice theoretic duality principle, those results which are valid for filters are also valid for ideals. But the dual of a PDL need not be a PDL. For this reason, it is necessary to provide proofs for similar results on ideals.

Lemma 11. *A subset $\mathcal{P}$ of a PDL V is a prime filter if and only if $V \backslash \mathcal{P}$ is a prime ideal.*

Proof. Let V be a PDL and $\mathcal{P} \subseteq V$. Assume that $\mathcal{P}$ is a prime filter of V. We have to prove that $V\backslash\mathcal{P}$ is a prime ideal of V. Let $x, y \in V\backslash\mathcal{P}$. Then $x \notin \mathcal{P}$ and $y \notin \mathcal{P}$. Since $\mathcal{P}$ is prime, we get $x \vee y \notin \mathcal{P}$. Therefore, $x \vee y \in V\backslash\mathcal{P}$. Again, let $a \in V$ and $x \in V\backslash\mathcal{P}$. If $x \wedge a \in \mathcal{P}$ then $x \vee (x \wedge a) \in \mathcal{P}$ and hence $x \in \mathcal{P}$. But $x \notin \mathcal{P}$. Therefore, $x \wedge a \notin \mathcal{P}$ and hence $a \wedge x \in V\backslash\mathcal{P}$. Thus $V\backslash\mathcal{P}$ is an ideal. Now we prove that $V\backslash\mathcal{P}$ is a prime ideal. Let $x, y \in V$ such that $x \wedge y \in V\backslash\mathcal{P}$. Suppose that $x \notin V\backslash\mathcal{P}$ and $y \notin V\backslash\mathcal{P}$. Then $x \in \mathcal{P}$ and $y \in \mathcal{P}$. Since $\mathcal{P}$ is a filter, $x \wedge y \in \mathcal{P}$ and hence $x \wedge y \notin V\backslash\mathcal{P}$. This is a contradiction. Therefore, we get either $x \in V\backslash\mathcal{P}$ or $y \in V\backslash\mathcal{P}$. Hence $V\backslash\mathcal{P}$ is a prime ideal of V.
Conversely, assume $V\backslash\mathcal{P}$ is a prime ideal of V. First we prove that $\mathcal{P}$ is a filter of V. Let $x, y \in \mathcal{P}$. Then $x \notin V\backslash\mathcal{P}$ and $y \notin V\backslash\mathcal{P}$. Since $V\backslash\mathcal{P}$ is a prime ideal, $x \wedge y \notin V\backslash\mathcal{P}$. Therefore, $x \wedge y \in \mathcal{P}$. Again, let $a \in V$ and $x \in \mathcal{P}$. Suppose $a \vee x \in V\backslash\mathcal{P}$. Then $(a \vee x) \wedge x \in V\backslash\mathcal{P}$ and hence $x \in V\backslash\mathcal{P}$. But $x \notin V\backslash\mathcal{P}$. Therefore, $a \vee x \notin V\backslash\mathcal{P}$. Thus $a \vee x \in \mathcal{P}$. Therefore, $\mathcal{P}$ is a filter of V. Now, we prove that $\mathcal{P}$ is a prime filter of V. Let $x \vee y \in \mathcal{P}$. Suppose $x \notin \mathcal{P}$ and $y \notin \mathcal{P}$. Then $x \in V\backslash\mathcal{P}$ and $y \in V\backslash\mathcal{P}$. Since $V\backslash\mathcal{P}$ is an ideal, $x \vee y \in V\backslash\mathcal{P}$ and hence $x \vee y \notin \mathcal{P}$. This is a contradiction. Thus we get either $x \in \mathcal{P}$ or $y \in \mathcal{P}$. Therefore $\mathcal{P}$ is a prime filter.

Theorem 6. *In a PDL, every maximal ideal is a prime ideal.*

Proof. Let V be a PDL and $\mathcal{G}$ be a maximal ideal of V. We have to prove that $\mathcal{G}$ is a prime ideal of V. Let $x, y \in V$ and $x \wedge y \in \mathcal{G}$. We have to prove that $x \in \mathcal{G}$ or $y \in \mathcal{G}$. Suppose $x \notin \mathcal{G}$, $y \notin \mathcal{G}$, we get $\mathcal{G} \vee (x] = V$ and $\mathcal{G} \vee (y] = V$. Now

$$\begin{aligned} 1 \in V &\implies 1 = s_1 \vee t \text{ for some } s_1 \in \mathcal{G} \text{ and } t \in (x] \\ &\implies 1 = s_1 \vee (x \wedge t) \\ &\implies 1 = (s_1 \vee x) \wedge (s_1 \vee t) \\ &\implies 1 = s_1 \vee x. \end{aligned}$$

