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The Block Topological Space and Block Topological Graph induced by Undirected Graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple undirected graph. A *block* of G is a maximal connected subgraph of G that contains no cut-vertices [11]. The family of vertex sets of blocks of G generates a unique topology. In this paper, we formally define the topology generated by the family of blocks in a graph called the *block topological space*. Moreover, we characterize and describe some special attributes of the block topological space. Finally, we associate a corresponding graph from a given block topological space by defining the *block topological graph*.

2020 Mathematics Subject Classifications: 37F20, 05C10, 54C35

Key Words and Phrases: Block topological space, Block topological graph, Graph, Block, Hausdorff space, Continuous

1. Introduction

There are many ways of associating topology from a graph, as seen in [1], [3], [5], [6], [7], and [8]. The most common method among these is by treating a collection of subsets of a nonempty set (e.g. vertex set or edge set) as a subbase to generate the desired topology which is reflected in the paper of Hassan and Abed in [7]. This topology is called the *independent topology* and is generated from the family of independent sets of each of the vertices in the graph. The same method was applied in the study of Abdu and Kilicman in [1] where they associated two topologies on the set of edges from a particular directed graph called *edge-compatible topology* and *edge-incompatible topology*. Another fascinating intercrossing of topology and graph theory is establishing an adjacency condition to obtain the desired graph from a given finite topological space. This idea was reflected in the paper of Alsanaia et.al. in [2] as they gave a formal definition of converting the finite discrete topological space using a suitable adjacency condition to obtain the graph called the *discrete topological graph*.

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Let $G = (V(G), E(G))$ be a simple undirected graph. For a nonempty subset S of $V(G)$, the subgraph induced by S, denoted by $G[S]$, has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. A subgraph H of a graph G is called an *induced subgraph* if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. A vertex v of G is an *isolated vertex* if it is not adjacent to any other vertices of G. Two vertices of G are *connected* if there is a path that connect them. If every two vertices of G are connected, then the graph G is connected. A *component* of G is a connected subgraph of G that is not contained in any larger connected subgraph of G. The number of components of G is denoted by $\kappa(G)$. A vertex v of G is a *cut-vertex* of G if $\kappa(G - v) > \kappa(G)$. If G is a nontrivial graph and v is a cut-vertex of component C_j of G, then the subgraph $C_j - v$ has m components $G_1, G_2, \cdots, G_m$ for $m \geq 2$ and the induced subgraphs $Br_i^j = G[V(G_i) \cup \{v\}]$ are connected and referred to as *branches* of C_j at v [4]. A *block* of a graph is a maximal connected subgraph that contains no cut-vertices [11]. The smallest possible block in a graph is a subgraph induced by a single vertex with a degree equal to zero. Moreover, two distinct blocks have at most one vertex in common and if they share the same vertex, then this vertex is a cut-vertex [4]. In addition, if $B_1, B_2, \cdots, B_k$ are the blocks of G, then $\bigcup_{i=1}^{k} V(B_i) = V(G)$.

A *topology* τ on a nonempty set X is a class of subsets of X that is closed under arbitrary union and finite intersection, and X and $\varnothing$ belong to τ. The member of τ is called an *open set* and the pair (X, τ) is called a *topological space.* The topology containing all the subsets of X is called the *discrete topology* on X and the topology containing exactly X and $\varnothing$ is called the *indiscrete topology* on X. A collection Γ of open sets is a *base* for a topology of X if each nonempty open is a union of sets belonging to Γ. A collection Σ of open sets is called a *subbase* if the set $\{A : A = \bigcap_{i=1}^{k} W_i, k \in \mathbb{Z}^+, W_i \in \Sigma\}$ is a base for a topology on X[10]. Any class $\mathscr{A}$ of subsets of X is a subbase of for a unique topology on X. That is, the finite intersection of sets in $\mathscr{A}$ form a base for a topology on X [9]. If (X, τ_1) and (Y, τ_2) are two topological space, then $f : (X, \tau_1) \to (Y, \tau_2)$ is *continuous* if the preimage of any open subset of Y is an open subset of X[10].

As seen in the above discussion, it is possible to exhaust the distinct blocks of a given graph and apply various methods of topologizing the family of the vertex sets of these blocks. It is with this motivation that we aim to introduce a novel approach to topologizing a graph using the blocks in a graph. The generated topology will then be called the block topological space of a graph. Moreover, we examine and investigate some elementary properties of sets in a block topological space. Finally, we introduce the notion of a block topological graph.

