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Two- dimensional coupled asymmetric van der Pol oscillator

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Abstract. A reliable algorithm based on an adaptation of the standard differential transform method (DTM) is presented, which is the multi-step differential transform method (Ms-DTM) since it may be difficult to directly apply differential transform method (DTM) to obtain the series solutions for the present two- dimensional coupled asymmetric van der Pol oscillator. The solutions of a two- dimensional coupled asymmetric van der Pol oscillator were obtained by Ms-DTM. Figurative comparisons between the Ms-DTM and the classical fourth order Runge-Kutta method (RK4) are given. The obtained results reveal that the proposed technique is a promising tool to solve the considered van der Pol oscillator and yield same information on the phase portrait confirming the stability of the system, effectively. It can be said that the considered approach can be easily extended to other nonlinear van der Pol oscillator systems and therefore is widely applicable in engineering and other sciences.

2020 Mathematics Subject Classifications: 34A45, 34C05, 65L05, 34C15, 34C25

Key Words and Phrases: Coupled van der Pol, two- dimension, phase portrait, semi analytical

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1. Introduction

The study of nonlinear oscillations is a crucial area of research in various fields of mechanical structures, physical science, and other mathematical sciences. Since most real-world systems are described by nonlinear differential equations, mechanical systems,

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mathematical physics, and engineering face several difficulties. Understanding the characteristics and methods for resolving nonlinear equations in mechanical systems has drawn more attention in recent years, with the aim of developing more accurate and efficient methods for analyzing and predicting their behavior. Nonlinear oscillator problems have recently been solved using a variety of analytical and numerical techniques, including homotopy perturbation method [7 − −10], energy balance method [2, 16, 27], frequency-amplitude formulation [5, 12, 13], parameter expanding method [11, 25, 26], and variational iteration method [14, 15], and recently accurate technique is used to find an approximate solution to the fractional-order Duffing-Van der Pol oscillators [18].

In chaos theory, the van der Pol equation is a typical reference. Early on in the model's creation, researchers in the biological and physical sciences made use of it. This formula is considered to be an example of an oscillator that damps nonlinearly. Several authors and scholars set out to find an analytical answer to the van der pol dilemma. The first person to simulate electrical circuits in a vacuum was Van der Pol [6]. The equation is often researched as a template for testing numerical techniques and for its extensive collection of chaotic solutions [3]. Deriving the closed-form solutions of a nonlinear differential equation lacks a general framework. As a result, there has been a growing emphasis on numerical techniques to get solutions (some of which are provided in [19]), perturbative approaches, approximated analytical approaches based on linearization [20], or adomian decomposition [17].

In order to examine the oscillatory behavior, the parameter is crucial. These parameters can be computed in a variety of ways [27]. Among of these techniques, the approximation Lie theorem was given by Baikov [5, 12, 13]. For the perturbed ODEs, Naeem and Mahomed [11, 26] proposed the partial Lagrangian technique [14, 25]. Many research works have been developed for the systems of approximation Hamiltonians.

The mathematical model for the system is a well- known second order ordinary differential equation with cubic nonlinearity – the Van der Pol equation. Since then, thousands of papers have been published achieving better approximations to the solutions occurring in such non- linear systems. The Van der Pol oscillator is a classic example of self-oscillatory system and is now considered as very useful mathematical model that can be used in much more complicated and modified systems. The Van der pol equation is so important to mathematicians, physicists and engineers and is still being extensively studied [4, 21 − −24]. Since its introduction in the 1920's, the Van der Pol equation has been a prototype for systems with self-excited limit cycle oscillations. The classical experimental setup of the system is the oscillator with vacuum triode. The investigations of the forced Van der Pol oscillator behaviour have carried out by many researchers. The equation has been studied over wide parameter regimes, from perturbations of harmonic motion to relaxation oscillations [1].

Let us discuss the exact solution of a Van der Pol like equation in two dimensions [8] as:

$$x'' + \omega^2 x + \varepsilon \left(\alpha x^2 + \beta y^2 - 1\right) x' = 0 \tag{1}$$

$$y'' + \omega^2 y - \varepsilon \left(\alpha x^2 + \beta y^2 - 1\right) y' = 0 \tag{2}$$

For $\alpha = \beta = 1$, the above equations reduce to:

$$x'' + \omega^2 x + \varepsilon \left(x^2 + y^2 - 1\right) x' = 0. \tag{3}$$

$$y'' + \omega^2 y - \varepsilon \left(x^2 + y^2 - 1\right) y' = 0. \tag{4}$$

Using $x = A \sin \omega t$ and $y = A \cos \omega t$, we have

$$x'' + \omega^2 x = 0. \tag{5}$$

$$y'' + \omega^2 y = 0. \tag{6}$$

The solution of (5&6) remains the same as our assumptions.