Similarly, we can find some $s_2 \in \mathcal{G}$ such that $1 = s_2 \vee y$. Now

$$(x \wedge y) \vee s_1 \vee s_2 = (x \vee s_1 \vee s_2) \wedge (y \vee s_1 \vee s_2) = 1.$$

Since, $s_1, s_2 \in \mathcal{G}$ and $x \wedge y \in \mathcal{G}$, we get $(x \wedge y) \vee s_1 \vee s_2 \in \mathcal{G}$ and hence $1 \in \mathcal{G}$. This is a contradiction. Therefore $x \in \mathcal{G}$ or $y \in \mathcal{G}$. Hence $\mathcal{G}$ is a prime ideal of V.

Theorem 7. *In a relatively complemented PDL V, every prime ideal of V is a maximal ideal.*

Proof. Let V be relatively complemented PDL and $\mathcal{F}$ be any prime ideal of V. Let $\mathcal{G}$ be an ideal of V such that $\mathcal{F} \subset \mathcal{G}$. Since $\mathcal{F} \neq \mathcal{G}$, there exists an element $x \in \mathcal{G}$ such that $x \notin \mathcal{F}$. Now, chose $t \in \mathcal{F}$ and consider the interval $\mathcal{I} = [t \wedge x, 1]$. Since V is a relatively complemented PDL, $\mathcal{I}$ is a dually complemented lattice and $x \in \mathcal{I}$. Therefore there exists some $z \in \mathcal{I}$ such that $z \vee x = 1$ and $z \wedge x = t \wedge x$. Since $\mathcal{F}$ is an ideal and $t \in \mathcal{F}$, we get $t \wedge x \in \mathcal{F}$ and hence $z \wedge x \in \mathcal{F}$. This gives $z \in \mathcal{F}$ or $x \in \mathcal{F}$. But $x \notin \mathcal{F}$. Therefore we get $z \in \mathcal{F} \subset \mathcal{G}$. Since $x \in \mathcal{G}$ and $z \in \mathcal{G}$, we get $z \vee x \in \mathcal{G}$. This gives $1 \in \mathcal{G}$. Since $\mathcal{G}$ is an ideal and $1 \in \mathcal{G}$, we get $\mathcal{G} = V$. Therefore $\mathcal{F}$ is a maximal ideal of V.

Theorem 8. *Let V be a PDL and $\mathcal{P}$ be a prime filter of V. Then $\mathcal{P}$ is a minimal prime filter of V if and only if $V \backslash \mathcal{P}$ is a maximal prime ideal of V.*

Proof. Assume that $\mathcal{P}$ is a minimal prime filter of V. Since $\mathcal{P}$ is a prime filter of V, $V \backslash \mathcal{P}$ is a prime ideal of V. Now we prove that $V \backslash \mathcal{P}$ is maximal ideal of V. Suppose $V \backslash \mathcal{P}$ is not a maximal ideal of V. Then there is a maximal ideal (prime ideal) $\mathcal{G}$ in V such that $V \backslash \mathcal{P} \subset \mathcal{G} \subseteq V$. This gives $V \backslash \mathcal{G} \subset \mathcal{P}$. That is $V \backslash \mathcal{G}$ is a prime filter of V contained in $\mathcal{P}$. Since $\mathcal{P}$ is minimal prime filter of V, $V \backslash \mathcal{G} \subset \mathcal{P}$ is not possible. Therefore, $V \backslash \mathcal{P}$ is a maximal ideal of V. Conversely, assume that $V \backslash \mathcal{P}$ is a maximal prime ideal of V. Then $\mathcal{P}$ is a prime filter of V. Now suppose $\mathcal{Q}$ is any prime filter of V such that $\mathcal{Q} \subset \mathcal{P}$. Then $V \backslash \mathcal{P} \subset V \backslash \mathcal{Q}$. That is, the prime ideal $V \backslash \mathcal{Q}$ is containing the maximal prime ideal $V \backslash \mathcal{P}$ of V. This is a contradiction. Therefore $\mathcal{P}$ is a minimal prime filter of V.

Theorem 9. *Every prime filter of V contains a minimal prime filter of V.*

Proof. Let $\mathcal{P}$ be a prime filter of V. Then $V \backslash \mathcal{P}$ is a prime ideal of V. Then by Zorn's Lemma, there is a maximal prime ideal $\mathcal{G}$ in V such that $V \backslash \mathcal{P} = \mathcal{F} \subseteq \mathcal{G}$. Thus $V \backslash \mathcal{G} \subseteq V \backslash \mathcal{F} = \mathcal{P}$. Therefore, $V \backslash \mathcal{G}$ is a minimal prime filter of V contained in the prime filter $\mathcal{P}$.