2. Steps in enumerating the blocks in a graph

General Assumption: Let G be a simple undirected graph with components $C_1, \cdots, C_j$ for some $j \in \mathbb{Z}^+$. The following steps are ways on enumerating the blocks in a graph.

Step 1: For any $1 \leq i \leq j$, if C_i contains no cut-vertices, then C_i is a block of G. Consequently, if G has no cut-vertex, then $C_1, C_2, \cdots, C_j$ are precisely the blocks of G.

Step 2: If C_i contains a cut-vertex v_{i_1}, then obtain the branches of C_i at v_{i_1}.

Step 3: If all the branches of C_i at v_{i_1} contains no cut-vertices, then of these branches is a block of G. Otherwise, take the branches of C_i at v_{i_1} with cut-vertices.

Step 4: For each branch of C_i with a cut-vertex, choose one cut-vertex v_{i_2} and obtain the sub-branches at v_{i_2}.

Step 5: Repeat the steps of separating the branch until all the resulting sub-branches contain no cut-vertices.

Step 6: Do these for all the components of G that contains cut-vertices.

Step 7: Collect all the components, branches, and sub-branches of G that contain no cut-vertices. These are precisely the blocks of G.

For example, consider the graph G in Figure 1.

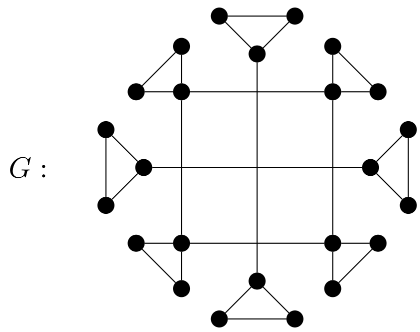

Figure 1: The graph G

The following are the components of G, each containing cut-vertices as shown in Figure 2.

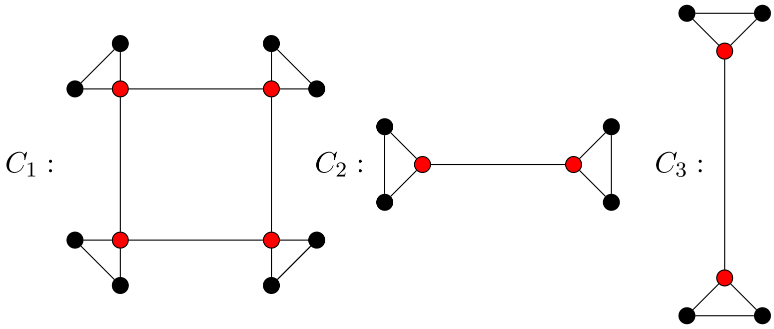

Figure 2: Components of G

For C_1, choose a cut-vertex v_{1_1} so that the branches of C_1 at v_{1_1} are shown in Figure 3.

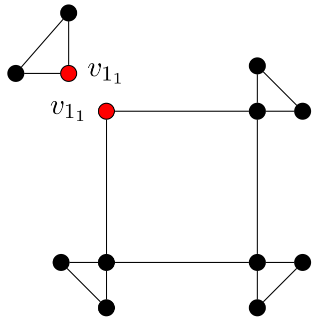

Figure 3: Branches of C_1 at v_{1_1}

Notice that one of the branches of C_1 at v_{1_1} has cut-vertices. Choose another cut-vertex v_{1_2} and obtain the sub-branches at v_{1_2}. The sub-branches at v_{1_2} are as shown in the Figure 4.

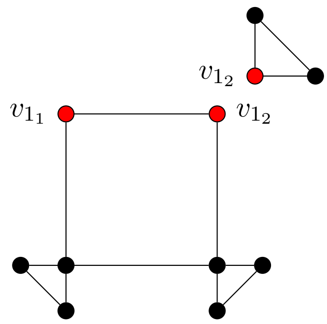

Figure 4: Sub-branches at v_{1_2}

Repeat the steps of separating the branch until the resulting sub-branches contain no cut-vertices. Thus, the following are the sub-branches at every cut-vertices of C_1 as shown in Figure 5. Furthermore, the blocks of G at C_1 are as shown in Figure 5.

