EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 2, 2024, 1294-1305
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Determination of the Fixed Point of Lotka-Volterra Function

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Abstract. The paper focuses on the Lotka-Volterra function in its discrete form. The purpose of the study was to determine the fixed points of the function. The study employs the Banach Fixed Point Theorem and Contraction Mapping in Metric Space on the function to demonstrate the uniqueness of the fixed points and its continuous stability after several iterations, using the fixed points as the initial conditions. The study has shown that $(0,0)$, $\left(0, \frac{\alpha-1}{\beta}\right)$, $\left(\frac{1+\gamma}{\delta}, 0\right)$ and $\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$are the fixed points of the function, with the initial pair serving as a trivial one and the other three solely depending on the parameter values for the behavior of the function. The outcome of the limit points of the function as the fixed points after several iterations forms a fixed orbit structure of the function, irrespective of the value of the parameter. The study also showed the uniqueness of the fixed points, demonstrating the stability and continuity of the function in its steady state.

2020 Mathematics Subject Classifications: 54H25, 47H10, 54E35

Key Words and Phrases: Lotka-Volterra, fixed point, parameter, solution, function,contraction mapping, fixed point theorem

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1. Introduction

Many researchers have studied about fixed point theorem. All their definitions seem to have one idea. That is fixed point theorem is where each fixed point of a function G must exist, $x \in X$ such that $G(x) = x$. [9] wrote a brief historical survey on fixed point theorem. Many papers were cited in that paper. Many interesting results on fixed point theorem were also given like generalization and extension of fixed point theorem through

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i2.5155

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different types of mappings. Suppose T represent a self-map on set X. A fixed point of the mapping T is referred to as an element x in X such that $T_x = x$. One of the most important theorems in fixed point theorem is the L.E. Brouwer's fixed point theorem in which it is said that each continual self-mapping of the sealed unit in the $n-$ dimensional Euclidean spaces R_n, possess a fixed point[5]. Advanced fixed point theorem for internal mapping by employing a known Ky Fan type outcome in a Hilbert space setting.[10] For two species (prey and predators) to exist, there is an equation which model the struggle. This model was brought up by two scientists: Lotka and Volterra.The Scientists came to a conclusion based on the problem they had in 1920 by Lotka and 1926 by Volterra. The conclusion was the same, that the interaction of the two species would give rise to periodic Oscillation in their populations.[2]
Many works have been done on the fixed-point theorem and the Lotka Volterra function, but our focus is on the determination of the fixed point of the Lotka Volterra function, applying the Banach fixed point theorem, and the contraction mapping on the Lotka Volterra function to find the fixed point of the Lotka Volterra function.

2. Preliminaries

Definition 1. *A fixed point is a point that remains the same after applying a map system of differential equations, etc. A point x_0 is referred to as a function's fixed point, if $g(x_0) = x_0$, [11].*

Definition 2. *The fixed point theorem in general term is stated as an outcome of a function having at least a fixed point under a certain condition [11]*

Definition 3. *Let U be a non-empty set. Then, the real function d (distance function) that assign any ordered pair $d(u,v)$ of element u, v and $w \in U$ is a metric space, if the following properties are satisfied:*

1. $d(u,v) \geq 0$ *and* $d(u,v) = 0$ *if and only if* $u = v$
2. $d(u,v) = d(v,u)$
3. $d(u,v) + d(v,w) \geq d(u,w)$.
   A function d satisfying the conditions $(1) - (3)$ is a metric on U[8]

Definition 4. *A dynamic system is one in which a function explains how a point's relationship to time changes with a given environment. Examples are the mathematical formulas that explain how water moves through a pipe, how many fish spawn in a lake each springtime etc.[4]*

Definition 5. *A steady state of $\dot{x} = f(x)$ is a point $x \in U$ as to which $f(x) = 0$.*
A steady state x^* is said to be Lyapunov stable if for any $\varepsilon > 0$, there exist $\delta > 0$ this way for all x_0 with $|x^* - x_0| < \delta$ we have $|\varphi(x_0,\ t) - x^*| < \varepsilon$ for all $t \geq 0$, [3].