Theorem 10. *If $\mathcal{P}$ is a prime filter in a PDL V, then the filter $\mathcal{O}(\mathcal{P})$ is the intersection of all the minimal prime filters of V contained in $\mathcal{P}$.*

Proof. Let $\mathcal{P}$ be a prime filter of V. Then $\mathcal{P}$ contains atleast one minimal prime filter of V. Let $\{\mathcal{Q}_\alpha | \alpha \in \Delta\}$ be the set of all minimal prime filters of V contained in $\mathcal{P}$. Now, we prove that $\mathcal{O}(\mathcal{P}) = \underset{\alpha \in \Delta}{\cap} \mathcal{Q}_\alpha$. Let $x \in \mathcal{O}(\mathcal{P})$. Then $a \vee x = 1$ for some $a \notin \mathcal{P}$. Now $a \notin \mathcal{P}$ gives that $a \notin \mathcal{Q}_\alpha$, for all $\alpha \in \Delta$. Since $\mathcal{Q}_\alpha$ is a prime filter for all $\alpha \in \Delta$ and $1 = a \vee x \in \mathcal{Q}_\alpha$, we get either $a \in \mathcal{Q}_\alpha$ or $x \in \mathcal{Q}_\alpha$. But $a \notin \mathcal{Q}_\alpha$. Therefore, we get $x \in \mathcal{Q}_\alpha$ for all $\alpha \in \Delta$ and hence $\mathcal{O}(\mathcal{P}) \subseteq \underset{\alpha \in \Delta}{\cap} \mathcal{Q}_\alpha$. Conversely, suppose $x \notin \mathcal{O}(\mathcal{P})$. Since $\mathcal{P}$ is a prime filter of V, $V \backslash \mathcal{P}$ is a prime ideal. Now, consider the ideal $\mathcal{A} = (x] \vee (V \backslash \mathcal{P})$. Since $x \notin \mathcal{O}(\mathcal{P})$, we get $y \vee x \neq 1$ for all $y \in V \backslash \mathcal{P}$. Hence $1 \notin \mathcal{A}$. Therefore, $\mathcal{A}$ is a proper ideal of V. Let $\mathcal{G}$ be any maximal ideal of V containing $\mathcal{A}$. Then $V \backslash \mathcal{G}$ is a minimal prime filter of V. Since $x \in \mathcal{A} \subseteq \mathcal{G}$ and $V \backslash \mathcal{P} \subseteq \mathcal{A} \subseteq \mathcal{G}$, we get $\mathcal{Q} = V \backslash \mathcal{G} \subseteq \mathcal{P}$ and $x \notin \mathcal{Q}$. That is, there is a minimal prime filter $\mathcal{Q}$ of V contained in $\mathcal{P}$ and $x \notin \mathcal{Q}$. Therefore, $x \notin \bigcap_{\alpha\in\Delta} \mathcal{Q}_\alpha$. Thus $\bigcap_{\alpha\in\Delta} \mathcal{Q}_\alpha \subseteq \mathcal{O}(\mathcal{P})$. Hence $\mathcal{O}(\mathcal{P}) = \bigcap_{\alpha\in\Delta} \mathcal{Q}_\alpha$.

If $f : V_1 \to V_2$ is a homomorphism and $\mathcal{I}$ is a filter of V_1 then $f(\mathcal{I})$ need not be a filter of V_2. But $[f(\mathcal{I}))$ is the filter generated by $f(\mathcal{I})$ in V_2. We denote this filter by $\mathcal{I}^e$. That is $\mathcal{I}^e = [f(\mathcal{I}))$. In the following Lemma, we can observe that that $f^{-1}(\mathcal{J})$ is a filter of V_1 if $\mathcal{J}$ is a filter of V_2. We denote this filter by $\mathcal{J}^c = f^{-1}(\mathcal{J})$.

Lemma 12. *Let V_1 and V_2 be two PDLs and $f : V_1 \to V_2$ be a homomorphism. Let $\mathcal{I}, \mathcal{I}_1$ be filters of V_1 and $\mathcal{J}, \mathcal{J}_1$ be filters of V_2. Then*

(1) *If $\mathcal{J}$ is a filter of V_2, then $\mathcal{J}^c = \{x \in V_1 | f(x) \in \mathcal{J} \subseteq V_2\}$ is a filter of V_1.*

(2) $\mathcal{I} \subseteq \mathcal{I}_1 \implies \mathcal{I}^e \subseteq \mathcal{I}_1^e$.