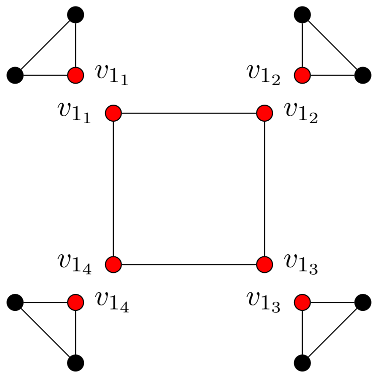

Figure 5: The blocks of G at C_1

Doing the preceding steps for all the components of G gives the distinct blocks of G as shown in the Figure 6.

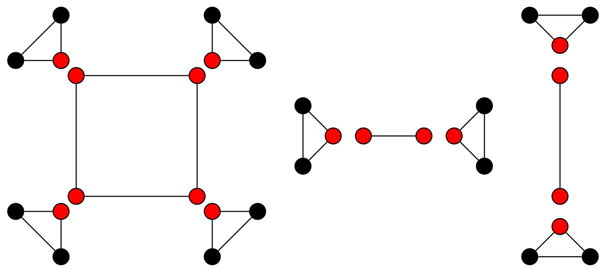

Figure 6: The blocks of G

3. The block topological space

Definition 1. *Let G be a graph and let $B_1, B_2, \cdots, B_k$ be the distinct blocks of G for some $k \in \mathbb{Z}^+$. Then the topology on $V(G)$ generated by the family*

$$\Sigma_B(G) = \{V(B_1), V(B_2), \cdots, V(B_k)\}$$

is called the ***block topology of*** *G, denoted by $\tau_B(G)$ and the pair $(V(G), \tau_B(G))$ is called the* ***block topological space of*** *G. Denote $\Gamma_B(G)$ to be the family of finite intersections of sets in $\Sigma_B(G)$. In this case, $\Gamma_B(G)$ is a* ***base*** *for $\tau_B(G)$. A subset A of $V(G)$ is $\tau_B(G)$-**open** if A belongs to $\tau_B(G)$ and if A^c belongs to $\tau_B(G)$, then A is $\tau_B(G)$-**closed**.*

Example 1. Consider the graph G in Figure 7. Observe that the blocks of G are given by B_1, B_2, and B_3 implying that $\Sigma_B(G) = \{\{v_1, v_2, v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_6\}\}$. By the Definition 1, taking the finite intersections of sets in $\Sigma_B(G)$ we obtain the family $\Gamma_B(G) =$

$\{\{v_1, v_2, v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_4\}\}$. Finally, by taking the arbitrary union of sets in $\Gamma_B(G)$ we obtain the block topology $\tau_B(G) = \{\varnothing, V(G), \{v_4\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_4, v_6\}\}$.

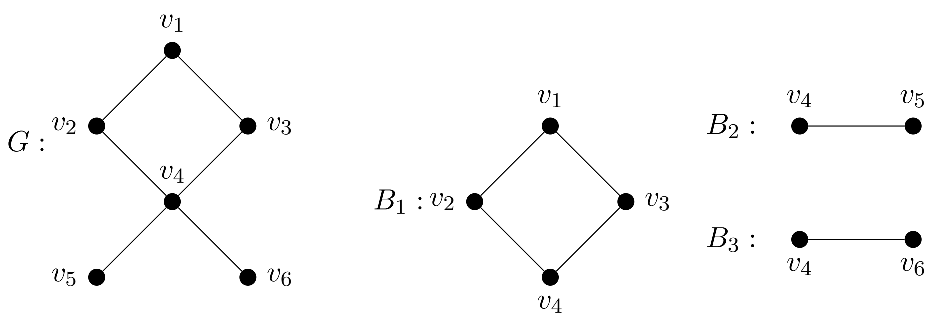

Figure 7: The blocks of G

By observation, two distinct blocks have at most one vertex in common and if they share a vertex, then this vertex is a cut-vertex. This means that the greater the number of cut-vertices in a graph the larger the number of blocks in a graph. However, increasing the number of blocks does not imply an increased number of cut-vertices (e.g. the case where we add an isolated vertex to the graph).