Definition 6. *An equilibrium x is said to be globally asymptotically stable in the set of all positive solutions.[7]*

Definition 7 (Lotka-Volterra Equations).

$$X_{n+1} = \alpha x_n - \beta x_n y_n \tag{1}$$
$$Y_{n+1} = \delta x_n y_n - \gamma y_n \tag{2}$$

Where $n = 0, 1, 2, \ldots, x$ is the quantity of preys, y is the quantity of predators, x_{n+1} and y_{n+1} are the two population growth rates and $\alpha, \beta, \delta, \gamma$ are positive real parameters defining the interaction between the two species [2].

Definition 8 (Population Equilibrium). *The model reaches population equilibrium when none of the population levels is shifting. When both derivatives equal* 0*, that is when it occurs.*

$$x_n\left(\alpha - \beta y_n\right) = 0 \tag{3}$$
$$y_n\left(\gamma - \delta x_n\right) = 0 \tag{4}$$

Hence, there are two solutions to the equations or the systems.
$\{y_n = 0,\ x_n = 0\}$ *and* $\left\{y_n = \frac{\alpha}{\beta} \quad x_n = \frac{\gamma}{\delta}\right\}$*, Hence, there are two equilibria.[7]*

Theorem 1 (Banach Contraction Principle). *Suppose*(P, d) *is a complete metric space and* $H : P \to P$ *is a mapping of contractions using the Lipschitz constant* $k < 1$*. Then, the fixed point* $\omega \in P$*, for all* $x \in P$ *is a unique point in* H*. That is;* $\lim_{n_+\infty} H^n(x) = \omega$*.Moreover, for each* $x \in P$*, we have* $d(H^n(x), \omega) \leq \frac{k^n}{1-k} d(H(x), x)$ *. [1]*

Theorem 2. *Given* (P, d) *as a complete metric space, and a mapping* $H : P \to P$ *for which* H^N *is a contraction mapping for* $N \geq 1$*. As a result,* H *has a distinct fixed point. In general, it is unclear if* H *has a fixed point whenever* H^N *has a fixed point. The term "periodic points of* H*" also applies to fixed point of* H^N*.[1].*

3. Main Work

Under this part we look for the main solutions of the Lotka Volterra function and also determine the fixed point of the function.

3.1. The zeros of the Lotka Volterra function

In this section, we look for the roots, or zeros, of the function. In determining the zeros or roots of the function, let $X_{n+1} = 0$, from equation (1), thus $X_{n+1} = \alpha x_n - \beta x_n y_n$ becomes

$$\alpha x_n - x_n \beta y_n = 0 \tag{5}$$
$$x_n(\alpha - \beta y_n) = 0 \tag{6}$$

It implies that $x_n = 0$ and $\alpha - \beta y_n = 0$ then $y_n = \frac{\alpha}{\beta}$
Hence, the root of $X_{n+1} = \alpha x_n - \beta x_n y_n$ thus (x_n, y_n) is $\left(0, \frac{\alpha}{\beta}\right)$

Also, let $y_{n+1} = 0$ then from equation (2) thus, $Y_{n+1} = \delta x_n y_n - \gamma y_n$ also becomes

$$\delta x_n y_n - \gamma y_n = 0 \tag{7}$$
$$y_n(\delta x_n - \gamma) = 0 \tag{8}$$

Then $y_n = 0$ and $\delta x_n - \gamma = 0$ implies $x_n = \frac{\gamma}{\delta}$
Similarly, the root of $Y_{n+1} = \delta x_n y_n - \gamma y_n$ thus (x_n, y_n) is $\left(\frac{\gamma}{\delta}, 0\right)$
Therefore, the roots are $(0,0)$, $\left(0, \frac{\alpha}{\beta}\right)$, $\left(\frac{\gamma}{\delta}, 0\right)$, and $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$

3.2. The solutions of the Lotka Volterra function

This section is mainly about the solutions of the function. Let

$$X_{n+1} = x_n \tag{9}$$
$$Y_{n+1} = y_n \tag{10}$$

Equating equation (1) and equation (9) becomes

$$\begin{aligned} &\alpha x_n - x_n \beta y_n = x_n \\ &\alpha x_n - x_n \beta y_n - x_n = 0 \\ &\text{implies} \quad x_n(\alpha - \beta y_n - 1) = 0 \end{aligned}$$