(3) $\mathcal{J} \subseteq \mathcal{J}_1 \implies \mathcal{J}^c \subseteq \mathcal{J}_1^c$.

(4) *For every $\mathcal{I} \in f(V_1)$, $\mathcal{I} \subseteq \mathcal{I}^{e^c}$ and $\mathcal{I} = \mathcal{I}^{e^c}$ if f is a bijection.*

(5) *For any $\mathcal{J} \in f(V_2)$, $\mathcal{J}^{c^e} \subseteq \mathcal{J}$ and $\mathcal{J}^{c^e} = \mathcal{J}$ if f is onto.*

Since the mapping $x \mapsto [x)$ is a homomorphism of the PDL V into the lattice $\mathcal{PF}(V)$ of all principal filters of V, we have the following theorem.

Theorem 11. (1) *For any filter $\mathcal{I}$ of V, $\mathcal{I}^e = \{[a) | a \in \mathcal{I}\}$ is a filter of $\mathcal{PF}(V)$. Moreover, $\mathcal{I}$ is prime if and only if $\mathcal{I}^e$ is prime.*

(2) *For any filter $\mathcal{K}$ of the lattice $\mathcal{PF}(V)$, $\mathcal{K}^c = \{a \in V | [a) \in \mathcal{K}\}$ is a filter of V. Further, $\mathcal{K}$ is prime if and only if $\mathcal{K}^c$ is prime.*

(3) *For any filters $\mathcal{I}_1, \mathcal{I}_2$ of V, $\mathcal{I}_1 \subseteq \mathcal{I}_2$ if and only if $\mathcal{I}_1^e \subseteq \mathcal{I}_2^e$.*

(4) *For any filters $\mathcal{K}_1, \mathcal{K}_2$ of $\mathcal{PF}(V)$, $\mathcal{K}_1 \subseteq \mathcal{K}_2$ if and only if $\mathcal{K}_1^c \subseteq \mathcal{K}_2^c$.*

(5) *$\mathcal{I}^{e^c} = \mathcal{I}$, for all filters $\mathcal{I}$ of V.*

(6) *$\mathcal{K}^{c^e} = \mathcal{K}$, for all filters $\mathcal{K}$ of $\mathcal{PF}(V)$.*

Lemma 13. *Let V be a PDL and $\mathcal{PF}(V)$ is the principle filter lattice of V. Then $\mathcal{P}$ is a minimal prime filter of V if and only if $\mathcal{P}^e$ is a minimal prime filter of $\mathcal{PF}(V)$.*

Proof. Let $\mathcal{P}$ be a minimal prime filter of V. Since $\mathcal{P}$ is a prime filter of V, $\mathcal{P}^e$ is a prime filter of $\mathcal{PF}(V)$. Now we prove that $\mathcal{P}^e$ is a minimal prime filter of $\mathcal{PF}(V)$ such that $\mathcal{Q} \subseteq \mathcal{P}^e$. Then, $\mathcal{Q}^c$ is a prime filter of V. Also $\mathcal{Q}^c \subseteq \mathcal{P}^{e^c} = \mathcal{P}$. But $\mathcal{P}$ is a minimal prime filter of V. Therefore we get $\mathcal{Q}^c = \mathcal{P}$ which gives $\mathcal{Q}^{c^e} = \mathcal{P}^e$ and hence $\mathcal{Q} = \mathcal{P}^e$. Therefore $\mathcal{P}^e$ is a minimal prime filter of V. Similarly, we can prove the converse.

Now, we define the notion of a normal paradistributive latticoid.

Definition 9. *A paradistributive latticoid V is called normal if every prime filter of V contains a unique minimal prime filter of V.*

Example 2. *Let $V = \{0, 1, 2, 3, 4, 5\}$ be a set with binary operations $\vee$ and $\wedge$ given in the following tables:*

$\vee$	0	1	2	3	4	5
0	0	1	1	0	0	0
1	1	1	1	1	1	1
2	1	1	2	2	1	2
3	0	1	2	3	0	3
4	4	1	1	4	4	4
5	4	1	2	5	4	5

$\wedge$	0	1	2	3	4	5
0	0	0	3	3	4	5
1	0	1	2	3	4	5
2	3	2	2	3	5	5
3	3	3	3	3	5	5
4	0	4	5	3	4	5
5	3	5	5	3	5	5

Then $(V, \vee \wedge 1)$ is a normal paradistributive latticoid. Here $\{1, 2\}$ and $\{1, 0, 4\}$ are the only prime filters of V.