Theorem 1. *Let G be a graph. For a vertex v of G, $\{v\}$ is $\tau_B(G)$-open if and only if v is a cut-vertex of G or v is an isolated vertex of G.*

Proof. ($\Rightarrow$) Let G be a graph with blocks $B_1, B_2, \cdots, B_k$. Suppose that $\{v\} \in \tau_B(G)$. Then $\{v\} \in \Gamma_B(G)$. This means that for some nonempty $\mathscr{A} \subseteq \{1, 2, \cdots, k\}$, $\{v\} = \bigcap_{i \in \mathscr{A}} V(B_i)$. If $|\mathscr{A}| = 1$, then $\{v\} \in \Sigma_B(G)$, which means that $G[\{v\}]$ is a block of G implying further that v is an isolated vertex. On the other hand, if $|A| > 1$, then $\{v\}$ is the intersection of two or more distinct blocks which further implies that v is a cut-vertex.
($\Leftarrow$) If v is an isolated vertex, then $G[\{v\}]$ is a block of G and by Definition 1, $\{v\}$ is a $\tau_B(G)$-open. Suppose that v is a cut-vertex and let C_j be a component of G containing v. Let Br_i, $i = 1, 2, \cdots, s$, be the branches of C_j at v. Note that for each $i = 1, 2, \cdots, s$, Br_i is composed of blocks of G which further means that each of the branch of C_j at v is $\tau_B(G)$-open. Hence, by Defintion 1, $\bigcap_{i=1}^{s} Br_i = \{v\}$ is $\tau_B(G)$-open. □

Theorem 2. *A graph G is an empty graph if and only if $\tau_B(G)$ is the discrete topology on $V(G)$.*

Proof. ($\Rightarrow$) Let G be an empty graph. Then $G[\{v\}]$ is a block of G and by Definition 1, any subset of $V(G)$ is $\tau_B(G)$-open. Hence, $\tau_B(G)$ is discrete.
($\Leftarrow$) Suppose that $\tau_B(G)$ is discrete. Then for all $v \in V(G)$, $\{v\}$ is $\tau_B(G)$-open. Suppose on the contrary that G is not an empty graph. Then $|E(G)| \geq 1$ and so let C_j be a component of G for some $j \in \mathbb{Z}^+$ such that $|V(C_j)| \geq 2$. In this case, C_j has at least two vertices that are not cut-vertices nor isolated vertices, say v and w. By Theorem

1, $\{v\}, \{w\} \notin \tau_B(G)$. A contradiction since $\tau_B(G)$ is discrete. Hence, G is an empty graph. □

Corollary 1. *Let G be a graph. If A is the collection of cut-vertices of G, then any subset of A is $\tau_B(G)$-open.*

Theorem 3. *A graph G is connected and contains no cut-vertices if and only if $\tau_B(G)$ is the indiscrete topology on $V(G)$.*

Proof. ($\Rightarrow$) Let G be a connected graph without cut-vertices. Then $V(G)$ is the only block of G and so $\Sigma_B(G) = \{V(G)\}$ which further implies thats $\tau_B(G) = \{\varnothing, V(G)\}$.
($\Leftarrow$) Suppose that $\tau_B(G)$ is indiscrete. Then any nonempty proper subset of $V(G)$ is not $\tau_B(G)$-open. Suppose on the contrary that either G disconnected or has cut-vertices. If G is disconnected, then G has two or more components. Let C_j be a component of G for some $j \in \mathbb{Z}^+$. Then C_j are composed of blocks of G and so $V(C_j)$ is the union of the vertex sets of these blocks implying that $V(C_j)$ is $\tau_B(G)$-open. But $V(C_j) \subsetneq V(G)$, a contradiction since $\tau_B(G)$ is indiscrete. Therefore, G must be connected. On the other hand, if G has a cut-vertices, then let v be a cut-vertex of G. By Theorem 1, $\{v\} \in \tau_B(G)$, a contradiction. Hence G is connected and contains no cut-vertices. □

Suppose that G is a graph and let $B_1, B_2, \cdots, B_k$ be the blocks of G. Denote $\mathscr{C}(G)$ to be the family of all the cut-vertices of G. Recall that two distinct blocks have at most one vertex in common and this vertex is a cut-vertex [4]. By Definition 1, the collection of the vertex set of each of the blocks of G together with all singletons containing the cut-vertices of G is a base for the block topological space of G. By this observation, the following theorem characterizes the $\tau_B(G)$-open sets.