However, for the case $\alpha \neq \beta \neq 1$ one has to use a specific analysis as given section 2 below.

2. Strong Coupling Limit for $\omega = 1, \alpha \neq \beta$, and $\varepsilon = 1$

In this section, we present some numerical analysis for the system given in (1 & 2) for the following case

Consider $\alpha = 1.546, \beta = 1.551, \omega = 1$ with $\varepsilon = 1$. In this case system (1&2) become:

$$x'' + x + \left(1.546x^2 + 1.551y^2 - 1\right) x' = 0. \tag{7}$$

$$y'' + y - \left(1.546x^2 + 1.551y^2 - 1\right) y' = 0. \tag{8}$$

The above system is a coupled nonlinear one. Numerical technique has to be used to solve it. Below we use fourth- order Runge- Kutta technique to simulate $x(t), y(t), x'(t)$ and $y'(t)$ vs time as shown in Figure 1 (a- c).

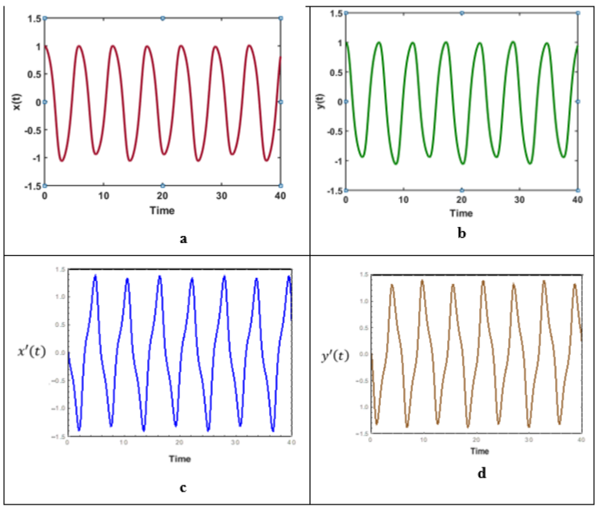
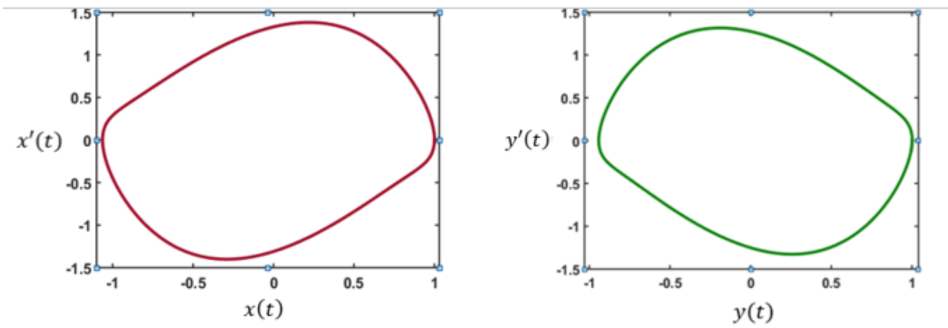


Figure 1: Simulation of system (7 & 8) using forth- order Runge- Kutta method (a) $x(t)$ vs time, (b) $y(t)$ vs time (c) $x'(t)$ vs time, and (d) $y'(t)$ vs time

The time-series of the dynamical variables ($x(t)$ and $y(t)$) is shown in figure 1 (a and b). As one can see, these solutions are periodic. It may be noted that the time evolution of the dynamical variables ($x(t)$ and $y(t)$) with the same initial conditions nearly show similar oscillatory behaviour, there are also minute changes in amplitudes. The same conclusion can be drawn for figure 1 (c and d) for the dynamical variables ($x'(t)$ and $y'(t)$).

Furthermore, In Figure 2(a& b) the phase portrait of $x'(t)$vs$x(t)$, and $y'(t)$ vs $y(t)$ has been plotted. These phase portraits are closed and show nearly rectangular shape. This implies that the suggested system a stable one.



a **b**

Figure 2: Phase portrait of system (7 & 8): (a) $x'(t)$ vs $x(t)$, and (b) $y'(t)$ vs $y(t)$.