Example 3. *Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be a set with binary operations $\vee$ and $\wedge$ given in the following tables:*

$\vee$	0	1	2	3	4	5	6	7	8
0	0	1	1	0	0	1	0	0	0
1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	2	2	2	2	2
3	0	1	2	3	3	2	3	3	3
4	0	1	2	3	4	2	3	4	4
5	1	1	5	5	5	5	5	5	5
6	0	1	5	6	6	5	6	6	6
7	0	1	5	6	7	5	6	7	7
8	0	1	5	6	8	5	6	8	8

$\wedge$	0	1	2	3	4	5	6	7	8
0	0	0	3	3	4	6	6	7	8
1	0	1	2	3	4	5	6	7	8
2	3	2	2	3	4	5	6	7	8
3	3	3	3	3	4	6	6	7	8
4	4	4	4	4	4	7	8	7	8
5	6	5	2	3	4	5	6	7	8
6	6	6	3	3	4	6	6	7	8
7	7	7	4	4	4	7	7	7	8
8	8	8	4	4	4	8	8	7	8

Then $(V, \vee \wedge 1)$ is not a normal paradistributive latticoid. Here $\{0, 1\}$, $\{1, 2, 5\}$ and $\{0, 1, 2, 3, 5, 6\}$ are the only prime filters of V.

Remark 1. *: If V is a PDL in which $a \vee b \neq 1$ for all $a \neq 1$ and $b \neq 1$, then V is normal (since $\{1\}$ becomes the unique minimal prime filter).*

Proposition 1. *Let V be a paradistributive latticoid with minimal elements. Then the following assertions are equivalent.*

(1) *Any two distinct minimal prime filters are co-maximal,*

(2) *V is normal,*

(3) *for any prime filter $\mathcal{P}$, $\mathcal{O}(\mathcal{P})$ is a minimal prime filter,*

(4) *for any $x, y \in V, x \vee y = 1 \implies (x)^\bullet \vee (y)^\bullet = V$,*

(5) *for any* $x, y \in V$, $(x \vee y)^{\bullet} = (x)^{\bullet} \vee (y)^{\bullet}$,

(6) *for any* $x, y \in V, x \vee y = 1$ *implies that there exists* $u \in (x)^{\bullet}$ *and* $v \in (y)^{\bullet}$ *such that* $u \wedge v$ *is minimal.*

Proof.

(1) ⇒ (2) Assume (1). Let $\mathcal{P}$ be a prime filter of V. Suppose that $\mathcal{Q}_1, \mathcal{Q}_2$ are two distinct minimal prime filters of V contained in $\mathcal{P}$. Then $\mathcal{Q}_1 \vee \mathcal{Q}_2 \subseteq \mathcal{P}$. But from our assumption, we have $\mathcal{Q}_1 \vee \mathcal{Q}_2 = V$. Therefore, we get $V \subseteq \mathcal{P}$. This is a contradiction. Thus $\mathcal{P}$ contains a unique minimal prime filter of V. Hence V is normal.

(2) ⇒ (3) Assume (2). Let $\mathcal{P}$ be a prime filter of V. Then $\mathcal{O}(\mathcal{P})$ is a filter of V contained in $\mathcal{P}$. Since V is normal, $\mathcal{P}$ contains a unique minimal prime filter, say $\mathcal{Q}$. We know that $\mathcal{O}(\mathcal{P})$ is the intersection of all the minimal prime filters of V contained in $\mathcal{P}$. Hence $\mathcal{O}(\mathcal{P}) = \mathcal{Q}$. Therefore $\mathcal{O}(\mathcal{P})$ is the minimal prime filter of V.

(3) ⇒ (4) Assume (3). Let $x, y \in V$ and $x \vee y = 1$. We have to prove that $(x)^{\bullet} \vee (y)^{\bullet} = V$. Suppose $(x)^{\bullet} \vee (y)^{\bullet} \neq V$. Then there exists a maximal filter $\mathcal{M}$ in V such that $(x)^{\bullet} \vee (y)^{\bullet} \subseteq \mathcal{M}$. Since $\mathcal{M}$ is a prime filter of V, from condition (3), we get $\mathcal{O}(\mathcal{M})$ is a prime filter of V. Now $x \vee y = 1 \in \mathcal{O}(\mathcal{M})$ implies that either $x \in \mathcal{O}(\mathcal{M})$ or $y \in \mathcal{O}(\mathcal{M})$. If $x \in \mathcal{O}(\mathcal{M})$ then $(x)^{\bullet} \nsubseteq \mathcal{M}$. This is not possible. Therefore $x \notin \mathcal{O}(\mathcal{M})$. Similarly, we get $y \notin \mathcal{O}(\mathcal{M})$. This is a contradiction. Therefore, $(x)^{\bullet} \vee (y)^{\bullet} = V$.