Theorem 4. *Let G be a graph and let $B_1, B_2, \cdots, B_k$ be the blocks of G. A set $A \subseteq V(G)$ is $\tau_B(G)$-open if and only if A is the union of sets in $\Sigma_B(G) \cup \{T : T \subseteq \mathscr{C}(G)\}$.*

Proof. ($\Rightarrow$) Let $B_1, B_2, \cdots, B_k$ be the distinct blocks of a graph G. Suppose A is a $\tau_B(G)$-open. Then A is the union of finite intersections of sets in $\Sigma_B(G)$. We now note that the intersection of two distinct blocks is at most one vertex which is a cut-vertex. Hence, the conclusion follows.
($\Leftarrow$) Let $A \in \Sigma_B(G) \cup \{T : T \subseteq \mathscr{C}(G)$. By Definition 1 and Theorem 1, A is $\tau_B(G)$-open. □

Remark 1. *Let G be a graph with more than one block. Then the union of all the nontrivial $\tau_B(G)$-open proper subsets of $V(G)$ equals to $V(G)$.*

Remark 2. *Let $(V(G), \tau_B(G))$ be a non indiscrete block topological space. Then the smallest possible number of $\tau_B(G)$-open sets is 4.*

Example 2. Consider the graph G in Figure 8. Here, $G[\{v_1, v_2\}]$ and $G[\{v_2, v_3\}]$ are the blocks of G and $\tau_B(G) = \{\varnothing, V(G), \{v_1, v_2\}, \{v_3, v_4\}\}$. Moreover, $\{v_1, v_2\} \cup \{v_3, v_4\} = V(G)$ and $|\tau_B(G)| = 4$.

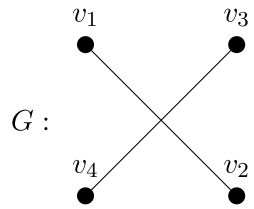

Figure 8: Graph G

Theorem 5. *Let G be a graph such that $|E(G)| \geq 1$. If v is not a cut-vertex adjacent to w, then every $\tau_B(G)$-open set containing v also contains w.*

Proof. Let G be a graph such that $|E(G)| \geq 1$. Suppose that vertex v is not a cut-vertex adjacent to w. Let $\mathscr{O}$ be a $\tau_B(G)$-open containing v. Suppose on the contrary that $w \notin \mathscr{O}$. Since v and w are adjacent in G and $E(G)$ is the disjoint union of the edges of B_is, then there exists a block B_j of G such that $v, w \in V(B_j)$. On the other hand, since v is not a cut-vertex, choose B_k as a block such that $v \in V(B_k) \subseteq \mathscr{O}$. In this case, B_k and B_j are distinct and $\{v\} \in V(B_k) \cap V(B_j)$. But, two distinct blocks intersect in at most one vertex which is a cut-vertex; hence v is a cut-vertex, a contradiction to our choice of v. Therefore, $w \in \mathscr{O}$. $\square$

A topological space is a **Hausdorff space** if for every two distinct elements in the mother set can be separated by two disjoint open sets [9]. The following Theorem characterizes the block topological space as being a Hausdorff space.

Theorem 6. *A $(V(G), \tau_B(G))$ is a Hausdorff space if and only if G is an empty graph.*

Proof. ($\Rightarrow$) Let $(V(G), \tau_B(G))$ be a Hausdorff space. Then every two distinct vertices in G can be separated by two disjoint $\tau_B(G)$-open sets. If G is not an empty graph, then $|E(G)| \geq 1$. In this case, choose a component C_j of G having more than one vertices. Here, C_j has a vertex v that is not a cut-vertex of G and v is adjacent to some vertex, say w. By Theorem 5, every $\tau_B(G)$-open set containing v also contains w. This is a contradiction since by assumption that $(V(G), \tau_B(G))$ is a Hausdorff space. Hence, G is an empty graph.
($\Leftarrow$) Let G be an empty graph. By Theorem 2, the generated block topology is discrete and thus the conclusion follows. $\square$

Example 3. Consider the graphs in Figure 9. Observe that vertex v is not a cut-vertex of G that is adjacent to w. By Theorem 5, any $\tau_B(G)$-open containing v also contains w. Hence, the block topological space of G is not a Hausdorff space. On the other hand, H is an empty graph and by Theorem 2, $\tau_B(H)$ is the discrete topology of $V(H)$. Hence, every singleton of $V(H)$ is $\tau_B(H)$-open implying further that the block topological space of H is a Hausdorff space.