Then $x_n = 0$ and $\alpha - \beta y_n - 1 = 0$
$\beta y_n = \alpha - 1$
Therefore, $y_n = \frac{\alpha - 1}{\beta}$
Also equating equation (2) and equation (10) becomes

$$\delta x_n y_n - \gamma y_n = y_n \tag{11}$$

Then $\delta x_n y_n - \gamma y_n - y_n = 0$
$y_n(\delta x_n - \gamma - 1) = 0$
Then $y_n = 0$ and $\delta x_n - \gamma - 1 = 0$
$\delta x_n = 1 + \gamma$
therefore, $x_n = \frac{1+\gamma}{\delta}$
Hence, the solutions of the function are $\left(0, \frac{\alpha-1}{\beta}\right)$ and $\left(\frac{1+\gamma}{\delta}, 0\right)$

3.3. Determination of the fixed point of the Lotka Volterra function

In this section, we will consider the two definitions of the function. That is; $x_{n+1} = \alpha x_n - x_n \beta y_n$ and $y_{n+1} = \delta x_n y_n - \gamma y_n$ and then work out for the fixed point of the function using the solutions, $(0,0)$, $\left(0, \frac{\alpha-1}{\beta}\right)$, $\left(\frac{1+\gamma}{\delta}, 0\right)$, and $\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$. We then apply the idea and the definition of the theorem of fixed points and fixed point. That is; a function G is fixed point theorem having a minimum of one fixed point $x \in X$ such that $G(x) = x$. A point x_0 is the fixed point of a function $g(x)$, such that $g(x_0) = x_0$.

1) Determining the fixed point of $X_{n+1} = \alpha x_n - x_n \beta y_n$
   At $(0,0)$
   $X_{n+1} = \alpha(0) - (0)\beta(0)$
   Implies $X_{n+1} = 0 - 0$
   $X_{n+1} = 0$
   Hence, $(0,0)$ is one of the fixed points but trivial.
   Also, at
   $\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$
   $X_{n+1} = \alpha\left(\frac{1+\gamma}{\delta}\right) - \left(\frac{1+\gamma}{\delta}\right)\beta\left(\frac{\alpha-1}{\beta}\right)$
   implies $X_{n+1} = \left(\frac{\alpha+\alpha\gamma}{\delta}\right) - \left(\frac{1+\gamma}{\delta}\right)(\alpha-1)$
   $= \left(\frac{\alpha+\alpha\gamma}{\delta}\right) - \left[\left(\frac{\alpha+\alpha\gamma}{\delta}\right) - \left(\frac{1+\gamma}{\delta}\right)\right]$
   $= \left(\frac{\alpha+\alpha\gamma}{\delta}\right) - \left(\frac{\alpha+\alpha\gamma}{\delta}\right) + \left(\frac{1+\gamma}{\delta}\right)$
   $= \left(\frac{1+\gamma}{\delta}\right)$
   Again, $\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$ is also a fixed point but its existence will depend on the values of the parameter of the function.

2) Determining the fixed point of $Y_{n+1} = \delta x_n y_n - \gamma y_n$
   Then at $(0,0)$
   $Y_{n+1} = \delta(0)(0) - \gamma(0)$
   $Y_{n+1} = 0 - 0$
   $Y_{n+1} = 0$

   Also, at $\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$
   $Y_{n+1} = \delta\left(\frac{1+\gamma}{\delta}\right)\left(\frac{\alpha-1}{\beta}\right) - \gamma\left(\frac{\alpha-1}{\beta}\right)$
   implies $Y_{n+1} = (1+\gamma)\left(\frac{\alpha-1}{\beta}\right) - \gamma\left(\frac{\alpha-1}{\beta}\right)$
   $= (1+\gamma)\left(\frac{\alpha-1}{\beta}\right) - \gamma\left(\frac{\alpha-1}{\beta}\right)$
   $= \left(\frac{\alpha-1}{\beta}\right) + \left(\frac{\gamma\alpha-\gamma}{\beta}\right) - \left(\frac{\gamma\alpha-\gamma}{\beta}\right)$
   $= \left(\frac{\alpha-1}{\beta}\right)$
   Hence, $(0,0)$, $\left(0, \frac{\alpha-1}{\beta}\right)$, $\left(\frac{1+\gamma}{\delta}, 0\right)$ and $\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$ are the fixed points of the function.

Final Results

In this section we impose the contraction mapping on the Lotka Volterra to see the outcome of its behaviour[6].