3. The Ms- DTM method

The multi-step differential transform technique, or Ms-DTM for short, is the method used to solve ODEs numerically, and it is the sole topic of this section. Let $[0, T]$ be the interval for the nonlinear initial value issue, where a finite series may be used to represent $f\left(t, x, x', \ldots, x^{(r)}\right) = 0$

$$x(t) = \sum_{n=0}^{N} a_n t^n, \quad t \in [0, T] \tag{9}$$

with an initial condition $x^{(r)}(0) = c_k$ for $k = 0, 1, \ldots, r-1$.

Using the nodes $t_m = mh$, we suppose that interval $[0, T]$ is split into M subintervals $[t_{m-1}, t_m]$ with $m = 1, 2, \ldots, M$ and step size $h = T/M$. Ms-DTM is a method for computational performance for values of h. The main idea of Ms-DTM is we first apply the differential transform method(in short, DTM) to ODE with initial value problem

$f\left(t, x, x', \ldots, x^{(r)}\right)=0$ over the interval $[0, t_1]$. For initial condition $x_1^{(k)}(0)=c_k$, following approximate solution denoted by

$$x_1(t)=\sum_{n=0}^{K} a_{1n} t^n, \quad t \in [0, t_1]. \tag{10}$$

For $m \geq 2$ and subinterval $[t_{m-1}, t_m]$ with initial conditions $x_m^{(k)}(t_{m-1})=x_{m-1}^{(k)}(t_{m-1})$, we apply the DTM to $f\left(t, x, x', \ldots, x^{(r)}\right)=0$, where t_0 in $Y(k)=\frac{1}{k!}\frac{d^k y(t)}{dt^k}\Big]_{t=t_0}$, which is called the differential transformation(DT, in short) of a function $y(t)$ is replaced by t_{m-1}. The process is repeated and generates a sequence of approximate solutions $x_m(t), m=1,2,\ldots, M$ with $N=K.M$ for the solution $x(t)$ and denoted by

$$x_m(t)=\sum_{n=0}^{K} a_{mn}\left(t-t_{m-1}\right)^n, \quad t \in [t_m, t_{m+1}]. \tag{11}$$

Finally, the Ms-DTM assumes the following solution denoted by $x(t)$

$$x(t)=\begin{cases} x_1(t), t \in [0, t_1] \\ x_2(t), t \in [t_1, t_2] \\ . \\ . \\ . \\ x_m(t), t \in [t_{M-1}, t_M]. \end{cases} \tag{12}$$

4. Solution of the system via Ms-DTM

In this section, we apply the Ms-DTM to Eqs. (7)-(8) to demonstrate the effectiveness of Ms-DTM as an approximate tool for solving the ODE. Let $x=u_1, x'=u_2, y=u_3$ and $y'=u_4$. Substituting back into the ODE (keeping in mind that $x''=u_2'$ and $y''=u_4'$) we get:

$$u_2'+u_2\left(1.546 u_1^2+1.551 u_3^2-1\right)+u_1=0. \tag{13}$$

and

$$u_3'-u_4\left(1.546 u_1^2+1.551 u_3^2-1\right)+u_3=0. \tag{14}$$

Thus Eqs. (7)-(8) can be expressed in the form of four simultaneous first-order differential equations in terms of u_1, u_2, u_3 and u_4, i.e.

$$\begin{aligned} & u_1'=u_2, \\ & u_2'=-u_2\left(1.546 u_1^2+1.551 u_3^2-1\right)-u_1 \\ & u_3'=u_4 \\ & u_4'=u_4\left(1.546 u_1^2+1.551 u_3^2-1\right)-u_3. \end{aligned} \tag{15}$$

Taking the DT of Eq. (15) by using the well-known differential transform formulas, we obtain

$$
\begin{aligned}
&U_1(k+1)=\frac{1}{k+1}U_2(k),\\
&U_2(k+1)=\frac{1}{k+1}\\
&\left(-1.546\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U_2(k_1)U_1(k_2-k_1)U_1(k-k_2)-1.551\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U_2(k_1)U_3(k_2-k_1)U_3(k-k_2)+U_2(k)-U_1(k)\right)\\
&U_3(k+1)=\frac{1}{k+1}U_4(k)\\
&U_4(k+1)=\frac{1}{k+1},\\
&\left(1.546\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U_4(k_1)U_1(k_2-k_1)U_1(k-k_2)+1.551\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U_4(k_1)U_3(k_2-k_1)U_3(k-k_2)-U_4(k)-U_3(k)\right)
\end{aligned}
\tag{16}
$$