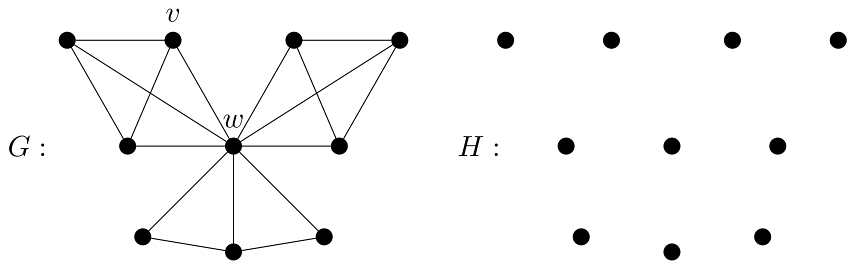

Figure 9: Graph of a non-Hausdorff space and Hausdorff space, respectively

Theorem 7. *Let G be a graph of order $n < 4$. Then G has no isolated vertices if and only if there exists a one-to-one correspondence from $(V(P_n), \tau_B(P_n))$ to $(V(G), \tau_B(G))$ that is continuous.*

Proof. ($\Rightarrow$) If G is a graph of order $n < 4$ without isolated vertices, then G is one of P_2, P_3, and K_3. For P_2 and P_3, use the identity mapping so that we arrive the desired conclusion. For K_3, since $|V(K_3)| = |V(P_3)|$, put f to be a one-to-one correspondence from $(V(P_3), \tau_B(P_3))$ to $(V(K_3), \tau_B(K_3))$. Since, $\tau_B(K_3) = \{\varnothing, V(K_3)\}$ and $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(V(K_3)) = V(P_3)$, f is continuous.
($\Leftarrow$) Let f be a continuous one-to-one correspondence from $(V(P_n), \tau_B(P_n))$ to $(V(G), \tau_B(G))$. Suppose that G has an isolated vertices.

Case 1: If $n = 2$, then $\tau_B(G)$ is discrete. Let w be an isolated vertex of G. Then $f^{-1}(\{w\})$ is a singleton subset of $V(P_2)$. Since by Theorem 3, $\tau_B(P_2)$ is indiscrete, $f^{-1}(\{w\})$ is not $\tau_B(P_2)$-open. This is a contradiction since f is continuous. Hence, G must not have an isolated vertex.

Case 2: Let w be an isolated vertex of G. Then for all $v \in V(G) \setminus \{w\}$, $wv \notin E(G)$. Also, $f^{-1}(w)$ must be a cut-vertex of P_3 since by assumption f is continuous.

Subcase 1: If v is an isolated vertex of G, then G must be an empty graph. This means that $\tau_B(G)$ is the discrete topology. Let u be a vertex in G such that $f^{-1}(u)$ is an end-vertex of P_3. Here, $\{u\}$ is $\tau_B(G)$-open and $f^{-1}(\{u\})$ is not a $\tau_B(P_3)$-open set. This is a contradiction since f is continuous.

Subcase2: If v is not an isolated vertex, then v must be adjacent to some vertex, say u. In this case, $G[\{v, u\}]$ is a block of G implying further that $\{v, u\}$ is $\tau_B(G)$-open. Since f is bijective, $f^{-1}(\{v, u\}) = f^{-1}(\{v\}) \cup f^{-1}(\{u\})$. Note that $f^{-1}(w)$ is a cut-vertex of P_3, and so $f^{-1}(v)$ and $f^{-1}(u)$ are end-vertices of P_3. By Theorem 1, $f^{-1}(\{v, u\})$ is not $\tau_B(P_3)$-open, a contradiction. $\square$

Remark 3. *If $G = P_1$, then the identity map satisfies the above theorem.*

Theorem 8. *Let G be a graph of order $n \geq 4$. Then G has at least two nonadjacent edges if and only if there is a one-to-one correspondence from $(V(P_n), \tau_B(P_n))$ to $(V(G), \tau_B(G))$ that is continuous.*

Proof. ($\Rightarrow$) Let G be a graph of order $n \geq 4$ such that G has at least two non-adjacent edges. Since G has at least 1 nontrivial component, there exists at least two vertices that are not cut-vertices and non-isolated vertices in that component. Denote the vertices of G by $\{v_1, v_2, \cdots, v_n\}$ such that v_1 and v_n are not cut-vertices nor isolated vertices of G and $v_1v_2, v_{n-1}v_n \in E(G)$. Let P_n be a path such that $V(P_n) = \{a_1, a_2, \cdots, a_n\}$ and $E(P_n) = \{a_ia_{i+1} : i = 1, 2, \cdots, n-1\}$. Now, define $f : (V(P_n), \tau_B(P_n)) \to (V(G), \tau_B(G))$ by $f(a_i) = v_i$. Obviously, f is a one-to-one correspondence from $(V(P_n), \tau_B(P_n))$ to $(V(G), \tau_B(G))$. It remains to show that f is continuous. Let A be $\tau_B(G)$-open. If A is empty, then $f^{-1}(A) = \varnothing \in \tau_B(P_n)$. Suppose that A is not empty.