Definition 9 (Contraction Mapping in Metric Space)**.** *Given (M,d) a metric space, a function $T : M \to M$ is said to be a contraction mapping if there is a constant a constant*

q with $q < 1$ such that for all $x, y \in M$

$$d(T(x), T(y)) \leq q \cdot d(x, y)$$

Applying the Banach fixed point theorem and the contraction mapping on the Lotka Volterra functions. Using Banach fixed point theorem to find the fixed point of Lotka Volterra functions.
From the definition of Banach fixed theorem, let $(M,\ d)$ be a complete metric space then every contraction has a unique fixed point.

If $T(x) = x, \quad T(y) = y$
then
$d(x, y) = d(T(x),\ T(y))$
$\leq q \cdot d(x, y)$
$q < 1$ so $d(x, y) = 0$ or $x = y$

To show that a fixed point exists, pick any $x \in M$. Setting $x \in x_0$, we define a sequence $\{x_i\}_{i \in z^+}$ by setting $x_{n+1} = T(x_n)$ $x_{n+1} = \alpha T\left(x_n\right) - \beta T\left(x_n\right)\left(y_n\right)$ $x_{n+1} = (\alpha - \beta(y_n))T(x_n)$
Rewriting the contraction formula we have

$$\begin{aligned}
&x_{n+1} = (\alpha - \beta(y_n))T(x_n)\\
&d(x_{n+2}, x_{n+1}) \leq (\alpha - \beta(y_n))qd(x_{n+1}, x_n)\\
&d(x_{n+2}, x_{n+1}) \leq (\alpha - \beta y_n)qd(x_{n+1}, x_n)\\
&d\left(x_{n+1},\ x_n\right) \leq \left(\alpha - \beta y_n\right) q^n d\left(x_1,\ x_0\right)\\
&d\left(x_{n+1},\ x_n\right) \leq \alpha q^n d\left(x_1,\ x_0\right) - \beta q^n y_n d\left(x, x_0\right)\\
&\leq q^n[\alpha d\left(x_1,\ x_0\right) - \beta y_n d\left(x_1,\ x_0\right)]\\
&d(x_{n+1},\ x_n) \leq q^n[(\alpha - \beta)y_n d\left(x_1,\ x_0\right)]\\
&\text{Assuming } n < m\\
&d\left(x_m,\ x_n\right) \leq d\left(x_m,\ x_{m-1}\right) + d\left(x_{m-1},\ x_{m-2}\right) + \ldots + \left(x_{n+1},\ x_n\right)\\
&d\left(x_m,\ x_n\right) \leq \left(q^{m-n-1} +\ q^{m-n-2} + \ldots + q + 1\right) d\left(x_{n+1}, x_n\right)\\
&\leq \left(\frac{1 - q^{m-n}}{1 - q}\right)\ d\left(x_{n+1},\ x_n\right)\\
&\leq \left(\frac{1 - q^{m-n}}{1 - q}\right) q^n\left(\alpha - \beta y_n\right) d\left(x_1,\ x_0\right)\\
&\text{since} \quad q^{m=n} < 1\\
&d(x_m, x_n) < \frac{q^n}{1 - q}(\alpha - \beta y_n)d(x_1, x_0)\\
&\text{Thus} \quad \{x_i\} \quad \text{is Cauchy.}\\
&\text{This shows that} \quad (x_n) \quad \text{is Cauchy sequence in X.}\\
&\text{Hence,} \quad (x_n) \text{ must be convergent, say } \lim_{n \to +\infty} x_n = x
\end{aligned}$$

Since T is continuous, we have $Tx = T\left(\lim_{n\to+\infty} x_n\right)$
$= \lim_{n\to+\infty} T(x_n)$
$= \lim_{n\to+\infty} x_{n+1}$
Since the limit of x_{n+1} is the same as that of (x_n)
Thus, x is a fixed point of T.