(4) ⇒ (5) Assume (4). Let $a \in (x \vee y)^{\bullet}$. Then $a \vee x \vee y = 1$. This gives $a \vee x \vee a \vee y = 1$. Therefore from condition (4), we have $(a \vee x)^{\bullet} \vee (a \vee y)^{\bullet} = V$. Now $a \in V$ implies that $a = t \wedge s$ where $t \in (a \vee x)^{\bullet}$ and $s \in (a \vee y)^{\bullet}$. So that $a = t \wedge s$, where $t \vee a \vee x = 1$ and $s \vee a \vee y = 1$. Hence $a = t \wedge s$, where $t \vee a \in (x)^{\bullet}$ and $s \vee a \in (y)^{\bullet}$. Therefore $a = t \wedge s = (t \wedge s) \vee a = (t \vee a) \wedge (s \vee a) \in (x)^{\bullet} \vee (y)^{\bullet}$. Thus $(x \vee y)^{\bullet} \subseteq (x)^{\bullet} \vee (y)^{\bullet}$. Hence $(x \vee y)^{\bullet} = (x)^{\bullet} \vee (y)^{\bullet}$.

(5) ⇒ (6) Let $x, y \in V$ and $x \vee y = 1$. Let m be a minimal element. Then $(1)^{\bullet} = (x \vee y)^{\bullet} = (x)^{\bullet} \vee (y)^{\bullet}$. This gives $V = (x)^{\bullet} \vee (y)^{\bullet}$. Now

$$\begin{aligned} m \in V &\implies m \in (x)^{\bullet} \vee (y)^{\bullet} \\ &\implies m = u \wedge v \text{ where } u \in (x)^{\bullet} \text{ and } v \in (y)^{\bullet} \\ &\implies m = u \wedge v \text{ where } u \vee x = 1 \text{ and } v \vee y = 1. \end{aligned}$$

Therefore, there exists $u \in (x)^{\bullet}$ and $v \in (y)^{\bullet}$ such that $u \wedge v = m$.

(6) ⇒ (1) Assume (6). Let $\mathcal{P}, \mathcal{Q}$ be two distinct minimal prime filters of V. Then $V \backslash \mathcal{P}$ is a maximal ideal of V. Then $V \backslash \mathcal{P}$ is a maximal ideal of V. Since $\mathcal{P}, \mathcal{Q}$ are distinct, chose $x \in \mathcal{P} \backslash \mathcal{Q}$. Clearly, $x \notin V \backslash \mathcal{P}$. Since $V \backslash \mathcal{P}$ is a maximal ideal of V, we get $(V \backslash \mathcal{P}) \vee (x] = V$. Now

$$1 \in V \implies 1 \in (V \backslash \mathcal{P}) \vee (x]$$

$$\implies 1 = a \vee b, \text{ where } a \in V \backslash \mathcal{P} \text{ and } b \in (x]$$
$$\implies 1 \vee x = a \vee b \vee x$$
$$\implies 1 = a \vee x.$$

Thus from condition (6), there exists $u \in (a)^{\bullet}$ and $v \in (x)^{\bullet}$ such that $u \wedge v$ is minimal. Hence $u \wedge v \in (a)^{\bullet} \vee (x)^{\bullet} \subseteq \mathcal{P} \vee \mathcal{Q}$. Therefore $\mathcal{P} \vee \mathcal{Q} = V$. Thus any two distinct minimal prime filters of V are co-maximal.

Let V be a bounded distributive lattice. Then V is said to be conormal provided that $x, y \in V, x \wedge y = 0$ implies there exist $u, v \in V$ such that $u \wedge x = v \wedge y = 0$ and $u \vee v = 1$. This property, discussed by Cornish[3] under the name of a normal lattice, aligns with the version of the definition presented by Simmons[9] in 1980 (specifically, Definition 4.3).