Case 1: If $v_1 \in A$ and $v_n \notin A$, then by Theorem 5, $v_2 \in A$. Observe that, if $A = \{v_1, v_2\}$, then $f^{-1}(A) = f^{-1}(\{v_1, v_2\}) = \{a_1, a_2\} \in \tau_B(P_n)$. Suppose that $A \smallsetminus \{v_1, v_2\} \neq \varnothing$. Then for all $v_j \in A \smallsetminus \{v_1, v_2\}$, $f^{-1}(v_j)$ is a cut-vertex of P_n implying further that $f^{-1}(A \smallsetminus \{v_1, v_2\})$ is a subset of $\mathscr{C}(P_n)$. Hence, $f^{-1}(A) = f^{-1}(\{v_1, v_2\}) \cup f^{-1}(A \smallsetminus \{v_1, v_2\})$ is $\tau_B(G)$-open. The argument follows when $v_n \in A$ and $v_1 \notin A$.

Case 2: If $v_1, v_n \in A$, then $v_2, v_{n-1} \in A$. Similarly, if $A = \{v_1, v_2, v_{n-1}, v_n\}$, then $f^{-1}(A) = f^{-1}(\{v_1, v_2, v_{n-1}, v_n\}) = \{a_1, a_2, a_{n-1}, a_n\} \in \tau_B(P_n)$. Also, if $A \smallsetminus \{v_1, v_2, v_{n-1}, v_n\} \neq \varnothing$, then $f^{-1}(\{v_1, v_2, v_{n-1}, v_n\}) \subseteq \mathscr{C}(P_n)$. Hence, $f^{-1}(A) = f^{-1}(\{v_1, v_2, v_{n-1}, v_n\}) \cup f^{-1}(A \smallsetminus \{v_1, v_2, v_{n-1}, v_n\})$ is $\tau_B(G)$-open.

Case 3: If $v_1, v_n \notin A$, then $f^{-1}(A) \subseteq \mathscr{C}(P_n)$ so that $f^{-1}(A)$ is $\tau_B(G)$-open.

Thus, f is continuous.
($\Leftarrow$) Let f be a one-to-one correspondence from $(V(P_n), \tau_B(P_n))$ to $(V(G), \tau_B(G))$ that is continuous. Suppose that G has no nonadjacent edges. Then G is one of the following: (1) $G = \overline{K_n}$; (2) $|E(G)| = 1$; or (3) The edges of G share a common vertex.

Case 1: Suppose $G = \overline{K_n}$. Let v be a vertex in G such that $f^{-1}(v)$ is an end vertex of P_n. Here, $\{v\}$ is τ_B-open and $\{f^{-1}(v)\}$ is not $\tau_B(P_n)$-open; a contradiction since f is continuous. Hence, G is not an empty graph.

Case 2: Suppose $|E(G)| = 1$. Let $vw \in E(G)$. Then $\{v, w\}$ is $\tau_B(G)$-open and so by the continuity of f, $f^{-1}(\{v, w\})$ is $\tau_B(P_n)$-open. Note that $f^{-1}(v)$ and $f^{-1}(w)$ cannot be both end-vertices of P_n; otherwise, $f^{-1}(\{v, w\})$ is not $\tau_B(P_n)$-open. Now, choose a vertex u in G different from v and w such that $f^{-1}(u)$ is an end-vertex of P_n. Note that $\{f^-1(u)\}$ is not a $\tau_B(P_n)$-open set. In this case, u is an isolated vertex and thus $\{u\}$ is $\tau_B(G)$-open. This is a contradiction since f is continuous. Hence, $|E(G)| > 1$.

Case 3: Suppose the edges of G share a common vertex and let x and y be vertices in G such that $f^{-1}(x)$ and $f^{-1}(y)$ are the end-vertices of P_n. Now, x and y are not isolated vertices of G nor cut-vertices of G, otherwise f is not continuous since $f^{-1}(\{x, y\})$ is not $\tau_B(P_n)$-open.