ILLUSTRATION 1

$X_{n+1} = g(x_n, y_n) = \alpha x_n - x_n\beta y_n$
Then, by considering the coordinate of y, that is $y_n = \dfrac{\alpha-1}{\beta}$
$\lim_{y_n\to\frac{\alpha-1}{\beta}} g(x_n, y_n) = \lim_{y_n\to\frac{\alpha-1}{\beta}} \beta(\alpha x_n - x_n\beta y_n)$
$= x_n \lim_{y_n\to\frac{\alpha-1}{\beta}} (\alpha - \beta y_n)$ where $n = 0, 1, 2, \ldots$
implies $\lim_{y_n\to\frac{\alpha-1}{\beta}} g(x_n, y_n) = x_n\left[\lim_{y_n\to\frac{\alpha-1}{\beta}} (\alpha) - \lim_{y_n\to\frac{\alpha-1}{\beta}} \beta y_n\right]$
$= x_n[\alpha - \beta(\alpha-1)]$
$= x_n \times 1$
$= x_n$
Hence, $\lim_{y_n\to\frac{\alpha-1}{\beta}} X_{n+1} = \lim_{y_n\to\frac{\alpha-1}{\beta}} g(x_n) = x_n$ irrespective of the values of the parameters.
This implies: when $n = 0, x_1 = g(x_0) = x_0$.
When $n = 1, X_2 = g(x_1) = x_1$.
When $n = 2, X_3 = g(x_2) = x_2$.
Hence, the orbit $\{x_0, x_1 = g(x_0) = x_0, x_2 = g(x_1) = x_1, x_3 = g(x_2) = x_2, \ldots\}$

Serve as the fixed points of the function since it is a repeated point that is unique and asymptotically stable throughout the iteration process.

Example 1. *Given* $f(x_n, y_n) = \alpha x_n - x_n\beta y_n$. *Let* $\gamma > 1$, $\delta > 1$, $\alpha > 1$, *and* $\beta > 1$. *At* $\gamma = 1.1$, $\delta = 2.1$, $\alpha = 1.2$, *and* $\beta = 2.5$.

$\left(\dfrac{1+\gamma}{\delta}, \dfrac{\alpha-1}{\beta}\right)$ *implies* (x_n, y_n)
Then, $\lim_{y_n\to\frac{\alpha-1}{\beta}} f(x_n, y_n) = \lim_{y_n\to\frac{\alpha-1}{\beta}} (\alpha x_n - x_n\beta y_n)$

$$At\ \alpha = 1.2, \quad \beta = 2.5$$
$$Implies \lim_{y_n \to \frac{\alpha-1}{\beta}} f(x_n, y_n) = x_n \lim_{y_n \to \frac{\alpha-1}{\beta}} (1.2 - 2.5y_n)$$
$$When\ n = 0, x_0 = \frac{1+\gamma}{\delta} = \frac{1+1.1}{2.1} = 1,$$
$$y_0 = \frac{\alpha - 1}{\beta} = \frac{1.2-1}{2.5} = 0.08$$
$$Implies\ (x_0, y_0) = (1, 0.08)$$
$$impliesX_1 = \lim_{y_n \to \frac{\alpha-1}{\beta}} f(x_0, y_0) = \lim_{y_0 \to 0.08} f(x_0, y_0) = \lim_{y_0 \to 0.08} f(1, 0.08)$$
$$= x_0 \lim_{y_0 \to 0.08} (1.2 - 2.5y_0)$$
$$= 1[1.2 - 2.5(0.08)]$$
$$This\ implies \quad X_1 = x_0 = 1$$

Now for X_2, we iterate the function again using $x_1 = 1$ and use different values for the parameters for y coordinate, that is;$\alpha = 10, \beta = 12$ at $n = 1$

$$\text{That is} \lim_{y_1 \to \frac{\alpha-1}{\beta}} f(x_1, y_1) = x_1 \lim_{y \to \frac{\alpha-1}{\beta}} (10 - 12y_1).$$
$$\text{where} \quad y_1 = \frac{10-1}{12} = \frac{9}{12} = 0.75.$$
$$\text{Hence} \quad (x_1, y_1) = (1, 0.75)$$
$$\text{Therefore} \quad X_2 = \lim_{y_n \to \frac{\alpha-1}{\beta}} f(x, y_1) = \lim_{y_1 \to 0.75} f(x_1, y_1).$$
$$= x_1 \lim_{y_1 \to 0.75} (10 - 12y_1)$$
$$= 1[10 - 12(0.75)]$$
$$\text{This implies} \quad X_2 = x_1 = 1$$

The iteration process so far indicates the limit of x_{n+1} is the same as that of (x_n) and keeps

repeating itself. Hence, $x_n = \dfrac{1+\gamma}{\delta}$ is a fixed point of the function.
Which implies that
when $n = 0, X_1 = g(x_0) = x_0$
when $n = 1, X_2 = g(x_1) = x_1$
when $n = 2, X_3 = g(x_2) = x_2$
$\vdots$
when $n = k, X_{k+1} = g(x_k) = x_k$