Theorem 12. *A PDL V is normal if and only if the bounded distributive lattice $\mathcal{PF}(V)$ is conormal.*

Corollary 2. *If V is a PDL with minimal element m then the following are equivalent.*

(1) *V is normal.*

(2) *Each maximal filter contains a unique minimal prime filter.*

(3) *For each maximal filter $\mathcal{M}$, $\mathcal{O}(\mathcal{M})$ is a prime filter.*

(4) *For any $x, y \in V$ if $x \vee y = 1$ then there exist $x_1, y_1 \in V$ such that $x_1 \vee x = 1$, $y_1 \vee y = 1$ and $x_1 \wedge y_1 = m$.*

We know that every filter of a PDL V is a sub PDL of V and every ideal is also a sub PDL of V. Now we prove the following.

Lemma 14. *A filter $\mathcal{J}$ of a PDL V is normal as a sub PDL of V if and only if $\mathcal{J}^e$ is normal as a sub lattice of $\mathcal{PF}(V)$.*

Proof. Let $\mathcal{J}$ be a filter of V. Assume that $\mathcal{J}$ is normal as a sub PDL of V. Let $\mathcal{P}$ be any prime filter of $\mathcal{J}^e$. Now we prove that $\mathcal{P}$ contains a unique minimal prime filter of $\mathcal{J}^e$. Suppose $\mathcal{Q}_1, \mathcal{Q}_2$ are two minimal prime filters of $\mathcal{J}^e$ such that $\mathcal{Q}_1 \subseteq \mathcal{P}$ and $\mathcal{Q}_2 \subseteq \mathcal{P}$. Then $\mathcal{Q}_1^c \subseteq \mathcal{P}^c$ and $\mathcal{Q}_2^c \subseteq \mathcal{P}^c$. Since $\mathcal{P}$ is a prime filter of $\mathcal{J}^e$, $\mathcal{P}^c$ is a prime filter of $\mathcal{J}^{e^c} = \mathcal{J}$. Since $\mathcal{J}$ is normal, the prime filter $\mathcal{P}^c$ of $\mathcal{J}$ contains a unique minimal prime filter. Therefore $\mathcal{Q}_1^c = \mathcal{Q}_2^c$. Hence $\mathcal{Q}_1 = \mathcal{Q}_1^{c^e} = \mathcal{Q}_2^{c^e} = \mathcal{Q}_2$. Therefore $\mathcal{J}^e$ is a normal as a sub lattice of $\mathcal{PF}(V)$.
Conversely, assume that $\mathcal{J}^e$ is normal as a sub lattice of $\mathcal{PF}(V)$. Now, we prove that the filter $\mathcal{J}$ is normal as a sub PDL of V. Let $\mathcal{P}$ be a prime filter of $\mathcal{J}$. Then $\mathcal{P}^e$ is a prime filter of $\mathcal{J}^e$. Since $\mathcal{J}^e$ is normal, $\mathcal{P}^e$ contains a unique minimal prime filter say $\mathcal{Q}$. Then $\mathcal{Q}^c$ is a minimal prime filter contained in $\mathcal{P}^{e^c} = \mathcal{P}$. Now, we prove that $\mathcal{Q}^c$ is unique. Let $\mathcal{Q}_1$ be any other minimal prime filter of $\mathcal{J}$ contained in $\mathcal{P}$. $\mathcal{Q}_1^e$ is a minimal prime filter

contained in the prime filter $\mathcal{P}^e$. Since $\mathcal{P}^e$ contains a unique minimal prime filter, we get $\mathcal{Q} = \mathcal{Q}_1^e$. This gives $\mathcal{Q}^c = (\mathcal{Q}_1^e)^c = \mathcal{Q}_1$. Therefore $\mathcal{Q}^c$ is unique.

Now, we recall that a relatively complemented PDL is a PDL in which every closed interval is a dually complemented lattice. Every relatively complemented PDL is associative. In the following result we prove that every relatively complemented PDL is a normal PDL.

Theorem 13. *Every relatively complemented PDL is a normal PDL.*

Proof. Let V be a relatively complemented PDL. Let $x, y \in V$ and $x \vee y = 1$. Let $z \in V$. Now consider the interval $[z \wedge y \wedge x, 1]$. Since $x \vee y = 1$, we have $x \wedge y = y \wedge x$. So that $x, y \in [z \wedge y \wedge x, 1]$. Therefore there exist complements x', y' of x, y, respectively in $[z \wedge y \wedge x, 1]$. Also, since $x \vee y = 1$, we have $x' \wedge y' = z \wedge y \wedge x$.

$$(z \vee x') \wedge (z \vee y') = z \vee (x' \wedge y') = z \vee (z \wedge y \wedge x) = z.$$

Since $z \vee x' \vee x = 1$ and $z \vee y' \vee y = 1$, we get $z \vee x' \in (x)^\bullet$ and $z \vee y' \in (y)^\bullet$. Therefore we get $z \in (x)^\bullet \vee (y)^\bullet$. Thus $V \subseteq (x)^\bullet \vee (y)^\bullet$ and hence V is a normal PDL.