where $U_1(k), U_2(k), U_3(k)$ and $U_4(k)$ are the differential transforms of u_1, u_2, u_3 and u_4, respectively. The DT of the initial conditions are given by $U_1(0) = 1, U_2(0) = 0, U_3(0) = 1$, and $U_4(0) = 0$. In view of $y(t) = \sum_{k=0}^{N} Y(k)\,(t-t_0)^n$, which is called the differential inverse transform, DTM series solution for Eqs. (7)-(8) can be obtained as,

$$
\begin{aligned}
u_1(t) &= \sum_{n=0}^{N} U_1(n)t^n,\\
u_2(t) &= \sum_{n=0}^{N} U_2(n)t^n,\\
u_3(t) &= \sum_{n=0}^{N} U_3(n)t^n,\\
u_4(t) &= \sum_{n=0}^{N} U_4(n)t^n.
\end{aligned}
\tag{17}
$$

Now, according to the Ms-DTM, taking $N = K.M$, the series solution for Eqs. (7)-(8) is given by,

$$
u_1(t)=\begin{cases}
\sum_{n=0}^{K} U_{11}(n)t^n, t\in[0,t_1]\\
\sum_{n=0}^{K} U_{21}(n)t^n, t\in[t_1,t_2]\\
\cdot\\
\cdot\\
\sum_{n=0}^{K} U_{M1}(n)t^n, t\in[t_{M-1},t_M]
\end{cases}
\tag{18}
$$

$$u_2(t)=\begin{cases}\sum_{n=0}^{K}U_{12}(n)t^n, t\in[0,t_1]\\ \sum_{n=0}^{K}U_{22}(n)t^n, t\in[t_1,t_2]\\ .\\ .\\ \sum_{n=0}^{K}U_{M2}(n)t^n, t\in[t_{M-1},t_M]\end{cases} \tag{19}$$

$$u_3(t)=\begin{cases}\sum_{n=0}^{K}U_{13}(n)t^n, t\in[0,t_1]\\ \sum_{n=0}^{K}U_{23}(n)t^n, t\in[t_1,t_2]\\ \vdots\\ .\\ \sum_{n=0}^{K}U_{M3}(n)t^n, t\in[t_{M-1},t_M]\end{cases} \tag{20}$$

and

$$u_4(t)=\begin{cases}\sum_{n=0}^{k}U_{14}(n)t^n, t\in[0,t_1]\\ \sum_{n=0}^{K}U_{24}(n)t^n, t\in[t_1,t_2]\\ \vdots\\ .\\ \sum_{n=0}^{K}U_{M4}(n)t^n, t\in[t_{M-1},t_M]\end{cases} \tag{21}$$

where U_{n1},U_{n2},U_{n3} and U_{n4}, for $n=1,2,\dots,M$, satisfy the following recurrence relations,

$$\begin{aligned}
&U_{n1}(k+1)=\frac{1}{k+1}U_{n2}(k),\\
&U_{n2}(k+1)=\frac{1}{k+1}\\
&\left(-1.546\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U_{n2}(k_1)U_{n1}(k_2-k_1)U_{n1}(k-k_2)-1.551\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U_{n2}(k_1)U_{n3}(k_2-k_1)U_{n3}(k-k_2)+U_{n2}(k)-U_{n1}(k)\right)\\
&U_{n3}(k+1)=\frac{1}{k+1}U_{n4}(k),\\
&U_4(k+1)=\frac{1}{k+1}\\
&\left(1.546\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U_{n4}(k_1)U_{n1}(k_2-k_1)U_{n1}(k-k_2)+1.551\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U_{n4}(k_1)U_{n3}(k_2-k_1)U_{n3}(k-k_2)-U_{n4}(k)-U_{n3}(k)\right)
\end{aligned} \tag{22}$$

such that $U_{n1}(0)=U_{(n-1)1}(0),U_{n2}(0)=U_{(n-1)2}(0),U_{n3}(0)=U_{(n-1)3}(0)$ and $U_{n4}(0)=U_{(n-1)4}(0)$.