Considering equation (12), that is $y_{n+1} = \delta x_n T(y_n) - \gamma T(y_n)$

We pick any $y \in \mathcal{M}$ setting $y = y_0$, we define a sequence $\{y_i\}_{i\in\mathbb{Z}^+}$ by setting $y_{n+1} = T(y_n)$

$y_{n+1} = \delta x_n T(y_n) - \gamma T(y_n)$

Rewriting the contraction formula we have $y_{n+1} = (\delta x_n - \gamma)T(y_n)$

$d(y_{n+2}, y_{n+1}) \leq (\delta x_n - \gamma)qd(y_{n+1}, y_n)$

or

$d(y_{n+1}, y_n) \leq (\delta x_n - \gamma)q^n d(y_1, y_0)$

$d(y_{n+1}, y_n) \leq q^n(\delta x_n - \gamma)d(y_1, y_0)$

Assuming $n < m$

$d(y_m, y_n) \leq d(y_m, y_{m-1}) + d(y_{m-1}, y_{m-2}) + \ldots + d(y_{n+1}, y_n)$

$d(y_m, y_n) \leq (q^{m-n-1} + q^{m-n-2} + \ldots + q + 1)d(y_{n+1}, y_n)$

$d(y_m, y_n) \leq \dfrac{1 - q^{m-n}}{1 - q} d(y_{n+1}, y_n)$

since $q^{m-n} < 1$

$d(y_m, y_n) < \dfrac{q^n}{1 - q}(\delta x_n - \gamma)d(y_1, y_0)$

Thus $\{y_i\}$ is Cauchy.

This shows that y_n is a Cauchy sequence in $\mathcal{M}$

Hence, (y_n) must be convergent, say $\lim\limits_{n\to+\infty} y_n = y$

Since T is continuous, we have $Ty = T\left(\lim\limits_{n\to+\infty} y_n\right)$

$= \lim\limits_{n\to+\infty} Ty_n$

$= \lim\limits_{n\to+\infty} y_{n+1}$

$= y$

Since the limit of y_{n+1} is the same as that of y_n

Thus, y is a fixed point of T.

ILLUSTRATION 2

Similarly, let $Y_{n+1} = h(x_n, y_n) = \delta x_n y_n - \gamma y_n$

Then, by considering the coordinate of x, that is $x_n = \dfrac{1+\gamma}{\delta}$

$\lim\limits_{x_n \to \frac{1+\gamma}{\delta}} h(x_n, y_n) = \lim\limits_{x_n \to \frac{1+\gamma}{\delta}} (\delta x_n y_n - \gamma y_n)$

$= y_n \lim\limits_{x_n \to \frac{1+\gamma}{\delta}} (\delta x_n - \gamma)$ where $n = 0, 1, 2, \ldots$

$= \delta\left(\dfrac{1+\gamma}{\delta}\right) y_n - \gamma y_n$

$= y_n \left[\delta\left(\frac{1+\gamma}{\delta}\right) - \gamma\right]$
$= y_n[(1+\gamma) - \gamma]$
$= y_n \times (1)$
$= y_n$
Hence, $Y_{n+1} = h(y_n) = y_n$ irrespective of the values of the parameters. $whitewhitespaceere$
when $n = 0, Y_1 = h(y_0) = y_0$
when $n = 1, Y_2 = h(y_1) = y_1$
when $n = 2, Y_3 = h(y_2) = y_2$
This forms the orbit $\{y_0, y_1 = h(y_0) = y_0, y_2 = h(y_1) = y_1, y_3 = h(y_2) = y_2, \ldots\}$ of the function that are equilibrium in nature, asymptotically stable and continuous after several iterations.

Example 2.