Theorem 14. *Let V be a PDL. Then V is normal if and only if each prime ideal in V is contained in a unique maximal ideal.*

Proof. Assume that each prime ideal in V is contained in a unique maximal ideal. Now we prove that V is a normal PDL. Let $\mathcal{P}$ be a prime filter of V. Then it is enough to prove that $\mathcal{P}$ contains a unique minimal prime filter of V. Since $\mathcal{P}$ is a prime filter, $V \backslash \mathcal{P}$ is a prime ideal of V. Therefore, $V \backslash \mathcal{P}$ is contained in a unique maximal ideal, say $\mathcal{G}$.
Now, $(V \backslash \mathcal{P}) \subseteq \mathcal{G}$ gives that $V \backslash \mathcal{G} \subseteq \mathcal{P}$. Clearly, we have $V \backslash \mathcal{G}$ is a minimal prime filter of V contained in $\mathcal{P}$. Let $\mathcal{Q}$ be a minimal prime filter of V such that $\mathcal{Q} \subseteq \mathcal{P}$. Then $V \backslash \mathcal{P} \subseteq V \backslash \mathcal{Q}$. Since $\mathcal{Q}$ is a minimal prime filter, $V \backslash \mathcal{Q}$ is a maximal prime ideal of V. That is, $V \backslash \mathcal{P}$ is a prime ideal which is contained in two maximal ideal $V \backslash \mathcal{Q}$ and $\mathcal{G}$. This is a contradiction to our assumption. Therefore $V \backslash \mathcal{G}$ is unique. Therefore, V is a normal PDL. Similarly, we can prove the converse.

Theorem 15. *Let V be a PDL. Then V is normal if and only if each minimal prime ideal in V is contained in a unique maximal ideal of V.*

Proof. Assume that V is a normal PDL. Let $\mathcal{F}$ be any minimal prime ideal in V. Then, the prime ideal $\mathcal{F}$ is contained in a unique maximal ideal of V. That is, every minimal prime ideal of V is contained in a unique maximal ideal of V. Conversely, assume that each minimal prime ideal in V is contained in a unique maximal ideal of V. Suppose, a prime filter $\mathcal{P}$ of V contains two minimal prime filters $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of V. Since $\mathcal{P}$ is a proper filter of V, $\mathcal{P}$ is contained in a maximal filter, say $\mathcal{M}$ of V. Then, we get $\mathcal{Q}_1 \subseteq \mathcal{P} \subseteq \mathcal{M}$ and $\mathcal{Q}_2 \subseteq \mathcal{P} \subseteq \mathcal{M}$. This gives $V \backslash \mathcal{M} \subseteq V \backslash \mathcal{Q}_1$ and $V \backslash \mathcal{M} \subseteq V \backslash \mathcal{Q}_2$. That is, the minimal prime ideal $V \backslash \mathcal{M}$ is contained in two maximal ideals $V \backslash \mathcal{Q}_1$ and $V \backslash \mathcal{Q}_2$. This is a contradiction. Therefore each prime filter in V contains a unique minimal prime filter of V. Hence V is normal.

4. Conclusions

In summary, we have defined $\mathcal{O}(\mathcal{P})$ for any filter $\mathcal{P}$ of a paradistributive latticoid and proved that $\mathcal{O}(\mathcal{P})$ is a filter when $\mathcal{P}$ is prime. Additionally, we have established that every minimal prime filter belonging to $\mathcal{O}(\mathcal{P})$ is contained in $\mathcal{P}$, and that $\mathcal{O}(\mathcal{P})$ is the intersection of all minimal prime filters contained in $\mathcal{P}$. The concept of a normal paradistributive latticoid has been introduced and characterized in terms of prime filters and minimal prime filters, with the proof that every relatively complemented paradistributive latticoid is normal. Our future work will delve into the introduction of relatively normal paradistributive latticoids. We also aim to explore the topological characterization of normal paradistributive latticoids. Furthermore, we plan to investigate $\mathcal{S}$-normal paradistributive latticoids, where $\mathcal{S}$ represents a sub paradistributive latticoid of a given paradistributive latticoid.

Conflicts of interest or competing interests

The authors declare that they have no conflicts of interest.

Informed Consent

The authors are fully aware and satisfied with the contents of the article.

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