Finally, if we start with $U_{01}(0) = 1, U_{02}(0) = 0, U_{03}(0) = 1$ and $U_{04}(0) = 0$, using the recurrence relations given in Eq. (11), then we can obtain the Ms-DTM solution given in Eqs. (7), (8), (9) and (10). Figure 3 shows the phase-portrait as obtained from the approximate solutions for Eqs. (7)-(8) obtained using Ms-DTM. From Figure 2 (a-b) and Figure 3 (a-b), we can observe that the phase-portrait trajectories obtained using Ms-DTM are in high agreement with the phase-portrait trajectories obtained using fourth-order Runge- Kutta method.

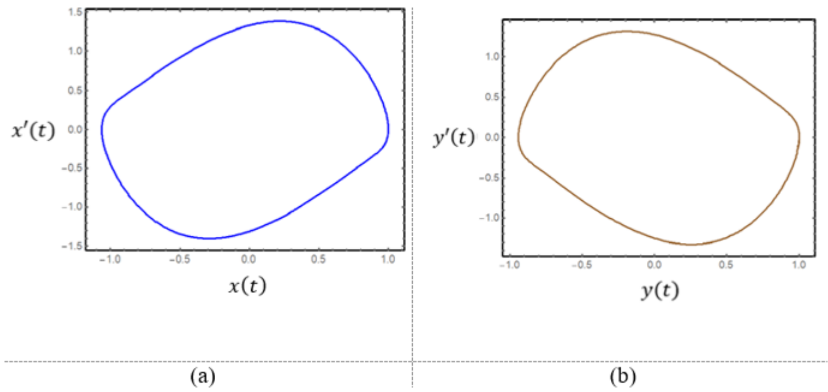


Figure 3: Phase portrait of system (7 & 8): (a) $x(t)$ vs $x'(t)$, and (b) $y(t)$ vs $y'(t)$ using Ms- DTM approach.

Figure 4 (a-d) below show the approximate solutions for Eqs. (7)-(8) obtained using Ms-DTM. One can see that there is a high agreement with the time-series of the dynamical variables obtained using fourth- order Runge- Kutta method shown in figure 1 (a-d).

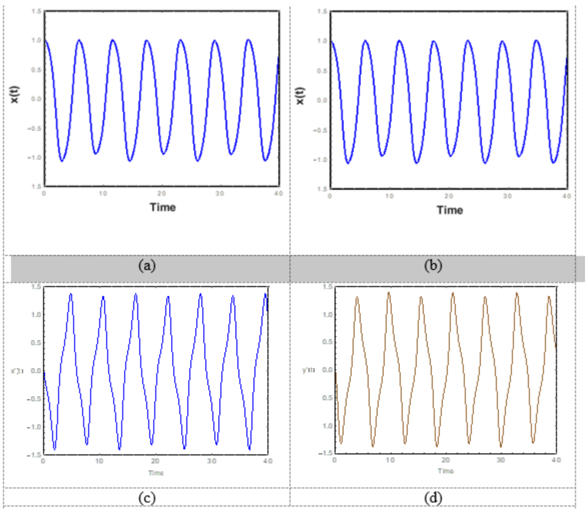


Figure 4: Simulation of system (7 & 8) using Ms-DTM (a) $x(t)$ vs time, (b) $y(t)$ vs time (c) $x'(t)$ vs time, and (d) $y'(t)$ vs time

5. Conclusion

Van der Pol oscillator is one of the paradigms of nonlinear dynamics and a common model for nonlinear phenomena in science and engineering. In this study, an algorithm for solving a two-dimensional coupled van der Pol like oscillator was introduced via Ms-DTM. A reliable accuracy solution was obtained via this algorithm. The considered two-dimensional coupled asymmetric van der Pol oscillator are discussed and focused it's quasi analytical solution of a coupled two-dimensional oscillator in strong coupling regime $\omega = \varepsilon = 1, \alpha \neq \beta$. For $\alpha = \beta = 1$, solution is a sinusoidal nature having elliptical nature phase portrait. However, here $\alpha \neq \beta \neq 1$ we get nearly rectangular type in both the cases, rectangular nature is asymmetric. Similarly, we also plot displacement vs time graph $(x \sim t, y \sim t)$ and velocity vs time graph $(x' \sim t, y' \sim t,\ )$We believe present two-dimensional van der Pol model will generate interest among people to look into coupling regime. The comparison between Ms-DTM solution and the fourth order Runge-Kutta method are discussed and given graphically. The solution via Ms-DTM is continuous on this domain and analytical at each subdomain. In this work all calculations are made by Mathematica Software.

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