Given $Y_{n+1} = h(x_n, y_n) = \delta x_n y_n - \gamma y_n$. *Let* $\gamma > 1, \delta > 1,$ *and* $\alpha > 1, \beta > 1$. *At* $\gamma = 1.1, \delta = 2.1, \alpha = 1.2, \beta = 2.5.$
$\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$ *implies* $(x_n, y_n))$
Then, taking the limit of the function Y_{n+1} *as* $x_n \to \frac{1+\gamma}{\delta}$ *for* $\gamma = 1.1, \delta = 2.1$
implies $\lim\limits_{x_n \to \frac{1+\gamma}{\delta}} h(x_n, y_n) = \lim\limits_{x_n \to \frac{1+\gamma}{\delta}} (\delta x_n y_n - \gamma y_n)$
$\lim\limits_{x_n \to \frac{1+\gamma}{\delta}} h(x_n, y_n) = \lim\limits_{x_n \to \frac{1+\gamma}{\delta}} (2.1 x_n y_n - 1.1 y_n)$
$= y_n \lim\limits_{x_n \to \frac{1+\gamma}{\delta}} (2.1 x_n - 1.1)$
Then for $x_n = \frac{1+\gamma}{\delta}$, *When* $n = 0,$ *implies* $x_0 = \frac{1+\gamma}{\delta} = \frac{1+1.1}{2.1} = 1,$
Also, for y_1 *implies* $Y_1 = \lim\limits_{x_o \to 1} h(x_0, y_0) = y_0 \lim\limits_{x_0 \to 1} (2.1 x_0 - 1.1)$
$= 0.008[2.1(1) - 1.1]$
$= 0.08$
Hence, $y_1 = y_0 = 0.08$

Now, using $y_1 = 0.08$ as the initial value for the next iteration Y_2 and taking different values for the parameters of x coordinate, that is; $\gamma = 10, \delta = 12$ at $n = 1$

$$\text{That is } \lim_{x_n \to \frac{1+\gamma}{\delta}} h(x_n, y_n) = \lim_{x_n \to \frac{1+\gamma}{\delta}} (12 x_n y_n - 10 y_n)$$

$$= y_n \lim_{x_n \to \frac{1+\gamma}{\delta}} (12x_n - 10)$$
$$x_1 = \frac{11}{12} = 0.92$$
$$(x_1, y_1) = (0.92, 0.08)$$
$$Y_2 = \lim_{x_1 \to 0.92} h(x_1, y_1) = y_1 \lim_{x_1 \to 0.92} (12x_1 - 10)$$
$$= 0.08[12(0.92) - 10]$$
$$= 0.08$$

Again the final value of $y_2 = y_1 = 0.08$
Hence, from the iteration process so far, the limit of Y_{n+1} is the same as that of (y_n) and keeps repeating itself. Thus,indicating that $y_n = \frac{\alpha-1}{\beta}$ is a fixed point of the function.

$$\begin{aligned}
&\text{Which implies that: when } n = 0, Y_1 = h(y_0) = y_0\\
&\text{when } n = 1, Y_2 = h(y_1) = y_1\\
&\text{when } n = 2, Y_3 = h(y_2) = y_2\\
&\vdots\\
&\text{when } n = k, Y_{k+1} = h(y_k) = y_k
\end{aligned}$$

Clearly, at a fixed value of $\gamma, \delta, \alpha, \beta$ for $h(x_n, y_n)$ and $f(x_n, y_n)$, i.e., $(\gamma = 1.1, \delta = 2.1, \alpha = 1.2, \beta = 2.5)$,The functions have fixed values, for instance, $(1, 0.08)$ irrespective of the number of successive iterations and the values for the parameters of the x and y coordinates. This indicates that the structure of the fixed orbits of the function is in equilibrium as it travels through time with a stable and continuous movement.

4. Conclusion

The Lotka-Volterra function has been studied, and it shows that the function has two sets of roots, or zeros. $(0, 0)$ and $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ where the latter depends on the parameters of the function. Again, there are four solutions of the function $(0, 0)$, $\left(0, \frac{\alpha-1}{\beta}\right)$, $\left(\frac{1+\gamma}{\delta}, 0\right)$ and $\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$, where $(0, 0)$ as a trivial solution always exits but the existence of $\left(\frac{1+\gamma}{\delta}, \frac{\alpha-1}{\beta}\right)$ depends on the parameters of the function.The study also shows that the solutions of the function are the fixed points of the function, and the limit points, as the fixed points of the function, are asymptotically stable and continuous after several iterations. The outcome of the fixed points after several iterations forms a fixed orbit structure of the function, irrespective of the value of the parameter. In addition, the uniqueness of the fixed points demonstrates the stability and continuity of the function in its steady state.

Conflicts of Interest

The authors declare no conflicts of interest and that authors are responsible for coauthors declaring their interests.

Data Availability

No data were used to support this study. Funding This study did not receive fund in any form.

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