Subcase 1: If $xy \in E(G)$, then there is another vertex v that is adjacent to both x and y since x and y are not cut-vertices. In this case, $G[\{x, v, y\}]$ is a block of G which further implies that $\{x, v, y\}$ is $\tau_B(G)$-open. Also, note that $f^{-1}(v)$ is a cut-vertex of P_n. Now, we have

$$\begin{aligned} f^{-1}(\{x, v, y\}) &= \{f^{-1}(x)\} \cup \{f^{-1}(v)\} \cup \{f^{-1}(y)\} \\ &= f^{-1}(\{x, v\}) \cup \{f^{-1}(y)\} \\ &= f^{-1}(\{x, y\}) \cup \{f^{-1}(v)\} \\ &= f^{-1}(\{v, y\}) \cup \{f^{-1}(x)\}. \end{aligned}$$

Thus, $f^{-1}(\{x, v, y\})$ is not $\tau_B(P_n)$-open by Theorem 4. This is a contradiction since f is continuous.

Subcase 2: If $xy \notin E(G)$, then there exists a vertex v that is adjacent to x and y. Here, $G[\{x, v\}]$ and $G[\{v, y\}]$ are blocks of G and so $\{x, v, y\}$ is $\tau_B(G)$-open. Similarly, $f^{-1}(\{x, v, y\}$ cannot be expressed as a union of vertex sets of blocks in P_n and a subset of $\mathscr{C}(P_n)$. This means that $f^{-1}(\{x, v, y\}$ is not $\tau_B(P_n)$-open. But f is continuous; hence a contradiction. $\square$

4. The block topological graph

Definition 2. *Let $(V(G), \tau_B(G))$ be block topological space where $\tau_B(G)$ is not the indiscrete topology on $V(G)$. A **block topological graph** of $(V(G), \tau_B(G))$ is a graph $G_{\tau_B(G)}$ with vertex set $V(G_{\tau_B(G)}) = \tau_B(G) \smallsetminus \{\varnothing, V(G)\}$ and edge set $E(G_{\tau_B(G)}) = \{AB : A \subseteq B, A, B \in V(G_{\tau_B(G)})\}$.*

Example 4. The corresponding block topological graph of G in Figure 7 is shown in Figure 10.

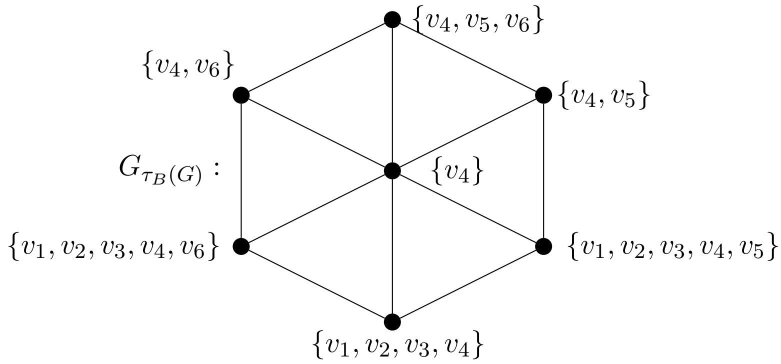

Figure 10: Block topological graph of G

Remark 4. *Let $G_{\tau_B(G)}$ be a block topological graph. Then $|V(G_{\tau_B(G)})| = |\tau_B(G)| - 2$.*

A connected graph containing no cut-vertices has no corresponding block topological graph. Recall that a block in a graph is not a subgraph to any other block in a graph. Hence, a block topological graph is never trivial nor complete.

5. Concluding remarks

The notion of block topological space induced by undirected simple graphs has been successfully introduced in this paper together with some important characterizations and special attributes of the resulting block topological space. Here, the authors presented an initial idea of the corresponding block topological graph. One definite extension of this research is the study of the block topological space and the block topological graph induced by special families of graphs and those graphs resulting from unary and binary operations that the authors had already started working on. Meanwhile, some possible and interesting direction for further study is on extending the idea of the block topology of a graph to various topological structures such as soft bitopological spaces [13], soft topological subspaces [12] and [14], fuzzy topological space [15], regular spaces, normal spaces, and completely regular spaces [9] or perhaps in looking into the block topology of a directed graph using the method of Hassan and Abed in [7].

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